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# Max-cut and extendability of matchings in distance-regular graphs



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## ABSTRACT

A connected graph  $G$  of even order  $v$  is called  $t$ -extendable if it contains a perfect matching,  $t < v/2$  and any matching of  $t$  edges is contained in some perfect matching. The extendability of  $G$  is the maximum  $t$  such that  $G$  is  $t$ -extendable. Since its introduction by Plummer in the 1980s, this combinatorial parameter has been studied for many classes of interesting graphs. In 2005, Brouwer and Haemers proved that every distance-regular graph of even order is 1-extendable and in 2014, Cioabă and Li showed that any connected strongly regular graph of even order is 3-extendable except for a small number of exceptions.

In this paper, we extend and generalize these results. We prove that all distance-regular graphs with diameter  $D \geq 3$  are 2-extendable and we also obtain several better lower bounds for the extendability of distance-regular graphs of valency  $k \geq 3$  that depend on  $k$ ,  $\lambda$  and  $\mu$ , where  $\lambda$  is the number of common neighbors of any two adjacent vertices and  $\mu$  is the number of common neighbors of any two vertices in distance two. In many situations, we show that the extendability of a distance-regular graph with valency  $k$  grows linearly in  $k$ . We conjecture that the extendability of a distance-regular graph of even order and valency  $k$  is at least  $\lfloor k/2 \rfloor - 1$  and we prove this fact for bipartite distance-regular graphs.

In course of this investigation, we obtain some new bounds for the max-cut and the independence number of distance-regular graphs in terms of their size and odd girth and we prove that our

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inequalities are incomparable with known eigenvalue bounds for these combinatorial parameters.

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## 1. Introduction

Our graph theoretic notation is standard (for undefined notions, see [8,26,48]). The adjacency matrix of a graph  $G = (V, E)$  has its rows and columns indexed after the vertices of the graph and its  $(u, v)$ th entry equals 1 if  $u$  and  $v$  are adjacent and 0 otherwise. If  $G$  is a connected  $k$ -regular graph of order  $v$ , then  $k$  is the largest eigenvalue of the adjacency matrix of  $G$  and its multiplicity is 1. In this case, let  $k = \lambda_1 > \lambda_2 \geq \dots \geq \lambda_v$  denote the eigenvalues of the adjacency matrix of  $G$ . If  $S$  and  $T$  are vertex disjoint subsets of  $G$ , let  $e(S, T)$  denote the number of edges with one endpoint in  $S$  and the other in  $T$ . If  $S$  is a subset of vertices of  $G$ , let  $S^c$  denote its complement. The max-cut of  $G$  is defined as  $\text{mc}(G) := \max_{S \subset V} e(S, S^c)$  and measures how close is  $G$  from being a bipartite graph. Given a graph  $G$ , determining  $\text{mc}(G)$  is a well-known NP-hard problem (see [23, Problem ND16, page 210] or [30]) and designing efficient algorithms to approximate  $\text{mc}(G)$  has attracted a lot of attention [1,18–20,27,28,38,44,47].

A set of edges  $M$  in a graph  $G$  is a matching if no two edges of  $M$  share a vertex. A matching  $M$  is perfect if every vertex is incident with exactly one edge of  $M$ . A connected graph  $G$  of even order  $v$  is called  $t$ -extendable if it contains at least one perfect matching,  $t < v/2$  and any matching of size  $t$  is contained in some perfect matching. Graphs that are 1-extendable are also called matching-covered (see Lovász and Plummer [36, page 113]). The extendability of a graph  $G$  of even order is defined as the maximum  $t < v/2$  such that  $G$  is  $t$ -extendable. This concept was introduced by Plummer [40] in 1980 and was motivated by work of Lovász [34] on canonical decomposition of graphs containing perfect matchings. Later on, Yu [49] expanded the definition of extendability to graphs of odd order. Zhang and Zhang [51] obtained an  $O(mn)$  algorithm to compute the extendability of a bipartite graph with  $n$  vertices and  $m$  edges, but the complexity of determining the extendability of a general graph is unknown at present time (see [41,42,50] for more details on extendability of graphs).

In this paper, we obtain a simple upper bound for the max-cut of certain regular graphs in terms of their odd girth (the shortest length of an odd cycle). In Section 2, we prove that if  $G$  is a non-bipartite distance-regular graph with  $e$  edges and odd girth  $g$ , then  $\text{mc}(G) \leq e(1 - \frac{1}{g})$ . As a consequence of this result, we show that if  $G$  is a non-bipartite distance-regular graph with  $v$  vertices, odd girth  $g$  and independence number  $\alpha(G)$ , then  $\alpha(G) \leq \frac{v}{2}(1 - \frac{1}{g})$ . We show that these bounds are incomparable with some spectral bounds of Mohar and Poljak [38] for the max-cut and of Cvetković (see [8, Theorem 3.5.1] or [26, Lemma 9.6.3]) and Hoffman (see [8, Theorem 3.5.2] or [26, Lemma 9.6.2]) for the independence number.

Holton and Lou [29] showed that strongly regular graphs with certain connectivity properties are 2-extendable and conjectured that all but a few strongly regular graphs are 2-extendable. Lou and Zhu [33] proved this conjecture and showed that every connected strongly regular graph of valency  $k \geq 3$  is 2-extendable with the exception of the complete 3-partite graph  $K_{2,2,2}$  and the Petersen graph. Cioabă and Li [15] proved that every connected strongly regular graph of valency  $k \geq 5$  is 3-extendable with the exception of the complete 4-partite graph  $K_{2,2,2,2}$ , the complement of the Petersen graph and the Shrikhande graph. Moreover, Cioabă and Li determined the extendability of many families of strongly regular graphs including Latin square graphs, block graphs of Steiner systems, triangular graphs, lattice graphs and all known triangle-free strongly regular graphs. For any such graph of valency  $k$ , Cioabă and Li proved that the extendability is at least  $\lceil k/2 \rceil - 1$  and conjectured that this fact should be true for any strongly regular graph.

In this paper, we extend and generalize these results and study the extendability of distance-regular graphs with diameter  $D \geq 3$ . Brouwer and Haemers [7] proved that distance-regular graphs

are  $k$ -edge-connected. Plesník ([35, Chapter 7] and [39]) showed that if  $G$  is a  $k$ -regular  $(k - 1)$ -edge-connected graph with an even number of vertices, then the graph obtained by removing any  $k - 1$  edges of  $G$  contains a perfect matching. These facts imply that every distance-regular graph of even order is 1-extendable. In Section 3, we improve this result and we show that all distance-regular graphs with diameter  $D \geq 3$  are 2-extendable. We prove that any distance-regular graph of valency  $k \geq 3$  with  $\lambda \geq 1$  is  $\lfloor \frac{k+1-\lambda+1}{2} \rfloor$ -extendable (when  $\mu = 1$ ),  $\lfloor \frac{1}{2} \lceil \frac{k+2}{2} \rceil \rfloor$ -extendable (when  $\mu = 2$ ) and  $\lfloor \frac{k}{3} \rfloor$ -extendable (when  $\mu \geq 3$  and  $k \geq 6$ ). We also show that any bipartite distance-regular graph of valency  $k$  is  $\lfloor \frac{k+1}{2} \rfloor$ -extendable. We also remark that our results for graphs of even order can be extended to graphs of odd order in similar fashion to what was done in [15], but for the sake of brevity and clarity, we will not include the details here.

