Eigenvalues, Expanders and Gaps between Primes

by

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I kept myself positive, by not getting all negative.

Yogi Berra

Abstract

We consider several problems regarding the eigenvalues of regular graphs, their connection with expansion and gaps between primes.

Using non-elementary methods, J.-P. Serre has proved several theorems regarding the extreme eigenvalues of regular graphs. In the first part, we present new and elementary proofs of some of Serre's results. We also discuss the eigenvalues of claw free regular graphs and answer a question of Linial.

In the second part, we improve a result of Greenberg regarding the behaviour of the extreme eigenvalues of irregular graphs.

The third part of the thesis is concerned with the Abelian Cayley graphs. We show that these graphs contain a large number of closed walks of even length. Using this result, we prove the nontrivial eigenvalues of Abelian Cayley graphs are large.

In the last part, we present a simple method of constructing new expanders from old. This method has connections with the study of gaps between consecutive primes. We show that for almost all the degrees, one can construct regular graphs with small nontrivial eigenvalues by modifying previous constructions of expanders.

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Cu dragoste, pentru Mel

Statement of originality

All the results with proofs presented in this thesis are original, unless otherwise stated. The results quoted from literature are presented as statements with indicated reference for their proof.

Contents

CHAPTER 1

Introduction

1.1. Eigenvalues of graphs

In this thesis, we discuss eigenvalues of graphs and their connections with expansion and gaps between primes. Graphs arise in many settings and can be used to solve many theoretical and practical problems. A *graph* consists of a *vertex* set V, an *edge* set E, and a relation that associates with each edge an unordered pair of vertices called its *endpoints*. A graph is *finite* if its vertex set and edge set are finite.

The *adjacency matrix* $A(X)$ of a graph X is the matrix with rows and columns indexed by the vertices of X with the uv-entry equal to the number of edges between vertices u and v. We denote by $\lambda_1(X) \geq \lambda_2(X) \geq \cdots \geq \lambda_n(X)$ the eigenvalues of $A(X)$ and we call them the eigenvalues of X from now on. Notice that since X is undirected, it follows that A is symmetric which implies the eigenvalues of X are real. The *spectrum* of a graph is the (multi)set of its eigenvalues.

Spectral graph theory is the study of the eigenvalues of graphs. It has a long history and is one of the most dynamic and fascinating subjects in graph theory with numerous applications in other fields, including communication networks [**1**], theoretical computer science [**58**], extremal graph theory [**65**], combinatorial optimization [**72**] and error-correcting codes [**89**]. The importance of spectral graph theory is also demonstrated by the large number of books in which eigenvalues are studied such as those by Biggs [**8**], Chung [**16**], Cvetkovic, Doob and Sachs [**23**], Davidoff, Sarnak and Vallete [**24**], Godsil and Royle [**37**] and Lubotzky [**62**].

To quote Fan Chung [**16**],

> Just as astronomers study stellar spectra to determine the make-up of distant stars, one of the main goals of spectral graph theory is to deduce the principal properties and structure of a graph from its graph spectrum.

It is also of great interest to determine how the structure of a graph influences the behaviour of its eigenvalues. Some of the connections between the eigenvalues and the structural properties of a graph will be described in this thesis.

Historically, the first relations between eigenvalues and a property of a graph, namely its chromatic number, were observed in 1967 by Wilf [**94**] and in 1970 by Hoffman [**43**]. Their results are presented in section 1.3.

In the mid 1980's, Alon and Milman [**3**] (see also [**1**]) and Tanner [**91**] found that eigenvalues play an important role in the study of expanders. These graphs have numerous applications in complexity theory, coding theory and error-correcting codes. We briefly describe the connections between eigenvalues and expanders in section 1.4.

A relation between the diameter of a graph and its eigenvalues was observed by Alon and Milman in 1985. Their result was later improved by Chung [**14**] in 1989. These theorems are presented in section 1.5.

The eigenvalues of a real symmetric matrix can be described as the solutions to a minimum and maximum problem by the Courant-Fisher theorem. This along with some of its consequences are described in the last section of this chapter.

Throughout this thesis, X will denote an undirected graph possibly with multiple edges, but without loops and having vertex set $V(X)$ of order n and edge set $E(X)$. Usually, the vertex set of X is $[n] = \{1, 2, \ldots, n\}$.

A graph X is *k-regular* if each vertex has exactly k neighbours. The *complete graph* K_n is the graph on n vertices in which any two distinct vertices are adjacent. It is obvious from the definition that K_n is $(n-1)$-regular. The *cycle* C_n is the graph on n vertices that can be arranged around a circle so that two vertices are adjacent if and only if they appear consecutively around the circle. Again, it follows from definition that C_n is 2-regular.

A *closed walk* in X of length $r \geq 0$ starting at $v_0 \in V(X)$ is a sequence $v_0, v_1, \ldots, v_r$ of vertices of X such that $v_r = v_0$ and v_{i-1} is adjacent to v_i for $1 \leq i \leq r$. We denote by $t_r(x)$ the number of closed walks of length r starting at the vertex x and by $\Phi_r(X) = \sum_{x \in V(X)} t_r(x)$ the number of closed walks of length r in X.

For $r \geq 0$, let $c_r(X)$ be the number of cycles of length r in a graph X. The *girth* of the graph X, denoted by $\mathrm{girth}(X)$, is the smallest r such that $c_r(X) > 0$ if X contains at least one cycle and $+\infty$ otherwise. The *odd girth* of X, denoted by $\mathrm{oddg}(X)$, is the smallest odd r such that $c_r(X) > 0$ if X contains at least one odd cycle and $+\infty$ otherwise. If X is a connected graph, then X is a *tree* if $\mathrm{girth}(X) = +\infty$. Also, a graph X is *bipartite* if $\mathrm{oddg}(X) = +\infty$.

The next result shows a simple connection between the eigenvalues of a graph X and the structure of X.

Proposition 1.1.1. *If X is a graph, then for each $r \geq 1$,*

$$\sum_{i=1}^{n} \lambda_i^r(X) = \mathrm{tr}\, A^r(X) = \Phi_r(X)$$

Proof. The first equality is clear because A^r has eigenvalues $\lambda_1^r, \lambda_2^r, \ldots, \lambda_n^r$. The second equality follows easily by induction on r. $\square$

The next result states the basic properties of the largest eigenvalues of regular graphs. We sketch its proof here.

Proposition 1.1.2. *Let X be a k-regular graph on n vertices. Then*

(i) $\lambda_1 = k$.

(ii) $|\lambda_i| \leq k$ *for each i with* $1 \leq i \leq n$.

(iii) *the multiplicity of λ_1 equals the number of components of X.*

Proof. (i) Let $\underline{1} \in \mathbb{R}^n$ be the vector with all the entries equal to 1. Then $A\underline{1} = k\underline{1}$ and thus, k is an eigenvalue of X.

(ii) Let $x \in \mathbb{R}^n$ be a unit eigenvector corresponding to some eigenvalue λ_i of X. Since $Ax = \lambda_i x$, then

$$|\lambda_i| = |x^t Ax| = |2 \sum_{uv \in E(X)} x_u x_v| \leq \sum_{uv \in E(X)} (x_u^2 + x_v^2) = k \sum_{u \in V(X)} x_u^2 = k.$$

(iii) The statement follows easily by noting that equality holds throughout the previous relation if and only if x is constant on the components of X. $\square$

Thus, the spectrum of every finite, connected, k-regular graph X is included in the interval $[-k, k]$. While k is always an eigenvalue for a k-regular graph X, $-k$ appears as an eigenvalue if and only if X is also bipartite. This result is contained in the following proposition. Its proof is also simple, see for example [**24**].

Proposition 1.1.3. *Let X be a connected, k-regular graph on n vertices. The following are equivalent:*

(i) *X is bipartite.*

(ii) *The spectrum of X is symmetric about* 0.

