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Addressing graph products and distance-regular graphs

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To the memory of David A. Gregory

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ABSTRACT

Graham and Pollak showed that the vertices of any connected graph G can be assigned t-tuples with entries in $\{0, a, b\}$, called addresses, such that the distance in G between any two vertices equals the number of positions in their addresses where one of the addresses equals a and the other equals b. In this paper, we are interested in determining the minimum value of such t for various families of graphs. We develop two ways to obtain this value for the Hamming graphs and present a lower bound for the triangular graphs.

1. Graph addressings

A t-*address* is a t-tuple with entries in $\{0, a, b\}$. An *addressing* of length t for a graph G is an assignment of t-addresses to the vertices of G so that the distance between two vertices is equal to the number of locations in the addresses at which one of the addresses equals a and the other address equals b. For example, we have a 3-addressing of a graph in Fig. 1. Graham and Pollak [13] introduced such addressings, using symbols $\{*, 0, 1\}$ instead of $\{0, a, b\}$, in the context of loop switching networks.

We are interested in the minimum t such that G has an addressing of length t. We denote such a minimum by $N(G)$. Graham and Pollak [13,14] showed that $N(G)$ equals the biclique partition number of the distance multigraph of G. Specifically, the *distance multigraph of* G, $\mathcal{D}(G)$, is the multigraph with the same vertex set as G where the multiplicity of any edge uv equals the distance between vertices u and v in G. The *biclique partition number* $\mathrm{bp}(H)$ of a multigraph H is the minimum number of complete bipartite subgraphs (bicliques) of H whose edges partition the edge set of H. This parameter and its covering variations have been studied by several researchers and appear in different contexts such as computational complexity or geometry (see for example, [8,13–16,19,21,22,25]). Graham and Pollak deduced that $N(G) \leq r(n-1)$ for any connected graph G of order n and diameter r and conjectured that $N(G) \leq n-1$ for any connected graph G of order n. This conjecture, also known as the *squashed cube conjecture*, was proved by Winkler [24].

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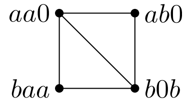

Fig. 1. A graph addressing of length 3.

To bound $N(G)$ below, Graham and Pollak used an eigenvalue argument on the adjacency matrix of $\mathcal{D}(G)$. Specifically, if M is a symmetric real matrix, let $n_+(M)$, $n_-(M)$, and $n_0(M)$ denote the number of eigenvalues of M (including multiplicity) that are positive, negative and zero, respectively. The *inertia* of M is the triple $(n_+(M), n_0(M), n_-(M))$. The adjacency matrix of $\mathcal{D}(G)$ will be denoted by $D(G)$; we will also refer to $D(G)$ as the *distance matrix* of G. The inertia of distance matrices has been studied by various authors for many classes of graphs [3,17,18,26]. Witsenhausen (cf. [13,14]) showed that

$$N(G) \geq \max\{n_+(D(G)), n_-(D(G))\}. \tag{1}$$

Letting J_n denote the all one $n \times n$ matrix and I_n denote the $n \times n$ identity matrix, and observing that $n_-(D(K_n)) = n_-(J_n - I_n)$, Graham and Pollak [13,14] used the bound (1) to conclude that

$$N(K_n) = n - 1. \tag{2}$$

Graham and Pollak [13,14] also determined $N(K_{n,m})$ for many values of n and m. The determination of $N(K_{n,m})$ for all values of n and m was completed by Fujii and Sawa [11]. A more general addressing scheme, allowing the addresses to contain more than two different nonzero symbols, was recently studied by Watanabe, Ishii and Sawa [23]. The parameter $N(G)$ has been determined when G is a tree or a cycle [14], as well as one particular triangular graph T_4 [25], described in Section 5. For the Petersen graph P, Elzinga, Gregory and Vander Meulen [10] showed that $N(P) = 6$. To the best of our knowledge, these are the only graphs G for which addressings of length $N(G)$ have been determined. We will say a t-addressing of G is *optimal* if $t = N(G)$. An addressing is *eigensharp* [19] if equality is achieved in (1).

In this paper, we study optimal addressings of Cartesian graph products and the distance-regular graphs known as triangular graphs. Let $H(n, q)$ be the Hamming graph whose vertices are the n-tuples over an alphabet with q letters with two n-tuples being adjacent if and only if their Hamming distance is 1. We give two different proofs showing that $N(H(n, q)) = n(q-1)$. This generalizes the Graham–Pollak result (2) since $H(1, q) = K_q$. We show that the triangular graphs are not eigensharp.