## 2. Max-cut of distance regular graphs

For notation and definitions related to distance-regular graphs, see [6,46]. We denote the intersection array of a distance-regular  $G$  of diameter  $D$  by  $\{b_0, \dots, b_{D-1}; c_1, \dots, c_D\}$  and we let  $k = b_0$  and  $a_i = k - b_i - c_i$  for  $0 \leq i \leq D$  as usual. Also, let  $\lambda = a_1$  and  $\mu = c_2$ . The following result gives a simple upper bound for the max-cut of a graph in terms of its odd girth under certain regularity conditions. Such regularity conditions will be satisfied by walk-regular graphs and distance-regular graphs. As pointed to us by one of the anonymous referees, the theorem below holds for any odd natural number  $g$  as long as the condition that every edge is in the same number of cycles of length  $g$ , is satisfied.

**Theorem 2.1.** *Let  $G$  be a non-bipartite graph with odd girth  $g$ . If every edge of  $G$  is contained in the same number of cycles of length  $g$ , then*

$$\text{mc}(G) \leq e \left( 1 - \frac{1}{g} \right). \quad (1)$$

**Proof.** Let  $\gamma$  be the number of cycles of length  $g$  containing some fixed edge of  $G$  and let  $\mathcal{C}$  be the set of cycles of length  $g$ . By counting pairs  $(e_0, C)$  with  $e_0 \in E(G)$ ,  $C \in \mathcal{C}$  with  $e_0$  contained in  $C$ , we get that  $|\mathcal{C}| = \frac{e\gamma}{g}$ . Let  $A$  be any subset of vertices and  $T$  be the set of the edges with both endpoints in  $A$  or in  $A^c$ . Every time we delete an edge in  $T$ , we destroy at most  $\gamma$  cycles in  $\mathcal{C}$ . Therefore  $|T| \geq \frac{|\mathcal{C}|}{\gamma} = \frac{e}{g}$ . Since  $e(A, A^c) = e - |T| \leq e(1 - \frac{1}{g})$ , this implies the desired conclusion.  $\square$

Our theorem can be applied to the family of  $m$ -walk regular graphs with  $m \geq 1$ . This family of graphs contains the distance-regular graphs. A connected graph  $G$  is  $m$ -walk-regular if the number of walks of length  $l$  between any pair of vertices only depends on the distance between them, provided that this distance does not exceed  $m$ . The family of  $m$ -walk-regular graphs was first introduced by Dalfó, Fiol, and Garriga [16,21].

Note that the upper bound of Theorem 2.1 is tight as shown for example by the blow up of an odd cycle  $C_g$ . Such a graph can be constructed from the odd cycle  $C_g$  by replacing each vertex  $i$  of  $C_g$  by a coclique  $A_i$  of size  $m$  for  $1 \leq i \leq g$  and adding all the possible edges between  $A_i$  and  $A_j$  whenever  $i$  and  $j$  are adjacent in  $C_g$ . The resulting graph which is also the lexicographic product of the cycle  $C_g$  with the empty graph of order  $m$  (see [26, Ex 26, p.17] for a definition), has  $gm$  vertices and  $gm^2$  edges. The odd girth of this graph is  $g$ , each edge of the graph is contained in the same number of cycles of length  $g$  and there is a cut of size  $e \left( 1 - \frac{1}{g} \right) = (g - 1)m^2$ .

Mohar and Poljak [38] showed that  $\text{mc}(G) \leq \frac{v\mu_{\max}}{4}$  for any graph  $G$  on  $v$  vertices whose largest Laplacian eigenvalue is  $\mu_{\max}$  (see also [1,18–20,47] for related results). Translated to regular graphs, their result implies the following inequality:

$$\text{mc}(G) \leq \frac{e}{2} \left( 1 - \frac{\lambda_v}{k} \right). \quad (2)$$

Note that the inequalities (1) and (2) are incomparable. This fact can be seen by considering the complete graphs and the odd cycles, but we give other examples of distance-regular graphs below. Also, a simple calculation yields that inequality (1) is better for graphs that in a spectral sense are closer to being bipartite (when  $\lambda_v \leq -k(1 - 2/g)$  more precisely).

The Hamming graph  $H(D, q)$  is the graph whose vertices are all the words of length  $D$  over an alphabet of size  $q$  with two words being adjacent if and only their Hamming distance is 1. The graph  $H(D, q)$  is distance-regular of diameter  $D$ , has eigenvalues  $(q - 1)D - qi$  for  $0 \leq i \leq D$  and is bipartite when  $q = 2$  [8, page 174]. When  $q \geq 3$ , inequality (1) always gives an upper bound  $\frac{2e}{3}$ . The upper bound from inequality (2) is  $\frac{e}{2}(1 + \frac{1}{q-1})$ . When  $q = 3$ , (1) is better. When  $q \geq 5$ , inequality (2) is better. When  $q = 4$ , both inequalities give the same upper bound.

The Johnson graph  $J(n, m)$  is the graph whose vertices are the  $m$ -subsets of a set of size  $n$  with two  $m$ -subsets being adjacent if and only if they have  $m - 1$  elements in common. The graph  $J(n, m)$  is distance-regular with diameter  $D = \min(m, n - m)$ , eigenvalues  $(m - i)(n - m - i) - i$ , where  $0 \leq i \leq D$  [8, page 175]. Inequality (1) always gives an upper bound  $\frac{2e}{3}$ . Inequality (2) is  $\frac{e}{2}(1 + \frac{D}{m(n-m)})$ . When  $\max(m, n - m) \geq 4$ , inequality (2) is better and in the other cases ( $m \in \{2, 3\}$  or  $n - m \in \{1, 2, 3\}$ ), (1) is better.

In the following examples, we compare (1) and (2) for other distance-regular graph with larger odd girth.

1. The Dodecahedron graph [6, page 417] is a 3-regular graph of order 20 and size 30. It has  $\lambda_v = -\sqrt{5}$  and  $g = 5$ . Inequality (1) gives  $mc(G) \leq 24$  and inequality (2) gives  $mc(G) \leq 26$ .
2. The Coxeter graph [6, page 419] is a 3-regular graph of order 28 and size 42. It has  $\lambda_v = -\sqrt{2} - 1 \approx -2.414$  and  $g = 7$ . Inequality (1) gives  $mc(G) \leq 36$  and inequality (2) gives  $mc(G) \leq 37$ .
3. The Biggs–Smith graph [6, page 414] is a 3-regular graph of order 102 and size 153. It has  $\lambda_v \approx -2.532$  and  $g = 9$ . Inequality (1) gives  $mc(G) \leq 136$  and inequality (2) gives  $mc(G) \leq 141$ .
4. The Wells graph [6, page 421] is a 5-regular graph of order 32 and size 80. It has  $\lambda_v = -3$  and  $g = 5$ . Inequality (1) gives  $mc(G) \leq 64$  and inequality (2) gives  $mc(G) \leq 64$ .
5. The Hoffman–Singleton graph [6, page 391] is a 7-regular graph of order 50 and size 175. It has  $\lambda_v = -3$  and  $g = 5$ . Inequality (1) gives  $mc(G) \leq 140$  and inequality (2) gives  $mc(G) \leq 125$ .
6. The Ivanov–Ivanov–Faradjev graph [6, page 414] is a 7-regular graph of order 990 and size 3465. It has  $\lambda_v = -4$  and  $g = 5$ . Inequality (1) gives  $mc(G) \leq 2772$  and inequality (2) gives  $mc(G) \leq 2722$ .
7. The Odd graph  $O_{m+1}$  [6, page 259–260] is the graph whose vertices are the  $m$ -subsets of a set with  $2m + 1$  elements, where two  $m$ -subsets are adjacent if and only if they are disjoint. Note that  $O_3$  is Petersen graph. The graph  $O_{m+1}$  is a distance-regular graph of valency  $m + 1$ , order  $v = \binom{2m+1}{m}$  and size  $e = \frac{m+1}{2} \binom{2m+1}{m}$ . It has  $\lambda_v = -m$  and  $g = 2m + 1$ . Inequality (1) gives  $mc(G) \leq e(1 - \frac{1}{2m+1})$  and inequality (2) gives  $mc(G) \leq e(1 - \frac{1}{2m+2})$ .