(iii) $\lambda_n = -k$.

The previous two results provide useful information regarding the largest and the smallest eigenvalue of a regular graph. Just by knowing the degree of a regular graph, we obtain the largest eigenvalue. However, the behaviour of the eigenvalues different from k and $-k$ is not as simple.

The eigenvalues k and $-k$ (if X bipartite) are called the *trivial* eigenvalues of X. If X is a connected k-regular graph, define $\lambda(X) = \max_{\lambda_i(X) \neq \pm k} |\lambda_i(X)|$. The difference $k - \lambda_2(X)$ is called the *spectral gap*[1] of X. Following Alon (see [**50**]), a graph X is called an (n, k, λ)*-graph* if it has n vertices, is k-regular, connected and $\lambda(X) \leq \lambda$.

Strongly regular graphs are a special class of regular graphs.

Definition 1.1.4. *A graph X on n vertices is strongly regular with parameters (n, k, a, c) if it is k-regular, every pair of adjacent vertices has a common neighbours and every pair of distinct nonadjacent vertices has c common neighbours.*

From the definition, it follows that if A is the adjacency matrix of an (n, k, a, c) strongly regular graph X, then

$$A^2 = kI + aA + c(J - I - A)$$

where J is the all one matrix. We deduce that the nontrivial eigenvalues of an (n, k, a, c) strongly regular graph X are

$$\theta = \frac{(a-c) + \sqrt{(a-c)^2 + 4(k-c)}}{2}$$

[1]Some authors define the spectral gap of X as $k - \lambda(X)$

and

$$\tau = \frac{(a-c) - \sqrt{(a-c)^2 + 4(k-c)}}{2}$$

We discuss some classes of strongly regular graphs in Section 4.

1.2. Thesis overview

In this thesis, we discuss the eigenvalues of graphs and their connections with expansion and gaps between consecutive primes.

J.-P. Serre [**24, 88**] has proved the following theorem regarding the largest eigenvalues of regular graphs. The known proofs of this theorem are non-elementary and use some properties of the Chebyschev polynomials. In Chapter 2, we present a new and elementary proof of Serre's theorem.

THEOREM 1.2.1 (Serre [**88**]). *For each $\epsilon > 0$, there exists a positive constant $c = c(\epsilon, k)$ such that for any k-regular graph X, the number of eigenvalues λ_i of X with $\lambda_i \geq (2-\epsilon)\sqrt{k-1}$ is at least $c|X|$.*

In Chapter 2, we also prove an analogue of Serre's theorem concerning the least eigenvalues of regular graphs. For $l \geq 1$, we denote by $\mu_l(X)$ the l-th smallest eigenvalue of a graph X.

THEOREM 1.2.2. *For any $\epsilon > 0$, there exist a positive constant $c = c(\epsilon, k)$ and a nonnegative integer $g = g(\epsilon, k)$ such that for any k-regular graph X with $\mathrm{oddg}(X) > g$, the number of eigenvalues μ_i of X with $\mu_i \leq -(2-\epsilon)\sqrt{k-1}$ is at least $c|X|$.*

This implies the following result of Winnie Li [**55**].

Theorem 1.2.3 (Li [**55**]). *Let* $(X_n)_{n\geq 0}$ *be a sequence of* k*-regular graphs such that* $\lim_{n\to+\infty} \operatorname{oddg}(X_n) = +\infty$*. Then*

$$\limsup_{n\to+\infty} \mu_1(X_n) \leq -2\sqrt{k-1}$$

Serre [**55**] has also proved by non-elementary means the following result that improves Li's theorem.

Theorem 1.2.4 (Serre [**55**]). *Let* $(X_n)_{n\geq 0}$ *be a sequence of* k*-regular graphs such that* $\lim_{n\to\infty} |V(X_n)| = +\infty$*. If*

$$\lim_{n\to+\infty} \frac{c_{2r+1}(X_n)}{|V(X_n)|} = 0 \tag{1.2.1}$$

for each $r \geq 1$*, then for each* $l \geq 1$

$$\limsup_{n\to\infty} \mu_l(X_n) \leq -2\sqrt{k-1}$$

In Chapter 2, we present a new and elementary proof of the previous result and provide examples showing that Theorem 1.2.4 fails if condition (1.2.1) is only satisfied for sufficiently large r.

In the same chapter, we answer a question of Linial [**57**] regarding the behaviour of the least eigenvalues of claw-free graphs. A graph is claw-free if no vertex has three pairwise non-adjacent neighbours.

Serre's theorem is related to a result of Greenberg that has not appeared in any journal as yet. In Chapter 3, we improve this result of Greenberg [**64**] involving the behaviour of the extreme eigenvalues of irregular graphs.

In Chapter 4, we show that Abelian Cayley graphs contain a large number of closed walks of even length. We use this result to prove the following Serre-type theorem for Abelian Cayley graphs. This shows that the Abelian Cayley graphs of degree k have many large nontrivial eigenvalues and implies that Abelian Cayley graphs are bad expanders.

THEOREM 1.2.5. *Given $k \geq 3$, for each $\epsilon > 0$, there exists a positive constant $C = C(\epsilon, k)$ such that for any multiplicative Abelian group G and for any symmetric set S of elements of G with $|S| = k$ and $1 \notin S$, the number of eigenvalues λ_i of the Cayley graph $X = X(G, S)$ such that $\lambda_i \geq k - \epsilon$ is at least $C \cdot |G|$.*

In the same chapter, we also discuss the chromatic and independence number of some finite analogues of the Euclidean graph.

In the last chapter, we present a simple method of constructing new expander graphs from old.

1.3. Chromatic and independence number

An r-colouring of a graph X is a function $f : V(X) \to \{1, 2, \ldots, r\}$. The colouring f is called *proper* if $f(x) \neq f(y)$ for any two adjacent vertices x and y. The *chromatic number* of X is the minimum r such that X admits a proper r-colouring and it will be denoted by $\chi(X)$. A subset S of $V(X)$ is called *independent* if no two distinct vertices of S are adjacent in X. The *independence number* of X is the size of the largest independent set of X and it will be denoted by $\alpha(X)$. Note that $\chi(X)\alpha(X) \geq |V(X)|$ for each graph X since for any proper colouring of X, each colour class is an independent set.

The problems of computing $\chi(X)$ and $\alpha(X)$ are NP-hard. The precise formulations of these statements and more details can be found in the monograph of Garey and Johnson [**36**].

The eigenvalues can provide useful bounds for $\chi(X)$ and $\alpha(X)$.

THEOREM 1.3.1. *Let X be a graph on n vertices. Then*

$$1+\frac{\lambda_1(X)}{-\lambda_n(X)} \leq \chi(X) \leq 1+\lambda_1(X)$$

The upper bound on $\chi(X)$ was proved by Wilf [**94**] and the lower bound was observed by Hoffman [**43**]. Hoffman also proved the following bound on the independence number.

THEOREM 1.3.2. *Let X be a k-regular graph on n vertices. Then*

$$\alpha(X) \leq \frac{-n\lambda_n(X)}{k-\lambda_n(X)}$$

Another spectral bound on the independence number was given by Cvetković [**21**].

THEOREM 1.3.3. *Let X be a graph with n vertices, n^+ positive eigenvalues and n^- negative eigenvalues. Then*

$$\alpha(X) \leq \min(n-n^+, n-n^-)$$

1.4. Expanders

Informally, expanders are highly connected sparse graphs. For $S, T \subseteq V(X)$, denote by $E(S,T)$ the set of all edges of X with one endpoint in S and the other in T. Then the *boundary* ∂S of a subset S of vertices, is $E(S, V(X)\setminus S)$, i.e. the set of edges with one endpoint in S and the other endpoint not in S. Obviously, $\partial S = \partial(V(X)\setminus S)$.