2. Addressing Cartesian products

Let $G = G_1 \square G_2 \square \cdots \square G_k$ denote the *Cartesian product* of graphs $G_1, G_2, \ldots, G_k$. Then G has vertex set $V(G) = \{(v_1, v_2, \ldots, v_k) \mid v_i \in V(G_i)\}$. Two vertices $v = (v_1, \ldots, v_k)$ and $u = (u_1, \ldots, u_k)$ of G are adjacent if for some index j, v_j is adjacent to u_j in G_j while $v_i = u_i$ for all remaining indices $i \neq j$. Thus, if d and d_i denote distances between pairs of vertices in G and G_i respectively, then for every $v, u \in V(G)$,

$$d(v, u) = \sum_{i=1}^{k} d_i(v_i, u_i). \tag{3}$$

It follows that if each G_i, $i = 1, \ldots, k$, is given an addressing, then each vertex x of G may be addressed by concatenating the addresses of its components x_i. Therefore, the parameter N is sub-additive on Cartesian products; that is, if

$$G = G_1 \square \cdots \square G_k \tag{4}$$

then

$$N(G) \leq N(G_1) + \cdots + N(G_k). \tag{5}$$

Note that $N(G_1) + \cdots + N(G_k) \leq \left(\sum_{i=1}^{k} n_i\right) - k \leq \left(\prod_{i=1}^{k} n_i\right) - 1 = n - 1$. Thus (5) can improve on Winkler's upper bound of $n - 1$ when G is a Cartesian product.

Question 2.1. *Must equality hold in (5) for all choices of G_i? Remark 3.4 might provide a possible counterexample.*

3. Distance matrices of cartesian products

In this section we determine $N(G)$ when G is the Cartesian product of complete graphs. We first develop some results about the inertia of the distance matrix of a Cartesian product.[3]

[3] The approach we take is due to the late D.A. Gregory.

If $v_1, \ldots, v_n$ denote the vertices of a connected graph G, the distance matrix $D(G)$ of G is the $n \times n$ matrix with entries $D(G)_{ij} = d(v_i, v_j)$. Because G is connected, its adjacency matrix $A(G)$ and its distance matrix $D(G)$ are irreducible symmetric nonnegative integer matrices and by the Perron–Frobenius Theorem (see [5, Proposition 3.1.1] or [12, Theorem 8.8.1]), the largest eigenvalue of each of these matrices has multiplicity 1. We call this largest eigenvalue the *Perron* value of the matrix and often denote it by ρ.

To obtain a formula for the distance matrix of a Cartesian product of graphs, we will use an additive analogue of the Kronecker product of matrices. Note that if A is an $n \times m$ matrix and $c \in \mathbb{R}$, then $c + A$ is the $n \times m$ matrix $cJ + A$ with J the all one $n \times m$ matrix. Further, recall that if A is an $n \times n$ matrix and B an $m \times m$ matrix, with $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, then the Kronecker products $A \otimes B$ and $x \otimes y$ are defined as

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1m}B \\ a_{21}B & a_{22}B & \cdots & a_{2m}B \\ \vdots & \vdots & & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mm}B \end{bmatrix} \quad \text{and} \quad x \otimes y = \begin{bmatrix} x_1 y \\ x_2 y \\ \vdots \\ x_n y \end{bmatrix}. \tag{6}$$

For the additive analogue, we use the symbol $\diamond$ and define $A \diamond B$ and $x \diamond y$ as

$$A \diamond B = \begin{bmatrix} a_{11}+B & a_{12}+B & \cdots & a_{1m}+B \\ a_{21}+B & a_{22}+B & \cdots & a_{2m}+B \\ \vdots & \vdots & & \vdots \\ a_{m1}+B & a_{m2}+B & \cdots & a_{mm}+B \end{bmatrix} \quad \text{and} \quad x \diamond y = \begin{bmatrix} x_1 + y \\ x_2 + y \\ \vdots \\ x_n + y \end{bmatrix}. \tag{7}$$

If $G = G_1 \Box G_2 \Box \cdots \Box G_k$, then the additive property (3) implies that

$$D(G) = D(G_1) \diamond D(G_2) \diamond \cdots \diamond D(G_k). \tag{8}$$

Note that $G_1 \Box G_2$ is isomorphic to $G_2 \Box G_1$ and, equivalently, $D(G_1) \diamond D(G_2)$ is permutationally similar to $D(G_2) \diamond D(G_1)$. Observe that

$$A \diamond B = A \otimes J_m + J_n \otimes B \text{ and } x \diamond y = x \otimes 1_m + 1_n \otimes y \tag{9}$$

where $1_n \in \mathbb{R}^n$ denotes the column vector whose entries are all one. Let $0_n \in \mathbb{R}^n$ denote the column vector with all zero entries. The following two lemmas are due to D.A. Gregory.