Theorem 2.1 can be used to obtain an upper bound for the independence number of certain regular graphs.

**Corollary 2.2.** *Let  $G$  be a non-bipartite regular graph with valency  $k$  and odd girth  $g$ . If every edge of  $G$  is contained in the same number of cycles of length  $g$ , then*

$$\alpha(G) \leq \frac{v}{2} \left( 1 - \frac{1}{g} \right). \tag{3}$$

**Proof.** Let  $S$  be an independent set of size  $\alpha(G)$ . Then  $k\alpha(G) = e(S, S^c) \leq \frac{vk}{2} (1 - \frac{1}{g})$  which implies the conclusion of the theorem.  $\square$

The Cvetković inertia bound (see [8, Theorem 3.5.1] or [26, Lemma 9.6.3]) states that if  $G$  is a graph with  $n$  vertices whose adjacency matrix has  $n_+$  positive eigenvalues and  $n_-$  negative eigenvalues, then

$$\alpha(G) \leq \min(n - n_-, n - n_+). \tag{4}$$

The Hoffman-ratio bound (see [8, Theorem 3.5.2] or [26, Lemma 9.6.2]) states that if  $G$  is a  $k$ -regular graph with  $v$  vertices, then

$$\alpha(G) \leq \frac{v}{1 + k/(-\lambda_v)}. \tag{5}$$

In the table below, we compare the bounds (3)–(5) for some of the previous examples. When the bounds obtained are not integers, we round them below. The exact values of the independence numbers below were computed using Sage.

Graph	$\alpha$	(3)	(4)	(5)
Dodecahedron	8	8	8	11
Coxeter	12	12	13	12
Biggs–Smith	43	45	58	46
Wells	10	12	13	12
Hoffman–Singleton	15	20	21	15

For the Hamming graph  $H(D, q)$  with  $D = 2$  and  $q \geq 3$ , (3) is better than (5). For the Hamming graph  $H(D, q)$  with  $D \geq 3$  and  $q \geq 3$ , (5) is better. For the Odd graph  $O_{m+1}$ , the inequalities (3) and (5) give the same bound that equals the independence number of  $O_{m+1}$ .

### 3. Extendability of matchings in distance-regular graphs

In this section, we will focus on the extendability of distance-regular graphs of even order. Similar results can be obtained for distance-regular graphs of odd order using the definition of extendability of Yu [49], but for the sake of simplicity we restrict ourselves to graphs of even order. A connected graph  $G$  of odd order  $v$  containing at least one matching of size  $\frac{v-1}{2}$  (a near perfect matching) is called  $t$ -near-extendable (or  $t1/2$ -extendable in the notation of Yu [49]) if  $t < \frac{v-1}{2}$  and for every vertex  $x$ , any matching of size  $t$  that does not cover  $x$ , is contained in some near perfect matching that misses  $x$ . Graphs that are 0-near-extendable are also called factor-critical or hypomatchable (see Lovász and Plummer [36, page 89]).

In Section 3.1, we describe the main tools which will be used in our proofs. In Section 3.2, we give various lower bounds for the extendability of distance-regular graphs. In Section 3.3, we show that all distance-regular graphs with diameter  $D \geq 3$  are 2-extendable.

#### 3.1. Main tools

Let  $o(G)$  denote the number of components of odd order in a graph  $G$ . If  $S$  is a subset of vertices of  $G$ , then  $G - S$  denotes the subgraph of  $G$  obtained by deleting the vertices in  $S$ . Let  $N(T)$  denote the set of vertices outside  $T$  that are adjacent to at least one vertex of  $T$ . When  $T = \{x\}$ , let  $N(x) = N(\{x\})$ . The distance  $d(x, y)$  between two vertices  $x$  and  $y$  of a connected graph  $G$  is the shortest length of a path between  $x$  and  $y$ . If  $x$  is a vertex of a distance-regular graph  $G$ , let  $N_i(x)$  denote the set of vertices at distance  $i$  from vertex  $x$  and  $k_i = |N_i(x)|$ ; the  $i$ th subconstituent  $\Gamma_i(x)$  of  $x$  is the subgraph of  $G$  induced by  $N_i(x)$ .

**Theorem 3.1** (Brouwer and Haemers [7]). *Let  $G$  be a distance-regular graph of valency  $k$ . Then  $G$  is  $k$ -edge-connected. Moreover, if  $k > 2$ , then the only disconnecting sets of  $k$  edges are the set of  $k$  edges on a single vertex.*

**Theorem 3.2** (Brouwer and Koolen [10]). *Let  $G$  be a distance-regular graph of valency  $k$ . Then  $G$  is  $k$ -connected. Moreover, if  $k > 2$ , then the only disconnecting sets of  $k$  vertices are the set of the neighbors of some vertex.*

The following lemma is an extension of previous work of the authors from [12] and [14] where similar results were obtained for strongly regular graphs.

**Lemma 3.3.** *Let  $G$  be a distance-regular graph with  $k \geq 4$ . If  $A \subset V$  with  $3 \leq |A| \leq k - 1$ , then  $e(A, A^c) \geq 3k - 6$ .*

**Proof.** If  $|A| \leq k - 2$ , then every vertex in  $A$  has at least  $k - (|A| - 1)$  many neighbors in  $A^c$  and consequently  $e(A, A^c) \geq |A|(k - |A| + 1) \geq 3(k - 2)$ . Let  $A \subset V$  with  $|A| = k - 1$ . If  $|N_1(x) \cap A| \leq k - 3$  for every  $x \in A$ , then  $e(A, A^c) \geq 3(k - 1)$ . Otherwise, let  $x \in A$  such that  $|N_1(x) \cap A| = k - 2$ . Denote  $N_1(x) \cap A^c = \{y, z\}$ . If  $\lambda = 0$ , then each vertex in  $N_1(x) \cap A$  has  $k - 1$  neighbors outside  $A$  and thus,  $e(A, A^c) \geq 2 + (k - 2)(k - 1) > 3k - 6$ . If  $\lambda \geq 1$ , then at least  $\lambda - 1$  of the  $\lambda$  common neighbors of  $x$  and  $y$  are contained in  $A$ . Therefore,  $y$  has at least  $\lambda$  neighbors in  $A$ . A similar statement holds for  $z$ . Thus,  $e(A, N_1(x) \cap A^c) \geq 2\lambda = 2(k - b_1 - 1)$ . Also,  $e(N_1(x) \cap A, N_2(x)) \geq (k - 2)b_1$  so  $e(A, A^c) \geq (k - 2)b_1 + 2(k - b_1 - 1) = 3k - 6 + (k - 4)(b_1 - 1) \geq 3k - 6$ .  $\square$

**Theorem 3.4** (Tutte [45]). *A graph  $G$  has a perfect matching if and only if  $o(G - S) \leq |S|$  for every  $S \subset V(G)$ .*

Yu [49, Theorem 2.2] obtained the following characterization of connected graphs that are not  $t$ -extendable using Tutte's theorem.

**Lemma 3.5** (Yu [49]). *Let  $t \geq 1$  and  $G$  be a connected graph containing a perfect matching. The graph  $G$  is not  $t$ -extendable if and only if it contains a subset  $S$  of vertices such that the subgraph induced by  $S$  contains  $t$  independent edges and  $o(G - S) \geq |S| - 2t + 2$ .*

The following necessary condition for a bipartite and connected graph not to be  $t$ -extendable, will be used later in our arguments.