DEFINITION 1.4.1. *The expansion constant of the graph X is*

$$h(X) = \min\left\{\frac{|\partial S|}{|S|} : S \subset V(X), |S| \leq \frac{|V(X)|}{2}\right\}$$

Note that $h(X) > 0$ if and only if X is connected. Also, the definition implies that for any subset S containing at most half of the vertices of X, $|\partial S| \geq h(X)|S|$. Thus, $h(X)$ is a measure of the connectivity of X. It is of great interest to find infinite sequences of graphs X with $h(X)$ bounded away from zero (see definition (1.4.3)).

In general, computing $h(X)$ is NP-hard (see [**71**] for more details). However, in some special cases such as complete graphs and cycles, it is easy to find an exact formula for the expansion constant.

PROPOSITION 1.4.2. *For $n \geq 3$, $h(K_n) = \lceil \frac{n}{2} \rceil$ and $h(C_n) = \frac{2}{\lfloor \frac{n}{2} \rfloor}$.*

The complete graphs are the best expanders, but the number of edges is not linear, but quadratic in n.

DEFINITION 1.4.3. *Let $(X_n)_{n\geq 1}$ be a family of finite, connected, k-regular graphs with $|V(X_n)| \to +\infty$ as $n \to +\infty$. We call $(X_n)_{n\geq 1}$ a family of expanders if there is a positive constant ϵ such that $h(X_n) \geq \epsilon > 0$ for each $n \geq 1$.*

The following theorem relates the expansion constant of a regular graph to its spectral gap. See [**3, 71, 72, 91**] for proofs and related results.

THEOREM 1.4.4. *Let X be a finite and connected k-regular graph. Then*

$$\frac{k-\lambda_2}{2} \leq h(X) \leq \sqrt{2k(k-\lambda_2)}$$

This result implies that an infinite family X_n of k-regular graphs is a family of expanders if and only if for each n, X_n is an $(|V(X_n)|, k, \lambda)$-graph, where λ is a constant, $\lambda < k$. We will use this spectral characterization of families of expanders to construct regular graphs with good expanding properties in Chapter 5.

The next proposition, also known as the Expander Mixing Lemma, implies that the edges of a k-regular graph with small nontrivial eigenvalues, are evenly distributed in the graph. See [**5, 58**] for a proof and related results.

PROPOSITION 1.4.5. *If X is an (n, k, λ)-graph, then*

$$\left| |E(S,T)| - \frac{k}{n}|S||T| \right| \leq \lambda\sqrt{|S||T|}$$

for each $S, T \subset V(X)$.

1.5. Diameter

The *distance* between two vertices u and v, denoted $d(u, v)$, is the length of a shortest path joining u and v in X. The *diameter* of X, denoted $\mathrm{diam}(X)$, is the maximum distance over all the pairs of vertices of X. If we regard a graph as a model for a communication network, then the diameter is a simple way of measuring how fast the information travels in the network. Obviously, the smaller the diameter, the better the network is.

If $f, g : \mathbb{N} \to \mathbb{R}$, we say $f(n) = \mathrm{O}(g(n))$ if there is a constant C such that $|f(n)| \leq C|g(n)|$ for all n. Also, we say that $f(n) = \mathrm{O}(g(n))$ as $n \to +\infty$ if there are constants C and $n_0 \in \mathbb{N}$ such that $|f(n)| \leq C|g(n)|$ whenever $n \geq n_0$.

The next result provides a well-known lower bound on the diameter of regular graphs.

PROPOSITION 1.5.1. *Let X be a connected k-regular graph, $k \geq 3$, with n vertices. Then*

$$\mathrm{diam}(X) \geq \log_{k-1} n - \mathrm{O}(1)$$

for all $n \geq 2$.

Proof. Let $D = \operatorname{diam}(X)$ and consider a vertex u of X. For $i \geq 0$, denote by $N_i(u)$ the set of vertices at distance i from u. Obviously, $N_0(u) = \{u\}, |N_1(u)| = k$ and for each $i \in \{2, \ldots, D\}, |N_i(u)| \leq k(k-1)^{i-1}$. Also, for $i > D, N_i(u) = \emptyset$. Since $V(X) = \cup_{i=0}^{D} N_i(u)$, it follows that

$$n = \sum_{i=0}^{D} |N_i(u)| \leq 1 + \sum_{i=1}^{D} k(k-1)^{i-1} = 1 + k\frac{(k-1)^D - 1}{k-2}$$

This implies

$$D \geq \log_{k-1} n + \log_{k-1}\left(1 - \frac{2(n-1)}{kn}\right)$$

which proves the claimed result. □

Note that the previous proposition is true for all k-regular connected graphs with $k \geq 3$ regardless of their eigenvalues. The spectrum of a graph can provide upper bounds on its diameter. This was observed first by Alon and Milman [**3**] in 1985. They proved that if X is a k-regular graph with n vertices, then

$$\operatorname{diam}(X) \leq 2\sqrt{\frac{2k}{k - \lambda_2(X)}} \log_2 n$$

In 1989, Chung [**14**] proved the following result. This improves Alon and Milman's result when λ is small.

Theorem 1.5.2 (Chung [**14**]). *If X is a nonbipartite (n, k, λ)-graph, then*

$$\operatorname{diam}(X) \leq \left\lceil \frac{\log n}{\log \frac{k}{\lambda}} \right\rceil.$$

By a minor modification of Chung's argument from [**14**], Quenell [**82**] obtained the following spectral upper bound on the diameter of a bipartite regular graph.

Theorem 1.5.3 (Quenell [**82**]). *If X is a bipartite (n, k, λ)-graph, then*

$$\operatorname{diam}(X) \leq \frac{\log \frac{n-2}{2}}{\log \frac{k}{\lambda}} + 2.$$

Notice that the smaller λ is, the smaller the upper bound will be. Further improvements of the previous two theorems were given by Chung, Faber and Manteuffel [**17**] in 1994 and by Kahale [**48**] in 1997.

1.6. The Courant-Fisher Theorem

Let $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A)$ denote the eigenvalues of a real symmetric matrix A of order n. The *Rayleigh-Ritz*[2] *ratio* or *quotient* of a non-zero vector $x \in \mathbb{R}^n$ with respect to A, is

$$R(A, x) = \frac{x^t A x}{x^t x}. \tag{1.6.1}$$

For $\alpha \in \mathbb{R}$, let $\mathcal{E}(A, \alpha) = \{x \in \mathbb{R}^n : Ax = \alpha x\}$. Obviously, $\mathcal{E}(A, \alpha) = \{0\}$ unless $\alpha = \lambda_i$ for some i with $1 \leq i \leq n$.

The eigenvalues of a real symmetric matrix can be described as the solutions to a minimum and maximum problem by the Courant-Fisher theorem. This theorem is true for Hermitian matrices (see [**45**], page 179), but we will state and prove it for real symmetric matrices only.

Theorem 1.6.1 (Courant-Fisher). *Let A be a real symmetric matrix of dimension n with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then for any integer k with $1 \leq k \leq n$,*

$$\lambda_k = \min_{w_1, w_2, \ldots, w_{k-1} \in \mathbb{R}^n} \max_{\substack{x \in \mathbb{R}^n, x \neq 0 \\ x \perp w_1, w_2, \ldots, w_{k-1}}} R(A, x) \tag{1.6.2}$$

and

$$\lambda_k = \max_{w_1, w_2, \ldots, w_{n-k} \in \mathbb{R}^n} \min_{\substack{x \in \mathbb{R}^n, x \neq 0 \\ x \perp w_1, w_2, \ldots, w_{n-k}}} R(A, x) \tag{1.6.3}$$

[2]Rayleigh and Ritz were two British physicists.