Lemma 3.1. *Let A and B be $n \times n$ and $m \times m$ real matrices respectively. If $Ax = \lambda x$ and $x^\top 1_n = \sum x_i = 0$, then $(A \diamond B)(x \diamond 0_m) = m\lambda(x \diamond 0_m)$. Also, if $1_m^T y = 0$ and $By = \mu y$, then $(A \diamond B)(0_n \diamond y) = n\mu(0_n \diamond y)$.*

Proof. We use properties of Kronecker products:

$$\begin{aligned} (A \diamond B)(x \diamond 0) &= (A \otimes J_m + J_n \otimes B)(x \otimes 1_m + 1_n \otimes 0) \\ &= Ax \otimes J_m 1_m + J_n x \otimes B 1_m + A 1_n \otimes J_m 0 + J_n \otimes B0 \\ &= Ax \otimes J_m 1_m = \lambda m (x \otimes 1_m) \\ &= \lambda m (x \otimes 1_m + 1_n \otimes 0) = \lambda m (x \diamond 0). \end{aligned}$$

A similar argument works for the vector $(0_n \diamond y)$. □

Throughout we will say a square matrix is *k-regular* if it has constant row sum k.

Lemma 3.2. *If A is ρ_A-regular and B is ρ_B-regular then $A \diamond B$ is $(m\rho_A + n\rho_B)$-regular.*

Proof. Using properties of Kronecker products,

$$\begin{aligned} (A \diamond B)(1_n \otimes 1_m) &= (A \otimes J_m + J_n \otimes B)(1_n \otimes 1_m) \\ &= A 1_n \otimes J_m 1_m + J_n 1_n \otimes B 1_m) \\ &= \rho_A m (1_n \otimes 1_m) + n\rho_B (1_n \otimes 1_m) \\ &= (\rho_A m + n\rho_B)(1_n \otimes 1_m). \end{aligned}$$

Thus $(A \diamond B)1 = (\rho_A m + n\rho_B)1$. □

Lemma 3.3. *If $G = G_1 \Box G_2 \Box \cdots \Box G_k$ and each $\mathcal{D}(G_i)$ is regular for $i = 1, \ldots, k$, then*

(a) $n_-(D(G)) \geq \sum_i n_-(D(G_i))$, *and*
(b) $n_+(D(G)) \geq 1 + \sum_i (n_+(D(G_i)) - 1)$.

Proof. Because $\mathcal{D}(G_i)$ is regular, we have $D(G_i)1_{n_i} = \rho_i 1_{n_i}$ where ρ_i is the Perron value of $D(G_i)$. Then 1_{n_i} is a ρ_i-eigenvector of $D(G_i)$ and $\mathbb{R}^{n_i}$ has an orthogonal basis of eigenvectors of $D(G_i)$ that includes 1_{n_i} as a member. Thus, using (8), Lemma 3.1 with $A = D(G_i)$ and $B = D(\Box_{j\neq i}G_j)$ implies that the $n_i - 1$ eigenvectors of $D(G_i)$ in the basis other than 1_{n_i} contribute $n_i - 1$ orthogonal eigenvectors to the matrix $D(G)$.

An eigenvector of $D(G_i)$ with eigenvalue $\lambda \neq \rho_i$ contributes an eigenvector of $D(G)$ with eigenvalue $\lambda(n_1 n_2 \cdots n_k)/n_i = \lambda n/n_i$. This eigenvalue has the same sign as λ if $\lambda \neq 0$. Also, if $i \neq j$, then each of the $n_i - 1$ eigenvectors contributed to $D(G)$ by $D(G_i)$ is orthogonal to each of the analogous $n_j - 1$ eigenvectors contributed to $D(G)$ by $D(G_j)$. Thus, the inequality (a) claimed for n_- follows. Also, by Lemma 3.1, 1_n is an eigenvector of $D(G)$ with a positive eigenvalue ρ, so the inequality (b) for n_+ follows. □

Remark 3.4 (*Observed by D.A. Gregory*)**.** The inequality in Lemma 3.3 need not hold if the regularity assumption is dropped. For example, suppose $G = G_1 \Box G_1$ where G_1 is the graph on 6 vertices obtained from $K_{2,4}$ by inserting an edge incident to the two vertices in the part of size 2. Then $n_-(D(G_1)) = 5$ but $n_-(D(G)) = 9 < 5 + 5$. Also, $N(G_1) = 5$, so $9 \leq N(G) \leq 10$ by (1) and (5). An affirmative answer to Question 2.1 would imply $N(G) = 10$.