**Lemma 3.6.** *Let  $G$  be a connected bipartite graph with color classes  $X$  and  $Y$ , where  $|X| = |Y| = m$ . If  $G$  is not  $t$ -extendable, then  $G$  has an independent set  $I$  of size at least  $m - t + 1$ , such that  $I \not\subset X$  and  $I \not\subset Y$ .*

**Proof.** Assume that  $G$  is not  $t$ -extendable. By Lemma 3.5, there is a vertex disconnecting set  $S$  such that the subgraph induced by  $S$  contains at least  $t$  independent edges and  $o(G - S) \geq |S| - 2t + 2$ . Let  $S$  be such a disconnecting set of maximum size. Our key observation is that  $G - S$  does not have non-singleton odd components. Indeed, note that any non-singleton odd component of  $G - S$  induces a bipartite graph with color classes  $A$  and  $B$ . Since  $|A| + |B|$  is odd, we get that  $|A| \neq |B|$  and assume that  $|A| > |B|$ . If  $S' = S \cup B$ , then  $S'$  is a vertex disconnecting set with  $|S'| > |S|$  and  $o(G - S') \geq |S'| - 2t + 2$ , contradicting to the maximality of  $|S|$ . By a similar argument,  $G - S$  contains no even components. Let  $I = V(G) \setminus S$ . Then  $I$  is an independent set of size at least  $m - t + 1$  since  $|I| + |S| = 2m$  and  $|I| \geq |S| - 2t + 2$ . Assume that  $I \subset X$ . Then  $S$  induces a bipartite graph with one partite set of size at most  $t - 1$ . This makes it impossible for the subgraph induced by  $S$  to contain  $t$  independent edges.  $\square$

Note that the study of such independent sets in regular bipartite graphs has been done by other authors in different contexts (see [17] for example).

**Lemma 3.7** (Lemma 6 [15]). *If  $G$  is a distance-regular graph of diameter  $D \geq 3$ , then for any  $x \in V(G)$ , the subgraph induced by the vertices at distance 2 or more from  $x$ , is connected.*

**Proof.** As  $G$  has diameter  $D \geq 3$ , then there are 4 vertices, which induce a  $P_4$ . It is known that  $P_4$  has spectrum  $\{\frac{1+\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}, \frac{-1-\sqrt{5}}{2}\}$ . By eigenvalue interlacing [8, Corollary 2.5.2],  $\lambda_2 \geq \frac{-1+\sqrt{5}}{2} > 0$ . Cioabă and Koolen [13, Theorem 3] proved that if the entry  $u_{i-1}$  of the standard sequence  $(u_0, u_1, \dots, u_D)$  corresponding to  $\lambda_2$ , is positive, then for all  $x \in V(G)$ ,  $\Gamma_{\geq i}(x)$  is connected, where  $\Gamma_{\geq i}(x)$  is the graph induced by the vertex set at distance at least  $i$  to vertex  $x$ . As  $u_1 = \lambda_2/k > 0$ , the conclusion follows.  $\square$

**Lemma 3.8** (Brouwer and Haemers [7]). Let  $G$  be a distance-regular graph. Let  $T$  be a disconnecting set of edges of  $G$ , and let  $A$  be the vertex set of a component of  $G - T$ . Fix a vertex  $a \in A$  and let  $t_i$  be the number of edges in  $T$  that join  $\Gamma_{i-1}(a)$  and  $\Gamma_i(a)$ . Then  $|A \cap \Gamma_i(a)| \geq (1 - \sum_{j=1}^i \frac{t_j}{c_j k_j}) k_i$  and

$$|A| \geq v - \sum_i \frac{t_i}{c_i k_i} (k_i + \dots + k_D).$$

If  $T$  is a disconnecting set of edges none of which is incident with  $a$ , then

$$|A| > v \left( 1 - \frac{|T|}{\mu k_2} \right).$$

**Lemma 3.9.** Let  $G$  be a distance-regular graph with  $\lambda \geq 1$ . If  $A$  is an independent set of  $G$ , then  $|N(A)| \geq 2|A|$ .

**Proof.** For any  $x \in N(A)$ ,  $N(x) \cap A$  is an independent set in the subgraph  $\Gamma_1(x)$ . As  $\Gamma_1(x)$  is  $\lambda$ -regular graph with  $k$  vertices, its independence number is at most  $k/2$ . Thus,  $|N(x) \cap A| \leq k/2$ . Therefore,  $|A|k = e(A, N(A)) = \sum_{x \in N(A)} |N(x) \cap A| \leq |N(A)|k/2$  which implies that  $|N(A)| \geq 2|A|$ .  $\square$

**Lemma 3.10.** Let  $G$  be a distance-regular graph with valency  $k \geq 3$ ,  $\lambda \geq 1$  and  $\mu \leq k/2$ . If  $A$  is an independent set of  $G$ , then  $|N(A)| \geq k + |A| - 1$ .

**Proof.** Let  $a = |A|$ . The case  $a = 1$  is trivial. If  $a \geq k - 1$ , Lemma 3.9 implies that  $|N(A)| \geq 2a \geq a + k - 1$ . Assume that  $2 \leq a \leq k - 2$ . If there are two vertices  $x, y \in A$ , such that  $N(x) \cap N(y) = \emptyset$ , then  $|N(A)| \geq |N(x) \cup N(y)| \geq 2k \geq k + a - 1$ . Assume that  $N(x) \cap N(y) \neq \emptyset$  for any  $x, y \in A$ . Since  $A$  is an independent set,  $|N(x) \cap N(y)| = \mu$  for any  $x \neq y \in A$ . For  $z \in N(A)$ , let  $d_z = |A \cap N(z)|$  and  $\bar{d} = \frac{\sum_{z \in N(A)} d_z}{|N(A)|}$ . Counting the edges between  $A$  and  $N(A)$ , we have  $ak = |N(A)|\bar{d}$ . Counting the 3-subsets of the form  $\{x, y, z\}$  such that  $x \neq y \in A, z \in N(A), x \sim z, y \sim z$  and then using Jensen's inequality for the function  $f(t) = \binom{t}{2}$ , we get that  $\binom{a}{2} \mu = \sum_{z \in N(A)} \binom{d_z}{2} \geq |N(A)| \binom{\bar{d}}{2}$ . Combining these facts, we obtain that  $(a - 1)\mu \geq k \left( \frac{ka}{|N(A)|} - 1 \right)$  which implies that  $|N(A)| \geq \frac{k^2 a}{k + a\mu - \mu}$ . As  $\mu \leq k/2$ , we have  $|N(A)| \geq \frac{k^2 a}{k + (a-1)k/2} = \frac{2ka}{a+1} = k + a - 1 + \frac{(a-1)(k-a-1)}{a+1} \geq k + a - 1$ .  $\square$

A distance-regular graph with intersection array  $\{k, \mu, 1; 1, \mu, k\}$  is called a *Taylor graph*. The following lemma due to Brouwer and Koolen (see [10, Lemma 3.14] and also [31, Proposition 5] for a generalization) gives a sufficient condition for a distance-regular graph to be a Taylor graph.

**Lemma 3.11** (Brouwer and Koolen [10]). Let  $G$  be a non-bipartite distance-regular graph with  $D \geq 3$ . If  $k < 2\mu$ , then  $G$  is a Taylor graph.

### 3.2. Lower bounds for the extendability of distance-regular graphs

In this subsection, we give some sufficient conditions, in terms of  $k, \lambda$  and  $\mu$ , for a distance-regular graph to be  $t$ -extendable, where  $t \geq 1$ .

**Theorem 3.12** (Chen [11]). Let  $t \geq 1$  and  $n \geq 2$  be two integers. If  $G$  is a  $(2t + n - 2)$ -connected  $K_{1,n}$ -free graph of even order, then  $G$  is  $t$ -extendable.