Proof. Let $u_1, u_2, \ldots, u_n \in \mathbb{R}^n$ be a basis of orthonormal eigenvectors of A, that is $Au_i = \lambda_i u_i, u_i^t u_i = 1$, for each $1 \le i \le n$ and $u_i \perp u_j$ if $i \ne j$. It follows easily that $A = U^t \Lambda U$, where U is the real n by n matrix whose i-th row is u_i^t and $\Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$.

We will prove (1.6.2) only, the proof of (1.6.3) being the same with A replaced by $-A$. First we show that

$$\lambda_k = \max_{\substack{x\in\mathbb{R}^n, x\ne 0\\ x\perp u_1,u_2,\ldots,u_{k-1}}} \frac{x^t A x}{x^t x}.$$

Let $x \in \mathbb{R}^n, x \ne 0$ with $x \perp u_1, u_2, \ldots, u_{k-1}$. Write x as $\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n$, where $\alpha_i \in \mathbb{R}$ for $1 \le i \le n$. Since the u_i's are orthonormal, it follows that $x^t u_i = \alpha_i u_i^t u_i = \alpha_i$, for each $1 \le i \le n$. Because $x \perp u_j$ for each $1 \le j \le k-1$, we deduce that $\alpha_j = 0$ for each $1 \le j \le k-1$. This implies

$$R(A,x) = \frac{x^t A x}{x^t x} = \frac{\sum_{j=k}^n \lambda_j \alpha_j^2}{\sum_{j=k}^n \alpha_j^2} \le \frac{\sum_{j=k}^n \lambda_k \alpha_j^2}{\sum_{j=k}^n \alpha_j^2} \le \lambda_k.$$

Thus, $\lambda_k \ge R(A,x)$, for each $x \ne 0, x \perp u_1, u_2, \ldots, u_{k-1}$. Let $y = u_k$. Then $y \ne 0, y \perp u_1, u_2, \ldots, u_{k-1}$ and $R(A,y) = \frac{y^t A y}{y^t y} = \frac{\lambda_k y^t y}{y^t y} = \lambda_k$. Hence,

$$\lambda_k = \max_{\substack{x\in\mathbb{R}^n, x\ne 0\\ x\perp u_1,u_2,\ldots,u_{k-1}}} R(A,x) \tag{1.6.4}$$

Let $w_1, w_2, \ldots, w_{k-1} \in \mathbb{R}^n$ and denote by $\mathcal{W}$ the vector space generated by the vectors $x \in \mathbb{R}^n$ that are perpendicular to $w_1, w_2, \ldots, w_{k-1}$. Let $\mathcal{U}_k$ be the vector space generated by $u_1, u_2, \ldots, u_k$. Then $\dim \mathcal{W} \ge n-k+1$ and $\dim \mathcal{U}_k = k$. It follows that $\mathcal{W} \cap \mathcal{U}_k$ contains a non-zero vector v. Write $v = \sum_{i=1}^k a_i u_i$. Then

$$\begin{aligned}\max_{\substack{x\in\mathbb{R}^n, x\ne 0\\ x\perp w_1,w_2,\ldots,w_{k-1}}} R(A,x) \ge R(A,v) &= \frac{\sum_{i=1}^k \lambda_i a_i^2}{\sum_{i=1}^k a_i^2}\\ &\ge \frac{\sum_{i=1}^k \lambda_k a_i^2}{\sum_{i=1}^k a_i^2} = \lambda_k.\end{aligned}$$

Thus,

$$\lambda_k \leq \max_{\substack{x\in\mathbb{R}^n, x\neq 0\\ x\perp w_1,w_2,\ldots,w_{k-1}}} R(A,x),$$

for any $w_1, w_2, \ldots, w_{k-1} \in \mathbb{R}^n$. Using (1.6.4), we deduce that

$$\lambda_k = \min_{w_1,w_2,\ldots,w_{k-1}\in\mathbb{R}^n} \max_{\substack{x\in\mathbb{R}^n, x\neq 0\\ x\perp w_1,w_2,\ldots,w_{k-1}}} R(A,x)$$

□

REMARK 1.6.2. *The cases $k = 1$ in (1.6.2) and $k = n$ in (1.6.3) form the Rayleigh-Ritz theorem.*

The next theorem is due to H.Weyl (see [**45**], page 181). It follows from the Courant-Fisher theorem.

THEOREM 1.6.3 (Weyl). *For any real symmetric matrices A and B of order n and for any $1 \leq i \leq n$, the following inequalities hold:*

$$\lambda_1(B) \geq \lambda_i(A+B) - \lambda_i(A) \geq \lambda_n(B) \tag{1.6.5}$$

PROOF. From the Rayleigh-Ritz theorem, we have

$$\lambda_1(B) \geq R(B,x) \geq \lambda_n(B)$$

for each $x \in \mathbb{R}^n$. Using the Courant-Fisher theorem, it follows that

$$\begin{aligned}
\lambda_i(A+B) &= \min_{w_1,w_2,\ldots,w_{i-1}\in\mathbb{R}^n} \max_{\substack{x\in\mathbb{R}^n, x\neq 0\\ x\perp w_1,w_2,\ldots,w_{i-1}}} R(A+B,x)\\
&= \min_{w_1,w_2,\ldots,w_{i-1}\in\mathbb{R}^n} \max_{\substack{x\in\mathbb{R}^n, x\neq 0\\ x\perp w_1,w_2,\ldots,w_{i-1}}} (R(A,x)+R(B,x))\\
&\geq \min_{w_1,w_2,\ldots,w_{i-1}\in\mathbb{R}^n} \max_{\substack{x\in\mathbb{R}^n, x\neq 0\\ x\perp w_1,w_2,\ldots,w_{i-1}}} (R(A,x)+\lambda_n(B))\\
&= \lambda_i(A) + \lambda_n(B),
\end{aligned}$$

for any integer i with $1 \le i \le n$. Also, from the Courant-Fisher theorem we deduce that

$$\begin{aligned}
\lambda_i(A+B) &= \max_{w_1,w_2,\ldots,w_{n-i}\in\mathbb{R}^n} \min_{\substack{x\in\mathbb{R}^n, x\neq 0 \\ x\perp w_1,w_2,\ldots,w_{n-i}}} R(A+B,x) \\
&= \max_{w_1,w_2,\ldots,w_{n-i}\in\mathbb{R}^n} \min_{\substack{x\in\mathbb{R}^n, x\neq 0 \\ x\perp w_1,w_2,\ldots,w_{n-i}}} (R(A,x)+R(B,x)) \\
&\le \max_{w_1,w_2,\ldots,w_{n-i}\in\mathbb{R}^n} \min_{\substack{x\in\mathbb{R}^n, x\neq 0 \\ x\perp w_1,w_2,\ldots,w_{n-i}}} (R(A,x)+\lambda_1(B)) \\
&= \lambda_i(A)+\lambda_1(B),
\end{aligned}$$

for each integer i with $1 \le i \le n$. This completes the proof of Weyl's theorem. □

In [**46**], Horn, Rhee and So completely describe the equality cases in Weyl's theorem.

THEOREM 1.6.4. *Let A and B be real symmetric matrices of order n and let i be between 1 and n. Then*

$$\lambda_1(B) = \lambda_i(A+B) - \lambda_i(A)$$

if and only

$$\mathcal{E}(B,\lambda_1) \cap \mathcal{E}(A+B,\lambda_i) \cap \mathcal{E}(A,\lambda_i) \neq \{0\} \tag{1.6.6}$$

and

$$\lambda_n(B) = \lambda_i(A+B) - \lambda_i(A)$$

if and only if

$$\mathcal{E}(B,\lambda_n) \cap \mathcal{E}(A+B,\lambda_i) \cap \mathcal{E}(A,\lambda_i) \neq \{0\} \tag{1.6.7}$$

We write $f(n) = \Omega(g(n))$ if $g(n) = \mathrm{O}(f(n))$. Also, we say $f(n) = \Theta(g(n))$ if $f(n) = \mathrm{O}(g(n))$ and $f(n) = \Omega(g(n))$.