If each $D(G_i)$ in (8) is regular, then Lemma 3.3 gives $1 + \sum_i(\text{rank } D(G_i) - 1) = 1 - k + \sum_i \text{rank } D(G_i)$ of the rank $D(G)$ nonzero eigenvalues of $D(G)$. The following results imply that if each $D(G_i)$ is regular then all of the remaining eigenvalues must be equal to zero. Equivalently, the results will imply that if each $D(G_i)$ in (8) is regular, then equality must hold in Lemma 3.3(a) and (b).

The next result (proved by D.A. Gregory) is obtained by exhibiting an orthogonal basis of $\mathbb{R}^{mn}$ consisting of eigenvectors of $A \diamond B$ when A and B are symmetric and regular.

Theorem 3.5. *Let A be a regular symmetric real $n \times n$ matrix with $A1_n = \rho_A 1_n$ with $\rho_A > 0$ and let B be a regular symmetric matrix of order m with $B1_m = \rho_B 1_m$ with $\rho_B > 0$. Then*

(a) $n_-(A \diamond B) = n_-(A) + n_-(B)$,
(b) $n_+(A \diamond B) = n_+(A) + n_+(B) - 1$, *and*
(c) $n_o(A \diamond B) = nm - n - m + 1 + n_o(A) + n_o(B)$.

Proof. As in Lemma 3.3, Lemma 3.1 can be used to provide eigenvectors that imply that $n_-(A \diamond B) \geq n_-(A) + n_-(B)$ and $n_+(A \diamond B) \geq n_+(A) + n_+(B) - 1$. It remains to exhibit an adequate number of linearly independent eigenvectors of $A \diamond B$ for the eigenvalue 0.

If $1_n^\top x = 0$ and $1_m^\top y = 0$, then

$$(A \diamond B)(x \otimes y) = (A \otimes J_m + J_n \otimes B)(x \otimes y) = Ax \otimes 0_m + 0_n \otimes By = 0_{nm}.$$

This gives at least $(n - 1)(m - 1) = nm - n - m + 1$ orthogonal eigenvectors of $A \diamond B$ with eigenvalue 0. Moreover, if $Au = 0$ then $1_n^\top u = 0$ and hence, by Lemma 3.1, $(A \diamond B)(u \diamond 0_m) = 0_{nm}$. Likewise, if $Bv = 0$ then $1_m^\top v = 0$ and by Lemma 3.1, $(A \diamond B)(0_n \diamond v) = 0_{nm}$. If each set of vectors x, each set of vectors y, each set of vectors u and each set of vectors v that occur above are chosen to be orthogonal, then the resulting vectors $x \otimes y$, $u \otimes 1_m$, $1_n \otimes v$ will be orthogonal. Thus, $n_o(A \diamond B) \geq nm - n - m + 1 + n_o(A) + n_o(B)$. Adding the three inequalities obtained above, we get

$$\begin{aligned} nm &= n_-(A \diamond B) + n_+(A \diamond B) + n_o(A \diamond B) \\ &\geq n_-(A) + n_-(B) + n_+(A) + n_+(B) - 1 + nm - n - m + 1 + n_o(A) + n_o(B) \\ &= nm. \end{aligned}$$

Thus equality holds in each of the three inequalities. □

Corollary 3.6. *If $G = G_1 \Box G_2 \Box \cdots \Box G_k$ and each $D(G_i)$ is regular for $i = 1, \ldots, k$, then*

(a) $n_-(D(G)) = \sum_i n_-(D(G_i))$, *and*
(b) $n_+(D(G)) = 1 + \sum_i (n_+(D(G_i)) - 1)$.

Remark 3.7. In the proof of Theorem 3.5, whether or not A and B are symmetric and regular, we always have $(A \diamond B)(x \otimes y) = 0_{nm}$ whenever $1_n^\top x = 0$ and $1_m^\top y = 0$. Thus,

$$Nul(A \diamond B) \geq (n - 1)(m - 1)$$

for all square matrices A and B of orders n and m, respectively.

To use Corollary 3.6, it is helpful to have conditions on a graph G that would imply that the distance matrix $D(G)$ is regular. Such a graph is called *transmission regular* (see for example [3]). The following remark gives a few examples of transmission regular graphs.

Remark 3.8 (*Transmission Regular Graphs*).

1. If G is either distance regular or vertex transitive, then $D(G)$ is ρ-regular where ρ is equal to the sum of all the distances from a particular vertex to each of the others.