**Corollary 3.13.** If  $G$  is a distance-regular graph with even order and  $\lambda \geq 1$ , then  $G$  is  $\lfloor \frac{1}{2} \lceil \frac{k+2}{2} \rceil \rfloor$ -extendable.

**Proof.** The graph  $G$  is  $K_{1, \lfloor k/2 \rfloor + 1}$ -free because  $\lambda \geq 1$ . Let  $t = \lfloor \frac{1}{2} \lceil \frac{k+2}{2} \rceil \rfloor$  and  $n = \lfloor k/2 \rfloor + 1$ . Then  $k \geq 2t + n - 2$ . The result follows from Theorems 3.2 and 3.12.  $\square$

We improve the previous result when  $\mu = 1$ .

**Theorem 3.14.** *If  $G$  is a distance-regular graph with even order,  $\lambda \geq 1$  and  $\mu = 1$ , then  $G$  is  $\lfloor \frac{k+1-\frac{k}{\lambda+1}}{2} \rfloor$ -extendable.*

**Proof.** The condition  $\mu = 1$  implies that  $\Gamma_1(x)$  is a disjoint union of cliques on  $\lambda + 1$  vertices, for any vertex  $x$  of  $G$ . Hence,  $\lambda + 1$  divides  $k$  and  $G$  is  $K_{1, \frac{k}{\lambda+1}+1}$ -free. Let  $t = \lfloor \frac{k+1-\frac{k}{\lambda+1}}{2} \rfloor$  and  $n = \frac{k}{\lambda+1} + 1$ . Then  $2t + n - 2 \leq k$ . The conclusion follows from [Theorems 3.2](#) and [3.12](#).  $\square$

The following theorem is an improvement of [Corollary 3.13](#) when  $3 \leq \mu \leq k/2$ .

**Theorem 3.15.** *Let  $G$  be a distance-regular graph with even order, and  $D \geq 3$ . If  $\lambda \geq 1$  and  $3 \leq \mu \leq k/2$ , then  $G$  is  $t$ -extendable, where  $t = \lceil \frac{(k-3)(k-1)}{3k-6} \rceil$ .*

**Proof.** Note that  $3 \leq \mu \leq k/2$  implies that  $k \geq 6$ . If  $G$  is not  $t$ -extendable, by [Lemma 3.5](#), there exists a disconnecting  $S$  with  $s$  vertices such that  $o(G-S) \geq s - 2t + 2$  (and in addition, the subgraph induced by  $S$  contains  $t$  independent edges). Let  $S$  be a disconnecting set with minimum cardinality such that  $o(G - S) \geq s - 2t + 2$ . Note that such  $S$  may not contain  $t$  independent edges. Let  $O_1, O_2, \dots, O_r$  be all the odd components of  $G - S$ , with  $r \geq s - 2t + 2$ . Let  $a \geq 0$  denote the number of singleton components among  $O_1, \dots, O_r$ .

We claim that  $e(A, S) \geq 3k - 6$  for any non-singleton odd component  $A$  of  $G - S$ .

Let  $A$  be a non-singleton odd component of  $G - S$  and  $B = (A \cup N(A))^c$ . If  $|A| \leq k - 1$ , the claim follows from [Lemma 3.3](#). Assume that  $|A| \geq k$ . Let  $S' := \{s \in N(A) \mid N(s) \subseteq A \cup N(A)\}$ . Then  $|S'| \leq 1$ . Otherwise, assume that  $x \neq y \in S'$ . Define  $S_0 = S \setminus \{x, y\}$  and  $A_0 = A \cup \{x, y\}$ . Then  $S_0$  is a disconnecting set with  $o(G - S') = o(G - S) \geq |S| - 2t + 2 > |S'| - 2t + 2$ , contradicting the minimality of  $|S|$ .

If we let  $A' := \{a \in A \mid d(a, b) = 2 \text{ for some } b \in B\}$ , then  $e(A, S) \geq \mu|A'|$ . If  $|A'| \geq k - 2$ , we get  $e(A, S) \geq \mu|A'| \geq 3(k - 2)$  and we are done. Otherwise, if  $|A'| < k - 2$ , then the set  $A' \cup S'$  is a disconnecting set with less than  $k - 1$  vertices, contradicting [Theorem 3.2](#). This finishes our proof of the claim.

Counting the number of edges between  $S$  and  $O_1 \cup \dots \cup O_r$ , we obtain the following

$$ks \geq e(S, O_1 \cup \dots \cup O_r) \geq ak + (r - a)(3k - 6) \geq ak + (s - 2t + 2 - a)(3k - 6). \tag{6}$$

This inequality is equivalent to

$$t \geq \frac{(k - 3)(s - a) + 3k - 6}{3k - 6} \tag{7}$$

and since  $s - a \geq k - 1$  ([Lemma 3.10](#)), we obtain that

$$t \geq \frac{(k - 3)(k - 1)}{3k - 6} + 1. \tag{8}$$

This is a contradiction with  $t = \lceil \frac{(k-3)(k-1)}{3k-6} \rceil$ .  $\square$

A straightforward calculation shows that  $\lceil \frac{(k-3)(k-1)}{3k-6} \rceil = \lfloor \frac{k}{3} \rfloor$  for  $k \geq 4$ .

**Theorem 3.16.** *Let  $G$  be a non-bipartite distance-regular graph with  $D \geq 3$  and  $\mu > k/2$ . Then  $\lambda \geq 1$  and  $G$  is  $t$ -extendable, where  $t = \lfloor k/3 \rfloor$ .*

**Proof.** [Lemma 3.11](#) implies that  $G$  is a Taylor graph with intersection array  $\{k, \mu, 1; 1, \mu, k\}$ . If  $\lambda = 0$ , then  $\mu = k - 1$  and  $G$  is obtained by deleting a perfect matching from  $K_{(k+1) \times (k+1)}$  (see [[6](#), Corollary 1.5.4]) which is a bipartite graph, contradiction.

Thus  $\lambda \geq 1$ . It is known that for any  $x \in V(G)$ ,  $\Gamma_1(x)$  is a strongly regular graph with parameters  $(k, \lambda, \frac{3\lambda-k-1}{2}, \frac{\lambda}{2})$  (see [[6](#), Section 1.5]). If  $\frac{3\lambda-k-1}{2} \geq 1$ , then [Lemma 3.9](#) implies that  $\alpha(\Gamma_1(x)) \leq k/3$ . If  $G$  is not  $t$ -extendable, then there is a vertex disconnecting set  $S$  containing  $t$  independent edges, such that  $G - S$  has at least  $s - 2t + 2 \geq k - 2t + 2 \geq 3$  odd components. Picking one vertex from each odd component yields an independent set  $I$  in  $G$ . If two vertices of this independent set were at distance 3, then the neighborhood of these two vertices will be formed by the remaining  $2k$  vertices of the graph



and therefore,  $G - S$  would have only two odd components, contradiction. Thus, any two vertices of this independent set are at distance 2 to each other. Pick a vertex  $x$  in this independent set. Any subset of  $k - 2t + 1$  vertices of  $I \setminus \{x\}$  will be an independent set in  $\Gamma_1(y)$ , where  $y$  is the antipodal vertex to  $x$ . Thus,  $k - 2t + 1 \leq k/3$ , contradiction with  $t = \lfloor k/3 \rfloor$ . If  $\frac{3\lambda - k - 1}{2} = 0$ , then  $\Gamma_1(x)$  has parameters  $(3\lambda - 1, \lambda, 0, \lambda/2)$ . If  $\lambda = 2$ ,  $\Gamma_1(x)$  is  $C_5$  which implies that  $k = 5$  and  $\mu = 2$ , contradiction with  $k/2 < \mu$ . If  $\lambda \geq 4$ , then  $\Gamma_1(x)$  must have integer eigenvalues implying that  $x^2 + \frac{\lambda}{2}x - \frac{\lambda}{2} = 0$  has integer roots. However,  $(\lambda/2)^2 + 2\lambda$  is not a perfect square, contradiction.  $\square$

In the end of this subsection, we will show that bipartite distance-regular graphs have high extendability.