In addition, $f(n) = \mathrm{o}(g(n))$ if

$$\lim_{n \to +\infty} \frac{f(n)}{g(n)} = 0.$$

Also, we write $f \sim g$ if

$$\lim_{n \to +\infty} \frac{f(n)}{g(n)} = 1.$$

CHAPTER 2

Eigenvalues of regular graphs

2.1. Preliminaries

In this chapter, we discuss various aspects of the distribution of the extreme eigenvalues of regular graphs. In Section 2.2, we present a new and elementary proof of a theorem of Serre concerning the largest eigenvalues of regular graphs. In Section 2.3, we prove an analogous theorem regarding the *least* eigenvalues of k-regular graphs: given $\epsilon > 0$, there exist a positive constant $c = c(\epsilon, k)$ and a nonnegative integer $g = g(\epsilon, k)$ such that for any k-regular graph X with no odd cycles of length less than g, the number of eigenvalues μ_i of X such that $\mu_i \leq -(2-\epsilon)\sqrt{k-1}$ is at least $c|V(X)|$. In Section 2.4, we present a new and elementary proof of a theorem of Serre regarding asymptotics of the least eigenvalues of regular graphs containing few odd cycles. In Section 2.5, we answer a question of Linial [**57**] regarding the smallest eigenvalues of claw-free graphs.

Perhaps the first result regarding the distribution of eigenvalues of regular graphs is due to McKay. In 1981, McKay [**68**] proved the following result. Recall that $c_r(X)$ denotes the number of cycles of length r in X.

THEOREM 2.1.1. *Let $(X_n)_{n\geq 1}$ be a sequence of k-regular graphs with $\lim_{n\to\infty} |V(X_n)| = +\infty$, such that, for each $r \geq 3$,*

$$\lim_{n\to+\infty} \frac{c_r(X_n)}{|V(X_n)|} = 0 \tag{2.1.1}$$

If

$$F(X_n, x) = \frac{1}{|V(X_n)|}|\{i : \lambda_i(X_n) \leq x\}|,$$

then

(2.1.2)

$$\lim_{n\to+\infty} F(X_n, x) = F(x) = \begin{cases} 0 & \text{if } x \leq -2\sqrt{k-1} \\ \int_{-2\sqrt{k-1}}^{x} \frac{k\sqrt{4(k-1)-y^2}}{2\pi(k^2-y^2)}\, dy & \text{if } -2\sqrt{k-1} < x < 2\sqrt{k-1} \\ 1 & \text{if } x \geq 2\sqrt{k-1} \end{cases}$$

Conversely, if $F(X_n, x)$ does not converge to $F(x)$ for some x, then the condition (2.1.1) *fails for some r.*

In 1986, Alon [**1**] stated the Alon-Boppana theorem.

THEOREM 2.1.2. *If $k \geq 3$ and X is a k-regular graph with n vertices, then*

$$\lambda_2(X) \geq 2\sqrt{k-1}\left(1 - \mathrm{O}\left(\frac{1}{\log_{k-1} n}\right)\right)$$

Theorem 2.1.1 implies that for each $\epsilon > 0$, a random k-regular graph on n vertices has $\mathrm{o}(n)$ eigenvalues greater than $2\sqrt{k-1} + \epsilon$. In [**1**], Alon also conjectured that for each $\epsilon > 0$, the second largest eigenvalue of a random k-regular graph is at most $2\sqrt{k-1} + \epsilon$[1]. This conjecture was recently proved by Friedman in [**32**].

Theorem 2.1.2 implies the following asymptotic Alon-Boppana theorem.

THEOREM 2.1.3. *Let $(X_m)_{m\geq 1}$ be a family of finite, connected, k-regular graphs with $|V(X_m)| \to +\infty$ as $m \to +\infty$. Then*

$$\liminf_{m\to+\infty} \lambda_2(X_m) \geq 2\sqrt{k-1}$$

[1]The exact statement of Alon's Second Eigenvalue Conjecture is the following: for each $\epsilon > 0$, if X is a random k-regular graph on n vertices, then the probability that $\lambda(X) \leq 2\sqrt{k-1} + \epsilon$ tends to 1 as n tends to infinity.

The first proof of Theorem 2.1.3 appears in a paper by Lubotzky, Phillips and Sarnak [**65**] and the first proof of Theorem 2.1.2 is given by Nilli (pseudonym for Alon) in [**78**]. The error term in Theorem 2.1.2 was improved by Friedman in [**30**] where the following result is proved.

THEOREM 2.1.4. *Let X be a k-regular graph with a subset of l points each of distance at least $2r$ from one another. Then*

$$\lambda_l(X) \geq 2\sqrt{k-1}\cos\frac{\pi}{r+1}$$

In particular, for $l = 2$ and $r = \lfloor \frac{\mathrm{diam}(X)}{2} \rfloor$, Friedman obtained in [**30**] the following improvement of Theorem 2.1.2. Note that $\cos x = 1 - \frac{x^2}{2} + \mathrm{O}(x^4)$.

THEOREM 2.1.5. *Let X be a k-regular graph with diameter* $\mathrm{diam}(X)$*. Then*

$$\lambda_2(X) \geq 2\sqrt{k-1}\left(1 - \frac{2\pi^2}{\mathrm{diam}^2(X)} + \mathrm{O}\left(\mathrm{diam}^{-4}(X)\right)\right)$$

To see that Friedman's result improves the error term from Theorem 2.1.2, recall Proposition 1.5.1 which states that $\mathrm{diam}(X) \geq \log_{k-1} n - \mathrm{O}(1)$. Thus, the error term in Theorem 2.1.5 is $\mathrm{O}\left(\frac{1}{\log^2_{k-1} n}\right)$ which is better than the error term $\mathrm{O}\left(\frac{1}{\log_{k-1} n}\right)$ from Theorem 2.1.2.

Friedman's method was simplified by Nilli in [**79**]. Nilli's theorem is slightly weaker than Theorem 2.1.4, but the proof is elementary.

THEOREM 2.1.6. *Let X be a k-regular graph with a subset of l points each of distance at least $2r$ from one another. Then*

$$\lambda_l(X) \geq 2\sqrt{k-1}\cos\frac{\pi}{r}$$

The behaviour of the least eigenvalues of regular graphs is less understood than that of the largest eigenvalues. Cycles seem to play an important role in the asymptotics of the least eigenvalues of regular graphs.

Bipartite graphs have symmetric spectra. Thus, the least eigenvalues of these graphs behave exactly the same as the largest eigenvalues only with a sign change. Intuitively, a graph with large girth looks locally like a tree and thus, one may expect that the least eigenvalues of graphs with high girth behave similarly to the negatives of the largest eigenvalues. This fact is described more precisely by the next result which follows from McKay's work (Theorem 2.1.1). Recall that we denote by $\mu_l(X)$ the l-th smallest eigenvalue of a graph X.

THEOREM 2.1.7. *Let $(X_n)_{n\geq 0}$ be a sequence of k-regular graphs such that $\lim_{n\to+\infty} \operatorname{girth}(X_n) = +\infty$. Then for each $l \geq 1$,*

$$\limsup_{n\to\infty} \mu_l(X_n) \leq -2\sqrt{k-1}.$$

In 1996, Li and Solé [**56**] prove error terms depending on the girth for the least eigenvalues of regular graphs.