2. If G is a regular graph of order n and the diameter of G is either one or two, then $D(G)$ is ρ-regular with $\rho = 2(n-1) - \rho_A$ where ρ_A is the Perron value of the adjacency matrix A of G. For if A is the adjacency matrix of G, then $D(G) = A + 2(J_n - I_n - A) = 2(J_n - I_n) - A$. This holds, for example, when G is the Petersen graph or $G = K_n$ (the complete graph on n vertices) or when $G = K_{m,m}$ (the complete balanced bipartite graph on $n = 2m$ vertices).

Theorem 3.9. *Let $G = G_1 \square G_2 \square \cdots \square G_k$. If G_i is transmission regular and $N(G_i) = n_-(D(G_i))$ for $i = 1, \ldots, k$, then $N(G) = \sum_{i=1}^{k} N(G_i)$.*

Proof. By the lower bound (1) and the sub-additivity property (5), $\sum_i N(G_i) \geq N(G) \geq n_- D(G)$. By Lemma 3.3(a), $n_-(D(G)) \geq \sum_i n_-(D(G_i)) = \sum_i N(G_i)$. $\square$

Example 3.10. The Cartesian product of complete graphs, $G = K_{n_1} \square K_{n_2} \square \cdots \square K_{n_k}$ is also known as a Hamming graph. By (2) and Theorem 3.9, it follows that $N(G) = \sum_{i=1}^{k}(n_i - 1)$. In the next section, we explore this result using a different description of the Hamming graphs.

4. Optimal addressing of Hamming graphs

Let $n \geq 1$ and $q \geq 2$ be two integers. The vertices of the Hamming graph $H(n, q)$ can be described as the words of length n over the alphabet $\{1, \ldots, q\}$. Two vertices $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ are adjacent if and only if their Hamming distance is 1. If $n = 1$, $H(1, q)$ is the complete graph K_q. The following result, can be derived from Example 3.10, but we provide another interesting and constructive argument.

Theorem 4.1. *If $n \geq 1$ and $q \geq 2$, then $N(H(n, q)) = n(q - 1)$.*

Proof. We first prove that the length of any addressing of $H(n, q)$ is at least $n(q - 1)$. For $0 \leq k \leq n$, let A_k denote the distance k adjacency matrix of $H(n, q)$. The adjacency matrix of the distance multigraph of $H(n, q)$ is $D(H(n, q)) = \sum_{k=1}^{n} kA_k$. The graph $H(n, q)$ is distance-regular and therefore, $A_1, \ldots, A_n$ are simultaneously diagonalizable. The eigenvalues of the matrices $A_1, \ldots, A_n$ were determined by Delsarte in his thesis [9] (see also [20, Theorem 30.1]).

Proposition 4.2. *Let $k \in \{1, \ldots, n\}$. The eigenvalues of A_k are given by the Krawtchouk polynomials:*

$$\lambda_{k,x} = \sum_{i=0}^{k} (-q)^i (q-1)^{k-i} \binom{n-i}{k-i} \binom{x}{i} \tag{10}$$

with multiplicity $\binom{n}{x}(q-1)^x$ for $x \in \{0, 1, \ldots, n\}$.

The Perron value of A_k equals $\binom{n}{k}(q-1)^k$. Thus, the Perron value of $D = D(H(n, q))$ equals $\sum_{k=1}^{n} \binom{n}{k} k(q-1)^k = nq^{n-1}(q-1)$ and has multiplicity one. The other eigenvalues of D are

$$\begin{aligned}
\mu_x &= \sum_{k=1}^{n} k\lambda_{k,x} = \sum_{k=1}^{n} k \sum_{i=0}^{k} (-q)^i (q-1)^{k-i} \binom{n-i}{k-i} \binom{x}{i} \\
&= \sum_{i=0}^{n} (-q)^i \binom{x}{i} \sum_{k=i}^{n} k(q-1)^{k-i} \binom{n-i}{k-i} = \sum_{i=0}^{n} (-q)^i \binom{x}{i} \sum_{t=0}^{n-i} (i+t)(q-1)^t \binom{n-i}{t} \\
&= \sum_{i=0}^{n} (-q)^i \binom{x}{i} \left(nq^{n-i} - (n-i)q^{n-i-1} \right) \\
&= q^{n-1} \sum_{i=0}^{n} \binom{x}{i} (-1)^i i = \begin{cases} -q^{n-1} & \text{if } x = 1 \\ 0 & \text{if } x \geq 2 \end{cases}
\end{aligned}$$

with multiplicity $\binom{n}{x}(q-1)^x$ for $1 \leq x \leq n$. Thus, the spectrum of D, with multiplicities, is

$$\begin{pmatrix} nq^{n-1}(q-1) & -q^{n-1} & 0 \\ 1 & n(q-1) & q^n - 1 - q(n-1) \end{pmatrix} \tag{11}$$

where the first row contains the distinct eigenvalues of D and the second row contains their multiplicities. Thus, $\max\{n_-(D), n_+(D)\} = n(q - 1)$ and Witsenhausen's inequality (1) imply that $N(H(n, q)) \geq n(q - 1)$.