**Theorem 3.17.** *If  $G$  is a bipartite distance-regular graph with valency  $k$ , then  $G$  is  $t$ -extendable, where  $t = \lfloor \frac{k+1}{2} \rfloor$ .*

**Proof.** Let  $X$  and  $Y$  be the color classes of  $G$ , where  $|X| = |Y| = m$ . Assume that  $G$  is not  $t$ -extendable. By Lemma 3.6,  $G$  has an independent set  $I$  of size at least  $m - t + 1$ , such that  $I \not\subseteq X$  and  $I \not\subseteq Y$ . Let  $A = I \cap X$ ,  $B = I \cap Y$ ,  $C = X \setminus A$ ,  $D = Y \setminus B$ . If  $|A| = a$ , then  $|B| \geq m - a - t + 1$ ,  $|C| = m - a$  and  $|D| \leq a + t - 1$ . As there are  $ak$  edges between  $A$  and  $D$ , and  $(a + t - 1)k \geq |D|k = e(D, X) = e(A, D) + e(C, D)$ , there are at most  $(t - 1)k$  edges between  $A$  and  $D$ . This implies that  $G$  has an edge cut of size at most  $(t - 1)k$ , which disconnects  $G$  into two vertex sets  $B \cup C$  and  $A \cup D$ . Without loss of generality, assume that  $|A \cup D| \leq m$ . By the second part of Lemma 3.8, we have

$$|A \cup D| > v \left( 1 - \frac{e(A \cup D, B \cup C)}{\mu k_2} \right) \geq 2m \left( 1 - \frac{(t - 1)k}{(k - 1)k} \right) \geq 2m(1 - 1/2) = m,$$

contradiction with  $|A \cup D| \leq m$ .  $\square$

### 3.3. The 2-extendability of distance-regular graphs of valency $k \geq 3$

Lou and Zhu [33] proved that any strongly regular graph of even order is 2-extendable with the exception of the complete tripartite graph  $K_{2,2,2}$  and the Petersen graph. Cioabă and Li [15] showed that any strongly regular graph of even order and valency  $k \geq 5$  is 3-extendable with the exception of the complete 4-partite graph  $K_{2,2,2,2}$ , the complement of the Petersen graph and the Shrikhande graph (see [7, page 123] for a description of this graph).

In this subsection, we prove that any distance-regular graph of diameter  $D \geq 3$  is 2-extendable. By Corollary 3.13, any distance-regular graph with  $\lambda \geq 1$  and  $k \geq 5$  is 2-extendable. Note also that any distance-regular graph of even order having valency  $k \leq 4$  and diameter  $D \geq 3$  must have  $\lambda = 0$  (see [5,9]). Theorem 3.17 implies that any bipartite distance-regular graph of valency  $k \geq 3$  is 2-extendable. Thus, we only need to settle the case of non-bipartite distance-regular graphs with  $\lambda = 0$ . We will need the following lemma.

**Lemma 3.18.** *If  $G$  is a non-bipartite distance-regular graph with valency  $k \geq 5$  and  $\lambda = 0$ , then  $\alpha(G) < v/2 - 1$ .*

**Proof.** If  $g$  is the odd girth of  $G$ , then  $v > 2g$  and Corollary 2.2 implies that  $\alpha(G) \leq \frac{v}{2} \left( 1 - \frac{1}{g} \right) < v/2 - 1$ .  $\square$

**Theorem 3.19.** *If  $G$  is a non-bipartite distance-regular graph with even order,  $D \geq 3$ , valency  $k \geq 3$  and  $\lambda = 0$ , then  $G$  is 2-extendable.*

**Proof.** We prove this result by contradiction and the outline of our proof is the following. We assume that  $G$  is not 2-extendable. Lemma 3.5 implies that there is a vertex disconnecting set  $S$ , such that the graph induced by  $S$  contains at least 2 independent edges and  $\alpha(G - S) \geq |S| - 2$ . Without loss of generality, we may assume that  $S$  is such a disconnecting set with the maximum size. We then prove that  $G - S$  does not have non-singleton components which implies that  $V(G) - S$  is an independent set of size at least  $v/2 - 1$ , contradiction to Lemma 3.18.

Assume  $k \geq 5$  first.

Note that any odd non-singleton component of  $G - S$  is not bipartite. Otherwise, assume there is a bipartite odd component of  $G - S$  with color classes  $X$  and  $Y$  such that  $|X| > |Y|$ . Let  $S' = S \cup Y$ . Then  $|S'| > |S|$  and  $o(G - S') \geq |S'| - 2$ , contradiction with  $|S|$  being maximum. Also,  $G - S$  has no even components. Otherwise, we can add one vertex of one such even component to  $S$  and creating a larger disconnecting set and an extra odd component, contradicting again the maximality of  $|S|$ . It is easy to see that  $G - S$  does not have any components with 3 vertices, because  $G$  is triangle free and any component with 3 vertices must be a path, hence bipartite.

Assume that  $A$  is an odd non-singleton component of  $G - S$ . If we can show that  $e(A, S) \geq 3k - 3$ , then using  $o(G - S) \geq |S| - 2$  and  $e(X, S) \geq k$  for any component  $X$  of  $G - S$  (from [Theorem 3.1](#)), we obtain the following contradiction by counting the edges between  $S$  and  $S^c$ :

$$k|S| - 4 \geq e(S, S^c) \geq 3k - 3 + k(|S| - 3) = k|S| - 3, \tag{9}$$

finishing our proof.

We now prove  $e(A, S) \geq 3k - 3$  whenever  $A$  is a non-singleton odd component of  $G - S$ .

If  $5 \leq |A| \leq 2k - 3$ , then as  $A$  has no triangle, Turán's theorem implies that  $e(A) \leq \frac{|A|^2 - 1}{4}$ , where  $e(A)$  denotes the number of edges with both endpoints in  $A$ . Thus,  $e(A, S) \geq k|A| - 2e(A) \geq k|A| - \frac{|A|^2 - 1}{2} \geq 3k - 4$ . The last equality is attained when  $A$  induces a bipartite graph  $K_{k-1, k-2}$ . This is impossible as the graph induced by  $A$  is not bipartite. Hence,  $e(A, S) \geq 3k - 3$ .

Let  $A$  be an odd component of  $G - S$  such that  $|A| \geq 2k - 1$ . If every vertex of  $A$  sends at least one edge to  $S$ , then we have two subcases:  $\mu \geq 2$  and  $\mu = 1$ .

If  $\mu \geq 2$ , then we can define  $S' := \{s \in N(A) \mid N(s) \subseteq A \cup N(A)\}$ . If  $|S'| \geq 3$ , then  $e(A, S) + 2e(S) \geq 3k + 1$ . This is because  $e(A, S) + 2e(S) = \sum_{x \in S} |N(x) \cap (A \cup S)|$ . As the graph induced by  $S$  contains at least 2 independent edges, the previous sum contains at least 4 positive terms, and at least 3 of such terms are equal to  $k$ . On the other hand, as in (9), counting the number of edges between  $S$  and  $S^c$ , we get that  $e(A, S) + (|S| - 3)k \leq e(S, S^c) = |S|k - 2e(S)$ . Thus,  $e(A, S) + 2e(S) \leq 3k$ , contradiction. If  $|S'| \leq 2$ , then let  $B = (A \cup N(A))^c$  and  $A' = \{a \in A \mid \exists b \in B \text{ such that } d(a, b) = 2\}$ . If  $A' = A$ , then  $|A'| \geq 2k - 1 \geq k - 2$ . If  $A \neq A'$ , then because  $A' \cup S'$  is a disconnecting set, [Theorem 3.2](#) implies that  $|A' \cup S'| \geq k$  and therefore,  $|A'| \geq k - 2$ . As each vertex in  $A'$  sends at least  $\mu$  edges to  $S$  and  $\mu \geq 2$ , we get that  $e(A, S) \geq 2k - 1 + (k - 2)(\mu - 1) \geq 3k - 3$ .