THEOREM 2.1.8. *Let X be a k-regular graph with girth g. Then*

$$\mu_1(X) \leq -2\sqrt{k-1}\cos\frac{\pi}{\lfloor\frac{g-1}{2}\rfloor+1} = -2\sqrt{k-1}\left(1-\frac{2\pi^2}{g^2}+\mathrm{O}(g^{-4})\right)$$

We mentioned earlier that graphs with high girth look locally like trees. Graphs with high odd girth are locally bipartite and thus, one might expect that if the odd girth gets larger, the bottom of the spectrum behaves similarly to the negative of the top part of it.

Already in 1993, Friedman [**30**] proved the following result connecting the least eigenvalues and the odd girth of a graph.

Theorem 2.1.9. *Let X be a k-regular graph that has a subset of l points each of distance at least $2r$ from one another, and that contains no odd cycle of length less than $2r$. Then*

$$\mu_l(X) \le -2\sqrt{k-1}\cos\frac{\pi}{2r+2}$$

A slightly weaker result, but with an elementary proof, is obtained by Nilli in [**79**]. All these theorems motivate the following definition which was used for the first time by Lubotzky, Phillips and Sarnak in 1986.

Definition 2.1.10. *A finite, connected, k-regular graph X is Ramanujan if*

$$|\lambda_i(X)| \le 2\sqrt{k-1} \tag{2.1.3}$$

for every nontrivial eigenvalue $\lambda_i(X)$.

In 1988, Lubotzky, Phillips and Sarnak [**65**] and independently, Margulis [**67**], applied deep number theoretic results of Eichler and Igusa on the Ramanujan conjectures and constructed for every prime $p \equiv 1 \pmod 4$, an infinite family of $(p+1)$-regular graphs that satisfy (2.1.3). By Theorem 2.1.3, it is easy to see that this is best possible.

We briefly describe here the construction of Lubotzky, Phillips and Sarnak. To our knowledge, the notion of Ramanujan graphs appeared for the first time in their paper [**65**].

We discuss Cayley graphs in more detail in Chapter 4. We just state their definition here since the graphs constructed in [**65**] are Cayley graphs of some matrix groups.

Definition 2.1.11. *Let G be a finite multiplicative group, with identity 1 and suppose $S = \{x_1, x_2, \ldots, x_s, x_1^{-1}, x_2^{-1}, \ldots, x_s^{-1}, y_1, \ldots, y_t\}$ is a subset of G such that*

$1 \notin S$, $x_i^2 \neq 1$ *for all* i *and* $y_j^2 = 1$ *for all* j. *The* (s,t)*-Cayley graph* $X = X(G,S)$ *is the simple graph with vertex set* G *and with* $x, y \in G$ *adjacent if* $xy^{-1} \in S$.

Notice that adjacency is well-defined since S is symmetric, i.e $a \in S$ if and only if $a^{-1} \in S$. Also, G is regular with valency $k = |S|$ and it contains no loops since $1 \notin S$. It is easy to see that X is connected if and only if S generates G.

Consider two distinct odd primes p and q such that $p, q \equiv 1 \pmod 4$. Denote by $\mathrm{PGL}(2, \mathbb{Z}/q\mathbb{Z})$ the factor group of the group of all two by two invertible matrices over $\mathbb{Z}/q\mathbb{Z}$ modulo its normal subgroup consisting of all scalar matrices. Also, denote by $\mathrm{PSL}(2, \mathbb{Z}/q\mathbb{Z})$ the factor group of the group of all two by two matrices over $\mathbb{Z}/q\mathbb{Z}$ with determinant 1 modulo its normal subgroup consisting of $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

Let u be an integer such that $u^2 \equiv -1 \pmod q$. By a classical theorem of Jacobi, there are $8(p+1)$ solutions $v = (a, b, c, d)$ such that $p = a^2 + b^2 + c^2 + d^2$. Thus, there are $p+1$ solutions such that $a > 0$ and b, c and d even. To each solution v, Lubotzky, Phillips and Sarnak associate the following matrix

$$\tilde{v} = \begin{bmatrix} a + ub & c + ud \\ -c + ud & a - ub \end{bmatrix}$$

Let S be the set of all these matrices. If p is a quadratic residue modulo q, then S is contained in $\mathrm{PSL}(2, \mathbb{Z}/q\mathbb{Z})$. Denote by $X^{p,q}$ the Cayley graph of $\mathrm{PSL}(2, \mathbb{Z}/q\mathbb{Z})$ with respect to S. If p is not a quadratic residue modulo q, then S is contained in $\mathrm{PGL}(2, \mathbb{Z}/q\mathbb{Z})$ and we denote by $X^{p,q}$ the Cayley graph of $\mathrm{PGL}(2, \mathbb{Z}/q\mathbb{Z})$ with respect to S. Since $|S| = p + 1$, it follows that $X^{p,q}$ is $(p+1)$-regular. Often, we will use the name LPS graphs when referring to the $X^{p,q}$'s.

Lubotzky, Phillips and Sarnak [**65**] show that for sufficiently large q, the graph $X^{p,q}$ is Ramanujan. By varying q, an infinite family of $(p+1)$-regular Ramanujan graphs is obtained. This construction was extended for all odd primes p in the book of Davidoff, Sarnak and Vallete [**24**]. It was also generalized for p a power of an odd prime by Morgenstern in [**74**].

Proving that the graphs $X^{p,q}$ are Ramanujan uses deep results from number theory, namely the Ramanujan conjecture proved by Eichler [**26**] and Igusa [**47**]. Let $r_q(n)$ denote the number of integral solutions of the equation $x_1^2+4q^2x_2^2+4q^2x_3^2+4q^2x_4^2 = n$. Jacobi's theorem mentioned earlier determines the exact value of $r_1(n)$. For general q and $n = p^k, k \geq 0$, there is no precise formula, but the Ramanujan conjecture states that for each $\epsilon > 0$, as k tends to $+\infty$

$$r_q(p^k) = C(p^k) + O_\epsilon(p^{\left(\frac{1}{2}+\epsilon\right)k})$$

where the main term $C(p^k)$ has an explicit known formula. Eichler's proof of the Ramanujan conjecture uses the Riemann hypothesis for finite curves proved by Weil [**92**].

The LPS graphs have other extremal properties. These are graphs with large girth and chromatic number. The following result was also proved in [**65**].

THEOREM 2.1.12. *Let p, q be two odd primes with $p, q \equiv 1 \pmod 4$. If p is a quadratic residue modulo q, then*

(i) *$X^{p,q}$ has $n = \frac{q(q^2-1)}{2}$ vertices and it is not bipartite.*
(ii) *$\mathrm{girth}(X^{p,q}) \geq 2\log_p q \sim \frac{2}{3}\log_p n$.*
(iii) *$\mathrm{diam}(X^{p,q}) \leq 2\log_p n + 2\log_p 2 + 1$.*
(iv) *$\alpha(X^{p,q}) \leq \frac{2\sqrt{p}}{p+1}n \sim \frac{2n}{\sqrt{p}}$.*
(v) *$\chi(X^{p,q}) \geq \frac{p+1}{2\sqrt{p}} \sim \frac{\sqrt{p}}{2}$.*

If p is not a quadratic residue modulo q, then

(i) *$X^{p,q}$ has $n = q(q^2-1)$ vertices and it is bipartite.*

(ii) $\mathrm{girth}(X^{p,q}) \geq 4\log_p q - \log_p 4 \sim \frac{4}{3}\log_p n$.

(iii) $\mathrm{diam}(X^{p,q}) \leq 2\log_p n + 2\log_p 2 + 1$.

2.2. An elementary proof of Serre's theorem

J.-P. Serre has proved the following theorem (see [**24, 27, 54, 88**]) using Chebyschev polynomials. The simplest self-contained proof of this theorem is given in [**24**], Section 1.4 and is highly non-elementary.