To show $n(q-1)$ is the optimal length of an addressing of $H(n, q)$, we describe a partition of the edge set of the distance multigraph of $H(n, q)$ into exactly $n(q-1)$ bicliques. For $1 \le i \le n$ and $1 \le t \le q-1$, define the biclique $B_{i,t}$ whose color classes are

$$\{(x_1, \dots, x_n) : x_i = t\}$$

and

$$\{(x_1, \dots, x_n) : x_i \ge t+1\}.$$

One can check easily that if u and v are two distinct vertices in $H(n, q)$, there are exactly $d_H(u, v)$ bicliques $B_{i,t}$ containing the edge uv. Thus, the $n(q-1)$ bicliques $B_{i,t}$ partition the edge set of the distance multigraph of $H(n, q)$ and $N(H(n, q)) \le n(q-1)$. This finishes our proof. □

We remark here that the spectrum of the distance matrix of $H(n, q)$ was also computed by Indulal [17], using a technique similar to what we presented in the previous section.

5. Triangular graphs

The *triangular graph* T_n is the line graph of the complete graph K_n on n vertices. (H is a *line graph* of G if the vertices of H are the edges of G with two vertices adjacent in H if the corresponding edges are incident to a common vertex in G.) When $n \ge 4$, the triangular graph T_n is a strongly regular graph with parameters $\left(\binom{n}{2}, 2(n-2), n-2, 4\right)$. The adjacency matrix of T_n has spectrum

$$\begin{pmatrix} 2(n-2) & n-4 & -2 \\ 1 & n-1 & \binom{n}{2}-n \end{pmatrix} \tag{12}$$

and therefore, the distance matrix $D(T_n)$ has spectrum

$$\begin{pmatrix} (n-1)(n-2) & 2-n & 0 \\ 1 & n-1 & \binom{n}{2}-n \end{pmatrix}. \tag{13}$$

Witsenhausen's inequality (1) implies that $N(T_n) = \mathrm{bp}(D(T_n)) \ge n-1$ for $n \ge 4$.

The problem of addressing T_4 is equivalent to determining the biclique partition number of the multigraph obtained from K_6 by adding one perfect matching. This formulation of the problem was studied by Zaks [25] and Hoffman [16] (see also item 5 in Section 6 below). Zaks proved that $N(T_4) = 4$ and hence T_4 is not eigensharp. We will reprove the lower bound of Zaks [25] in Lemma 5.2 using a technique from [10]. The argument of Lemma 5.2 will then be used to show that T_n is not eigensharp for any $n \ge 4$ in Theorem 5.3.

The *addressing matrix* of a t-addressing is the $n \times t$ matrix $M(a, b)$ where the ith row of $M(a, b)$ is the address of vertex i. $M(a, b)$ can be written as a function of a and b:

$$M(a, b) = aX + bY,$$

where X and Y are matrices with entries in $\{0, 1\}$. Elzinga et al. [10] use the addressing matrix, along with results from Brandenburg et al. [4] and Gregory et al. [15], to create the following theorem:

Theorem 5.1 (*[10]*). *Let $M(a, b)$ be the address matrix of an eigensharp addressing of a graph G. Then for all real scalars a, b, each column of $M(a, b)$ is orthogonal to the null space of $D(G)$. Also, the columns of $M(1, 0)$ are linearly independent, as are the columns of $M(0, 1)$.*

In [10, Theorem 3], Elzinga et al. use Theorem 5.1 to show that the Petersen graph does not have an eigensharp addressing. We will use a similar approach to study the triangular graphs.

Lemma 5.2. *The triangular graph T_4 is not eigensharp, that is, $N(T_4) \ge 4$.*

Proof. Suppose T_4 is eigensharp. Let $D = D(T_4)$. By Theorem 5.1, all vectors in the null space of D are orthogonal to the columns of a 6×3 addressing matrix $M(a, b)$.

We can construct null vectors of D in the following manner, referring to the entries of the null vector as *labels*: choose any two non-adjacent vertices, and label them with zeros. The remaining four vertices form a 4-cycle, which will be alternatingly labeled with 1 and -1, as in Fig. 2.