If  $\mu = 1$ , then the graph induced by  $A$  contains no triangles and four-cycles. If  $|A| \geq 3k - 3$ , then  $e(A, S) \geq 3k - 3$ , as every vertex of  $A$  sends at least one edge to  $S$ . If  $|A| \leq 3k - 4$ , then  $e(A) \leq \frac{|A|\sqrt{|A|-1}}{2}$  since the graph induced by  $A$  contains no triangles and four-cycles (see [[24](#), [Theorem 2.2](#)] or [[48](#), [Theorem 4.2](#)]). Since also  $2k - 1 \leq |A| \leq 3k - 4$ , we get that  $e(A, S) = k|A| - 2e(A) \geq |A|(k - \sqrt{|A|-1}) \geq (2k - 1)(k - \sqrt{3k - 5}) \geq 3k - 3$ .

The only case remaining is when  $|A| \geq 2k - 1$  and  $A$  has a vertex  $x$  having no neighbors in  $S$  (such a vertex is called a deep point in [[9](#)]). Note that  $A^c$  always has a deep point because every vertex in  $V(G) \setminus (A \cup S)$  is a deep point of  $A^c$ . We have two cases:

1. When  $k \geq 6$ , we will show that  $e(A, S) \geq 3k - 3$ . Otherwise, by [Lemma 3.8](#),

$$|A| > v \left( 1 - \frac{3k - 4}{\mu k_2} \right) = v \left( 1 - \frac{3k - 4}{k(k - 1)} \right) \geq v/2. \tag{10}$$

The last inequality is true since  $k \geq 6$ . As  $A^c$  always has a deep point, by [Lemma 3.8](#) again, we get that  $|A^c| > v/2$ , contradiction.

2. When  $k = 5$ , we do not have inequality (10) so we need a different proof. If  $\mu \geq 3 > k/2$ , by [Theorem 3.16](#),  $\lambda \geq 1$ , contradiction. So, we must have  $1 \leq \mu \leq 2$ .

We first show that  $A$  is the only non-singleton component of  $G - S$ . Assume that there are at least two non-singleton components in  $G - S$ . Let  $B$  be another non-singleton component of  $G - S$ . Then  $B$  has a deep point, by previous arguments. If  $e(A, S) \geq 2k - 1$  and  $e(B, S) \geq 2k - 1$ , then  $k|S| - 4 \geq e(S, S^c) \geq 2(2k - 1) + (|S| - 4)k = k|S| - 2$ , contradiction. Without loss of generality, assume that  $e(A, S) \leq 2k - 2$ . By [Lemma 3.8](#),  $|A| > v \left( 1 - \frac{2k - 2}{\mu k_2} \right) = v \left( 1 - \frac{2}{k} \right) = \frac{3v}{5}$ . On the other hand, [Lemma 3.8](#) also implies that  $|A^c| > \frac{3v}{5}$ , contradiction.

Thus,  $A$  is the only non-singleton component in  $G - S$ . Recall that  $|A| \geq 2k - 1$  and  $A$  has a deep point  $x$ . If  $e(A, S) \leq 3k - 5$ , by Lemma 3.8,  $|A| > v \left(1 - \frac{3k-5}{\mu k_2}\right) = v \left(1 - \frac{10}{20}\right) \geq v/2$ . Lemma 3.8 also implies that  $|A^c| > v/2$ , contradiction. If  $e(A, S) = 3k - 4 = 11$ , by counting the edges between  $S$  and  $S^c$ , we know that  $S$  contains exactly two independent edges. Also,  $o(G - S) = |S| - 2$ . Let  $X$  be the set of singleton components of  $G - S$ . We have  $|X| = |S| - 3$ . By Theorem 3.2,  $|S| \geq k + 1 = 6$  and  $|X| \geq 3$ .

Now, we have two subcases:

- (i) Assume that  $\mu = 2$ . Let  $W = \{a \in A \mid \exists s \in S, a \sim s\}$ . Note that  $W \subset A$  and  $W$  is a disconnecting set of  $G$ . By Theorem 3.2,  $|W| \geq 5$  and the only disconnecting sets of 5 vertices are the neighbors of some vertex. If  $|W| = 5$ , then we have  $W = N(x)$  for some vertex  $x$ . By Lemma 3.7, the subgraph induced by the vertices at distance 2 or more from  $x$  is connected. In other word,  $W$  disconnects  $G$  into two components,  $x$  and  $V \setminus (W \cup \{x\})$ . Since  $|A^c| > 1$ , we must have  $A^c = V \setminus (W \cup \{x\})$  and  $A \setminus W = \{x\}$ . Hence,  $|A| = 6$ , contradicting to  $|A|$  is odd. So,  $|W| \geq 6$ .

We claim that for any  $x \in W$ , there exists  $t \in X$  such that  $d(x, t) = 2$ . Assume otherwise. Then there is  $s \in S$  such that  $N(s) \subset A \cup S$ . Since the graph induced by  $S$  contains exactly two independent edges,  $s$  has at most one neighbor in  $S$  and at least four neighbors in  $A$ . If we let  $A' = A \cup \{s\}$  and  $S' = S \setminus \{s\}$ , then  $e(A', S') \leq 8$ . By Lemma 3.8,  $|A'| > v \left(1 - \frac{8}{\mu k_2}\right) = v \left(1 - \frac{8}{20}\right) = \frac{3v}{5}$ . On the other hand, Lemma 3.8 also implies that  $|(A')^c| > \frac{3v}{5}$ , contradiction.

As  $\mu = 2$ , each vertex in  $W$  has at least 2 neighbors in  $S$  and  $e(A, S) \geq 12$ , which is also a contradiction.

- (ii) Assume that  $\mu = 1$ . We will first prove that  $a_2 \leq 1$ . If for every  $s \in S$ ,  $|N(s) \cap X| \leq 2$ , by counting the edges between  $S$  and  $X$ , we have  $5|X| = e(S, X) \leq 2|S|$ . On the other hand,  $|X| = |S| - 3 \geq \frac{5}{2}|X| - 3$ , thus  $|X| \leq 2$ , contradicting to  $|X| \geq 3$ . Hence, there exists  $s \in S$  such that  $|N(s) \cap X| \geq 3$ . Let  $x, y, z \in N(s) \cap X$ . As  $\mu = 1$ ,  $N(x) \cap N(y) = N(y) \cap N(z) = N(x) \cap N(z) = \{s\}$ . Let  $U = (N(x) \cup N(y) \cup N(z)) \setminus \{s\}$ . It is easy to check that  $U \subset N_2(s)$ ,  $|U| = 12$ ,  $|N_2(s)| = 20$ , and  $\Gamma_2(s)$  is  $a_2$ -regular. Since there are at most two edges inside  $U$ ,  $12a_2 - 4 \leq e(U, N_2(s) \setminus U) \leq 8a_2$  and thus  $a_2 \leq 1$ .