THEOREM 2.2.1 (Serre). *For each $\epsilon > 0$ and $k \geq 1$, there exists a positive constant $c = c(\epsilon, k)$ such that for any k-regular graph X, the number of eigenvalues λ_i of X with $\lambda_i \geq (2-\epsilon)\sqrt{k-1}$ is at least $c|X|$.*

Here we present an elementary proof of Serre's result. For the proof of this theorem we require the next lemma which follows from [**68**], Lemma 2.1. For the sake of completeness, we include a brief proof of the lemma here.

LEMMA 2.2.2. *Let v_0 be a vertex of a k-regular graph X. Then the number of closed walks of length $2s$ in X starting at v_0 is greater than or equal to $\frac{1}{s+1}\binom{2s}{s}k(k-1)^{s-1}$.*

PROOF. The number of closed walks of length $2s$ from v_0 to itself is at least the number of closed walks of length $2s$ from some vertex u_0 to itself in the infinite k-regular tree. To each closed walk in the infinite k-regular tree, there corresponds a sequence of nonnegative integers $\delta_0 = 0, \delta_1, \ldots, \delta_{2s}$, where δ_i is the distance from u_0 after i steps. Note that $|\delta_{i+1} - \delta_i| = 1$ for each i between 0 and $2s-1$. The number of such sequences is the s-th Catalan number $\frac{1}{s+1}\binom{2s}{s}$. For each sequence of distances, there are at least $k(k-1)^{s-1}$ closed walks of length $2s$ since for each step away from u_0 there are at least $k-1$ choices in the tree (k if the walk is at u_0). $\square$

By Stirling's bound on $s!$ or by a simple induction argument it is easy to see that $\binom{2s}{s} \geq \frac{4^s}{s+1}$, for any $s \geq 1$. Hence, for any k-regular graph X and for any $s \geq 1$, we have by Lemma 2.2.2

$$\Phi_{2s}(X) \geq |V(X)| \frac{1}{s+1} \binom{2s}{s} k(k-1)^{s-1} > |V(X)| \frac{1}{(s+1)^2} (2\sqrt{k-1})^{2s} \tag{2.2.1}$$

We present now a simple proof of Theorem 2.2.1.

Proof. Let X be a k-regular graph of order n with eigenvalues $k = \lambda_1 \geq \cdots \geq \lambda_n \geq -k$. Given $\epsilon > 0$, let m be the number of eigenvalues λ_i of X with $\lambda_i \geq (2-\epsilon)\sqrt{k-1}$. Then $n-m$ of the eigenvalues of X are less than $(2-\epsilon)\sqrt{k-1}$. Thus

$$\begin{aligned}
\operatorname{tr}[(kI + A)^{2s}] &= \sum_{i=1}^{n} (k + \lambda_i)^{2s} \\
&< (n-m)(k + (2-\epsilon)\sqrt{k-1})^{2s} + m(2k)^{2s} \\
&= m((2k)^{2s} - (k + (2-\epsilon)\sqrt{k-1})^{2s}) + n(k + (2-\epsilon)\sqrt{k-1})^{2s}
\end{aligned}$$

On the other hand, the binomial expansion and (2.2.1) give

$$\begin{aligned}
\operatorname{tr}[(kI + A)^{2s}] &= \sum_{i=0}^{2s} \binom{2s}{i} k^i \Phi_{2s-i}(X) \\
&\geq \sum_{j=0}^{s} \binom{2s}{2j} k^{2j} \Phi_{2s-2j}(X) \\
&> \frac{n}{(s+1)^2} \sum_{j=0}^{s} \binom{2s}{2j} k^{2j} (2\sqrt{k-1})^{2s-2j} \\
&= \frac{n}{2(s+1)^2} ((k + 2\sqrt{k-1})^{2s} + (k - 2\sqrt{k-1})^{2s}) \\
&> \frac{n}{2(s+1)^2} (k + 2\sqrt{k-1})^{2s}
\end{aligned}$$

Thus,

$$\frac{m}{n} > \frac{\frac{1}{2(s+1)^2}(k + 2\sqrt{k-1})^{2s} - (k + (2-\epsilon)\sqrt{k-1})^{2s}}{(2k)^{2s} - (k + (2-\epsilon)\sqrt{k-1})^{2s}}$$

for any $s \geq 1$. Since

$$\lim_{s\to\infty}\left(\frac{(k+2\sqrt{k-1})^{2s}}{2(s+1)^2}\right)^{\frac{1}{2s}} = k+2\sqrt{k-1}$$
$$> k+(2-\epsilon)\sqrt{k-1} = \lim_{s\to\infty}\left(2(k+(2-\epsilon)\sqrt{k-1})^{2s}\right)^{\frac{1}{2s}}$$

it follows that there exists $s_0 = s_0(\epsilon, k)$ such that for all $s \geq s_0$

$$\frac{(k+2\sqrt{k-1})^{2s}}{2(s+1)^2} > 2(k+(2-\epsilon)\sqrt{k-1})^{2s}$$

so that

$$\frac{(k+2\sqrt{k-1})^{2s}}{2(s+1)^2} - (k+(2-\epsilon)\sqrt{k-1})^{2s} > (k+(2-\epsilon)\sqrt{k-1})^{2s}$$

Hence, if

$$c(\epsilon,k) = \frac{(k+(2-\epsilon)\sqrt{k-1})^{2s_0}}{(2k)^{2s_0}-(k+(2-\epsilon)\sqrt{k-1})^{2s_0}}$$

then $c(\epsilon, k) > 0$ and $m > c(\epsilon, k)n$. □

The proofs of Serre's theorem given in [**24, 27, 54**] are very complicated and don't allow an easy estimation of the constant $c(\epsilon, k)$ in terms of ϵ and k. We should note that Serre's theorem can be also deduced from the previous results of Friedman (see Theorem 2.1.4) or Nilli (see Theorem 2.1.6). Their results imply an estimate of $\left(\frac{1}{2}\right)^{O\left(\frac{\log k}{\arccos(1-\epsilon)}\right)} \sim \left(\frac{1}{2}\right)^{O\left(\frac{\log k}{\sqrt{2\epsilon}}\right)}$ for the proportion of the eigenvalues that are at least $(2-\epsilon)\sqrt{k-1}$. Our proof of Serre's theorem is much simpler than Friedman's or Nilli's methods, but their results provide better bounds on $c(\epsilon, k)$ than ours. From our proof of Serre's theorem, we obtain that a proportion of $\left(\frac{1}{2}\right)^{O\left(\frac{\sqrt{k}}{\epsilon}\log\left(\frac{\sqrt{k}}{\epsilon}\right)\right)}$ of the eigenvalues are at least $(2-\epsilon)\sqrt{k-1}$. This is because in Theorem 1 we pick s_0 such that $\frac{s_0}{\log s_0} = \Theta\left(\frac{\sqrt{k}}{\epsilon}\right)$.

Theorem 2.2.1 has the following consequence regarding the asymptotics of the largest eigenvalues of regular graphs.

COROLLARY 2.2.3. *Let $(X_n)_{n\geq 0}$ be a sequence of k-regular graphs such that $\lim_{n\to+\infty} |V(X_n)| = +\infty$. Then for each $l \geq 1$,*

$$\liminf_{n\to+\infty} \lambda_l(X_n) \geq 2\sqrt{k-1}$$

This corollary has also been proved directly by Serre in an appendix to [**55**] using the eigenvalue distribution theorem in [**88**]. When $l = 2$, we get Theorem 2.1.3.

2.3. An analogue for the least eigenvalues of regular graphs

The analogous result to Theorem 2.2.1 for the least eigenvalues of a k-regular graph is not true. This can be seen if we consider the line graphs. These have no eigenvalues less than -2. In the last section of this chapter, we present a simple proof of this well-known fact and also discuss the eigenvalues of claw free graphs.