Let $w(a, b)$ be any column of $M(a, b)$. We claim that $w(a, b)$ has at least three a-entries, and at least three b-entries. For convenience, we will refer to vertices corresponding to the a-entries of $w(a, b)$ as *a-vertices*. If there are no a-vertices, then since $M(a, b) = aX + bY$, one of the columns of X is the zero vector. It would follow that the columns of $M(1, 0)$ are linearly dependent, contradicting Theorem 5.1. Thus $w(a, b)$ has at least one a and at least one b entry.

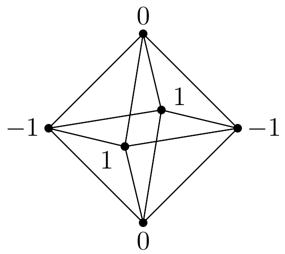

Fig. 2. T_4 with a $D(T_4)$ null-vector labeling.

Suppose $w(a, b)$ has at most 2 a-entries. We will consider three cases (in the next paragraph): there are two adjacent a-vertices, there are two non-adjacent a-vertices, or there is exactly one a-vertex. In each case, we will construct a null vector x of D which is not orthogonal to $u = w(1, 0)$, contradicting Theorem 5.1. We will use the labeling in Fig. 2.

Suppose $w(a, b)$ has two adjacent a-vertices. By labeling one of the a-vertices with a zero and the adjacent a-vertex with 1, we can construct a null vector x (as in Fig. 2) with $x^T u = 1 \neq 0$ for $u = w(1, 0)$. Suppose $w(a, b)$ has two non-adjacent a-vertices. Label the two a-vertices with 1 to get a null vector x with $x^T u = 2 \neq 0$. Suppose there is only one a-vertex in w. Label the a-vertex with 1 to get a null vector x with $x^T u = 1 \neq 0$.

Therefore, at least three positions of $w(a, b)$ have the value a. Similarly, at least three positions of $w(a, b)$ must have value b.

Since each column of $M(a, b)$ has at least three a-entries and three b-entries, there are at least nine a, b pairings corresponding to each column. Since $M(a, b)$ has three columns, there are 27 a, b column-wise pairs in total. However, the number of column-wise a, b pairs in the addressing matrix $M(a, b)$ is simply the number of edges in $\mathcal{D}(T_4)$, namely 18. This contradiction implies that T_4 is not eigensharp. □

Theorem 5.3. *The triangular graph T_n is not eigensharp for any $n \geq 4$, that is, $N(T_n) \geq n$ for all $n \geq 4$.*

Proof. Note that T_4 is an induced subgraph of T_n since K_4 is an induced subgraph of K_n. Let T be an induced subgraph of T_n isomorphic to T_4.

Suppose T_n is eigensharp. Let $M(a, b)$ be an eigensharp addressing matrix of T_n. By Theorem 5.1, the columns of $M(a, b)$ are orthogonal to any null vector of $D(T_n)$. Let w be one of the columns of $M(a, b)$. We can construct a null vector y of $D(T_n)$ by labeling the vertices corresponding to T as described in Fig. 2 and labeling the remaining vertices of T_n with zeros. In [10], it is described that the columns of an addressing matrix correspond to bicliques that partition the edge set of the distance multigraph $\mathcal{D}(T_n)$. Every biclique decomposition of $\mathcal{D}(T_n)$ induces a decomposition of $\mathcal{D}(T)$, an induced subgraph of $\mathcal{D}(T_n)$. Lemma 5.2 tells us that at least 4 bicliques are needed to decompose $\mathcal{D}(T)$. Therefore, there must be at least four columns of $M(a, b)$ whose 6 entries corresponding to T have at least one a and one b. The proof of Lemma 5.2 guarantees that each of these 4 vectors, restricted to the vertices of T, has at least three a entries and three b entries. Since $\mathcal{D}(T)$ is an induced subgraph of $\mathcal{D}(T_n)$, there are the same number of edges between the corresponding vertices in the two graphs. However, a contradiction occurs: the eigensharp addressing implies that there are at least 36 edges in $\mathcal{D}(T)$, but there are in fact 18. Therefore T_n is not eigensharp. □

For the triangular graph T_5 (the complement of the Petersen graph), the following six bicliques partition the edge set of $\mathcal{D}(T_5)$:

$$\{12, 13, 14, 15\} \cup \{23, 24, 25, 34, 35, 45\}$$
$$\{12, 25\} \cup \{13, 14, 34, 35, 45\}$$
$$\{23, 24\} \cup \{15, 25, 34, 35, 45\}$$
$$\{13, 23, 35\} \cup \{14, 24, 45\}$$
$$\{15\} \cup \{12, 13, 14, 34\}$$
$$\{34\} \cup \{25, 35, 45\}.$$

Thus, by Theorem 5.3, we know that $5 \leq N(T_5) \leq 6$.