Note that  $\mu = 1$  and  $a_2 \leq 1$  imply that  $b_2 \geq 3$ . If there exists  $r \in S$ , such that  $N(r) \subset X$ , then  $d(r, A) \geq 3$ . By Lemma 3.8,  $|A^c| > v \left(1 - \frac{3k-4}{k_2 b_2}\right) \geq v \left(1 - \frac{11}{60}\right) = \frac{49v}{60}$ .

On the other hand, Lemma 3.8 also implies that  $|A| > \left(1 - \frac{3k-4}{\mu k_2}\right) = v \left(1 - \frac{3k-4}{k(k-1)}\right) = \frac{9v}{20}$ , contradiction. Thus, for all  $r \in S$ , we have  $N(r) \not\subset X$ . Consider the edges between  $X$  and  $S$ . We have  $5|X| = e(X, S) \leq 4|S|$  and therefore,  $|X| = |S| - 3 \geq 5|X|/4 - 3$ . Thus,  $|X| \leq 12$ ,  $|S| \leq 15$ ,  $27 \geq |A^c| > 9v/20$  and  $v < 60$ . Note that there is no distance-regular graph with  $v < 60$ ,  $k = 5$ ,  $\lambda = 0$ ,  $\mu = 1$  and  $a_2 \leq 1$ , see the table [5,22] (where it was shown that there exists no distance-regular graph with intersection array  $\{5, 4, 3; 1, 1, 2\}$ ).

This finishes the proof of the case  $k \geq 5$ .

When  $k = 4$ , all the distance-regular graph with even order are bipartite [9] so we are done by Theorem 3.17.

When  $k = 3$ , there are 3 non-bipartite triangle-free distance-regular graphs with even order (see [3] or [6, Chapter 7]): the Coxeter graph (intersection array  $\{3, 2, 2, 1; 1, 1, 1, 2\}$ ), the Dodecahedron graph (intersection array  $\{3, 2, 1, 1, 1; 1, 1, 1, 2, 3\}$ ) and the Biggs–Smith graph (intersection array  $\{3, 2, 2, 2, 1, 1, 1, 1; 1, 1, 1, 1, 1, 3\}$ ). We will show that each one of them is 2-extendable.

Let  $G$  be the Coxeter graph. Then  $G$  has 28 vertices, girth 7 and independence number 12 (see [2] for example). If  $G$  is not 2-extendable, there is a disconnecting set  $S$  of maximum size, such that the graph induced by  $S$  contains 2 independent edges and  $o(G - S) \geq |S| - 2$ . As  $|S| \geq 4$ , we have  $o(G - S) \geq 2$ . Assume that  $G - S$  contains a non-singleton component  $A$ . As  $|A^c| \geq |S| + 1 \geq 5$ , we have that  $3 \leq |A| \leq v - |A^c| \leq 23$ . If  $3 \leq |A| \leq 5$ , the graph induced by  $A$  is bipartite as the girth of  $G$  is 7. As in the case  $k \geq 5$ , we can construct a larger disconnecting set contradicting the maximality of  $S$ . If  $|A^c| = 5$ , then we have that  $|S| = 4$  and there is one singleton component  $\{x\}$  in  $A^c$ . Since  $S$  contains two independent edges and  $x$  has three neighbors in  $S$ , we obtain  $\lambda \neq 0$ , contradiction.

Let  $G$  be the Dodecahedron graph. Then  $G$  has 20 vertices, girth 5 and independence number 8 (see [26, pp. 6 and pp. 67] for example). If  $G$  is not 2-extendable, there is a disconnecting set  $S$  of maximum size, such that the graph induced by  $S$  contains 2 independent edges and  $o(G - S) \geq |S| - 2$ . As  $|S| \geq 4$ , we have  $o(G - S) \geq 2$ . Assume that  $G - S$  contains a non-singleton component  $A$ . As  $|A^c| \geq |S| + 1 \geq 5$ , we have  $3 \leq |A| \leq 15$ . We will prove that  $|A| \neq 3, 5, 7, 9$  and  $|A^c| \neq 5, 7, 9$ . By maximality of  $|S|$ , the graph induced by  $A$  is not bipartite. So,  $|A| \neq 3$ . If  $|A| = 7$ , then the graph induced by  $A$  contains at most one cycle. Thus,  $e(A) \leq 7$  and  $e(A, A^c) = 3|A| - 2e(A) \geq 7$ . If  $|A| = 9$ , then the graph induced by  $A$  contains at most two cycles. Thus,  $e(A) \leq 10$  and  $e(A, A^c) = 3|A| - 2e(A) \geq 7$ . In either case, we will obtain a contradiction by inequality (9). Using the same argument, we can show that  $|A^c| \neq 7, 9$ . If  $|A^c| = 5$ , then we have that  $|S| = 4$  and there is one singleton component  $\{x\}$  in  $A^c$ . Since  $S$  contains two independent edges and  $x$  has three neighbors in  $S$ , we obtain  $\lambda \neq 0$ , contradiction.

Let  $G$  be the Biggs–Smith graph. Then  $G$  has girth 9 and 102 vertices. If  $G$  is not 2-extendable, there is a disconnecting set  $S$  of maximum size, such that the graph induced by  $S$  contains 2 independent edges and  $o(G - S) \geq |S| - 2$ . Assume that  $G - S$  contains a non-singleton component  $A$ . By similar argument as the previous cases, we can assume that  $5 \leq |A| \leq 97$ . When  $5 \leq |A| \leq 7$ ,  $e(A) = |A| - 1$  and  $e(A, A^c) = 3|A| - 2e(A) = |A| + 2 \geq 7$ . When  $9 \leq |A| \leq 15$ ,  $e(A) \leq |A|$  and  $e(A, A^c) = 3|A| - 2e(A) \geq |A| \geq 9$ . When  $17 \leq |A| \leq 51$ ,  $e(A, A^c) \geq \frac{(3-2.56155)|A|(102-|A|)}{102} \geq 6.21134 > 6$  (see [8, Corollary 4.8.4] or [37]). If  $e(A, S) \geq 3k - 3 = 6$ , we will obtain a contradiction by inequality (9). Using the same argument, we can obtain a contradiction when  $5 \leq |A^c| \leq 51$ . Thus, all the components of  $G - S$  are singletons. Therefore,  $\alpha(G) \geq o(G - S) \geq \max\{102 - |S|, |S| - 2\} \geq 50$ , contradiction with  $\alpha(G) = 43$  (see the table on page 6).  $\square$

#### 4. Final remarks

Note that some of the bounds in this paper may be improved if one obtains better lower bound for  $e(A, A^c)$  with  $k \leq |A| \leq v - k$ . We make the following conjecture which is still open for strongly regular graphs [15].

**Conjecture 4.1.** *If  $G$  is a distance-regular graph of valency  $k$ , even order  $v$  and diameter  $D \geq 3$ , then the extendability of  $G$  is at least  $\lceil k/2 \rceil - 1$ .*

A stronger property than  $m$ -extendability is the property  $E(m, n)$  introduced by Porteous and Aldred [43]. A connected graph with at least  $2(m + n + 1)$  vertices is said to be  $E(m, n)$  if for every pair of disjoint matchings  $M, N$  of  $G$  of size  $m$  and  $n$ , respectively, there exists a perfect matching in  $F$  such that  $M \subseteq F$  and  $F \cap N = \emptyset$ . It would be interesting to investigate this property for distance-regular graphs and graphs in association schemes. Godsil [25] conjectured that the edge-connectivity of a connected class of an association scheme equals its valency and Brouwer [4] made the stronger conjecture that the vertex-connectivity equals the valency. Brouwer's conjecture has been proved by Brouwer and Koolen [10] for distance-regular graphs, but both Godsil and Brouwer's conjectures are open in the other cases. Godsil's conjecture would imply that any connected class in an association scheme of even order, has a perfect matching. To our knowledge, this is not known at present time.

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