6. Open problems

We conclude this paper with some open problems.

1. Must equality hold in (5) for all choices of G_i?

2. It is known that determining $\mathrm{bp}(G)$ for a graph G is an NP-hard problem (see [19]). This problem is NP-hard even when restricted to graphs G with maximum degree $\Delta(G) \le 3$ (see [7]). To show $\mathrm{bp}(G)$ is NP-hard to compute, one does a reduction from the minimum vertex-cover problem by subdividing each edge by two vertices as this ensures the only bicliques in the subdivided graph are stars. Such a reduction cannot be used when trying to compute $N(G) = \mathrm{bp}(\mathcal{D})$ as the distance multigraph $\mathcal{D}(G)$ will contain all possible kinds of bicliques. Thus, a different reduction is needed. What is the complexity of finding $N(G)$ for general graphs G? How about graphs with $\Delta(G) \le 3$, or other families of graphs?
3. What is $N(T_n)$ for $n \ge 5$?
4. The triangular graph T_n is a special case of a Johnson graph. For $n \ge m \ge 2$, the Johnson graph $J(n,m)$ has as its vertex set the m-subsets of an n-set, with two m-subsets being adjacent if and only if their intersection has size $m-1$. The Johnson graph is distance-regular and its eigenvalues were determined by Delsarte in his thesis [9] (see also [20, Theorem 30.1]). Atik and Panigrahi [3] computed the spectrum of the distance matrix $D(J(n,m))$:
$$\begin{pmatrix} s & 0 & -\frac{s}{n-1} \\ 1 & \binom{n}{m} - n & n-1 \end{pmatrix} \tag{14}$$
where $s = \sum_{j=1}^{m} j \binom{m}{j}\binom{n-m}{j}$. Inequality (1) implies that $N(J(n,m)) \ge n-1$. What is $N(J(n,m))$?
5. Finding an optimal addressing of the complete multipartite graph $K_{2,\dots,2}$ with m color classes of size 2 is a highly non-trivial open problem. It is equivalent to finding the biclique partition number of the multigraph obtained from the complete graph K_{2m} by adding a perfect matching. Motivated by questions in geometry involving nearly-neighborly families of tetrahedra, this problem was studied by Zaks [25] and Hoffman [16]. The best current results for $N(K_{2,\dots,2}) = \mathrm{bp}(\mathcal{D}(K_{2,\dots,2}))$ are due to these authors (the lower bound is due to Hoffman [16] and the upper bound is due to Zaks [25]):
$$m + \lfloor \sqrt{2m} \rfloor - 1 \le N(K_{2,\dots,2}) \le \begin{cases} 3m/2 - 1 & \text{if } m \text{ is even} \\ (3m-1)/2 & \text{if } m \text{ is odd.} \end{cases} \tag{15}$$
6. The Clebsch graph is a strongly regular graph with parameters $(16, 5, 0, 2)$ that is obtained from the 5-dimensional cube by identifying antipodal vertices. The eigenvalue bound gives $N \ge 11$ and the connection with the 5-dimensional cube might be useful to find a good biclique decomposition of the distance multigraph of this graph.
7. What is $N(G)$ if G is a random graph? Winkler's work [24], Witsenhausen inequality (1) and the Wigner semicircle law imply that $n - 1 \ge N(G) \ge n/2 - c\sqrt{n}$ for some positive constant c. Chung and Peng [6] have shown for a random graph $G \in \mathcal{G}_{n,p}$ with $p \le 1/2$ and $p = \Omega(1)$, almost surely
$$n - o((\log_{\frac{1}{p}} n)^{3+\epsilon}) \le \mathrm{bp}(G) \le n - 2\log_{\frac{1}{1-p}} n \tag{16}$$
for any positive constant ϵ. Here $\mathcal{G}_{n,p}$ is the Erdős-Rényi random graph model. Alon [1] proved that there is a positive constant $c > 0$ such that $\mathrm{bp}(G) = n - \Theta\left(\frac{\log(np)}{p}\right)$ for any $p \in (2/n, c)$. Recently, Alon, Bohman and Huang [2] extended some work of Alon [1] and proved that there exists a positive constant $c' > 0$ such that almost surely, $\mathrm{bp}(G) \le n - (1 + c')\alpha(G)$ for $G \in \mathcal{G}_{n,1/2}$.

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