However, by adding an extra condition to the hypothesis of Theorem 2.2.1, we can prove an analogue of Serre's theorem for the least eigenvalues of a k-regular graph. Recall that $\mathrm{oddg}(X)$ is the length of the shortest odd cycle in X.

THEOREM 2.3.1. *For any $\epsilon > 0$ and integer $k \geq 1$, there exist a positive constant $c = c(\epsilon, k)$ and a nonnegative integer $g = g(\epsilon, k)$ such that for any k-regular graph X with $\mathrm{oddg}(X) > g$, the number of eigenvalues μ_i of X with $\mu_i \leq -(2-\epsilon)\sqrt{k-1}$ is at least $c|X|$.*

PROOF. Let X be a k-regular graph of order n with eigenvalues $-k \leq \mu_1 \leq \mu_2 \leq \cdots \leq \mu_n = k$. Given $\epsilon > 0$, let m be the number of eigenvalues μ_i of X with $\mu_i \leq -(2-\epsilon)\sqrt{k-1}$. Then $n - m$ of the eigenvalues of X are greater than

$-(2-\epsilon)\sqrt{k-1}$. Thus

$$\begin{aligned}
\operatorname{tr}[(kI-A)^{2s}] &= \sum_{i=1}^{n}(k-\mu_i)^{2s} \\
&< (n-m)(k+(2-\epsilon)\sqrt{k-1})^{2s} + m(2k)^{2s} \\
&= m((2k)^{2s} - (k+(2-\epsilon)\sqrt{k-1})^{2s}) + n(k+(2-\epsilon)\sqrt{k-1})^{2s}
\end{aligned}$$

In the previous section, we proved that there exists $s_0 = s_0(\epsilon, k)$ such that for all $s \geq s_0$

$$\frac{(k+2\sqrt{k-1})^{2s_0}}{2(s_0+1)^2} - (k+(2-\epsilon)\sqrt{k-1})^{2s_0} > (k+(2-\epsilon)\sqrt{k-1})^{2s_0}$$

Let $g(\epsilon, k) = 2s_0$. If $\operatorname{oddg}(X) > 2s_0$, then for $0 \leq j \leq s_0 - 1$, the number of closed walks of length $2s_0 - 2j - 1$ in X is 0. Hence, $\Phi_{2s_0-2j-1}(X) = 0$, for $0 \leq j \leq s_0 - 1$. Thus

$$\begin{aligned}
\operatorname{tr}[(kI-A)^{2s_0}] &= \sum_{j=0}^{s_0}\binom{2s_0}{2j}k^{2j}\Phi_{2s_0-2j}(X) - \sum_{j=0}^{s_0-1}\binom{2s_0}{2j+1}k^{2j+1}\Phi_{2s_0-2j-1}(X) \\
&= \sum_{j=0}^{s_0}\binom{2s_0}{2j}k^{2j}\Phi_{2s_0-2j}(X) \\
&> \frac{n}{2(s_0+1)^2}(k+2\sqrt{k-1})^{2s_0}
\end{aligned}$$

where the last inequality follows as in the proof of Theorem 2.2.1. Thus, if

$$c(\epsilon, k) = \frac{(k+(2-\epsilon)\sqrt{k-1})^{2s_0}}{(2k)^{2s_0} - (k+(2-\epsilon)\sqrt{k-1})^{2s_0}}$$

then $c(\epsilon, k) > 0$ and $m > c(\epsilon, k)n$. □

The next result is an immediate consequence of Theorem 2.3.1.

Corollary 2.3.2. *Let $(X_n)_{n\geq 0}$ be a sequence of k-regular graphs such that $\lim_{n\to+\infty} \operatorname{oddg}(X_n) = +\infty$. Then for each $l \geq 1$*

$$\limsup_{n\to+\infty} \mu_l(X_n) \leq -2\sqrt{k-1}$$

When $l = 1$, we get the main result from [**55**]. Also, Corollary 2.3.2 holds when $l = 1$ and $\lim_{i\to+\infty} \operatorname{girth}(X_i) = +\infty$. This special case of Corollary 2.3.2 was proved in [**56**] using orthogonal polynomials. It is also a consequence of Theorem 2.1.7.

2.4. Odd cycles and eigenvalues

A stronger theorem than Corollary 2.3.2 has been proved by Serre in [**55**] using the eigenvalue distribution results from [**88**]. The results and proofs in [**88**] are highly non-elementary. Roughly speaking, Serre's next theorem says that if a k-regular graph has few odd cycles, then its least eigenvalues are asymptotically less than $-2\sqrt{k-1}$, i.e. they behave like the largest eigenvalues only with a sign change. Next, we state and present a new and elementary proof of Serre's result.

Theorem 2.4.1. *Let $(X_n)_{n\geq 0}$ be a sequence of k-regular graphs such that $\lim_{n\to+\infty} |V(X_n)| = +\infty$. If*

$$\lim_{n\to+\infty} \frac{c_{2r+1}(X_n)}{|V(X_n)|} = 0 \tag{2.4.1}$$

for each $r \geq 1$, then for each $l \geq 1$

$$\limsup_{n\to+\infty} \mu_l(X_n) \leq -2\sqrt{k-1}$$

Proof. Modifying some ideas developed by McKay in [**68**], we first show that

$$\lim_{n\to+\infty} \frac{\Phi_{2r+1}(X_n)}{|V(X_n)|} = 0$$

for each $r \geq 1$.

For a graph X and $r \geq 1$, let $n_{2r+1}(X)$ denote the number of vertices v_0 in the graph X such that the subgraph of X induced by the vertices at distance at most r from v_0 is bipartite. Thus, $|X| - n_{2r+1}(X)$ is the number of vertices u_0 of X such that the subgraph of X induced by the vertices at distance at most r from u_0 contains at least one odd cycle. Since each such vertex is no further than r from each of the vertices of an odd cycle of length at most $2r+1$, it follows that

$$|V(X)| - n_{2r+1}(X) \leq \sum_{l=1}^{r-1} k^r c_{2l+1}(X)$$

where k^r is an upper bound on the number of vertices in any ball of radius r in X. Thus, we have the following inequalities

$$1 - \sum_{l=1}^{r-1} k^r \frac{c_{2l+1}(X_i)}{|V(X_i)|} \leq \frac{n_{2r+1}(X_i)}{|V(X_i)|} \leq 1$$

for all $r \geq 1, i \geq 0$. Hence, for each $r \geq 1$

(2.4.2) $$\lim_{i \to +\infty} \frac{n_{2r+1}(X_i)}{|V(X_i)|} = 1$$

For $i \geq 0$ and $r \geq 1$, we have

(2.4.3) $$\Phi_{2r+1}(X_i) \leq n_{2r+1}(X_i) \cdot 0 + (|V(X_i)| - n_{2r+1}(X_i))k^{2r+1}$$

since k^{2r+1} is an upper bound on the number of closed walks of length $2r+1$ starting at a fixed vertex of a k-regular graph. From (2.4.2) and (2.4.3), we obtain that for each $r \geq 1$

(2.4.4) $$\lim_{i \to +\infty} \frac{\Phi_{2r+1}(X_i)}{|V(X_i)|} = 0$$

By using relation (2.2.1), it follows that for each $r \geq 1$

(2.4.5) $$\liminf_{i \to +\infty} \frac{\Phi_{2r}(X_i)}{|V(X_i)|} \geq \frac{(2\sqrt{k-1})^{2r}}{(r+1)^2}$$

Let $l \geq 1$. If $A_i = A(X_i)$, then

$$\operatorname{tr}(kI - A_i)^{2s} = \sum_{j=1}^{|V(X_i)|} (k - \lambda_j(X_i))^{2s} < (|V(X_i)| - l)(k - \mu_l(X_i))^{2s} + l(2k)^{2s}$$

Once again, the binomial expansion gives us

$$\operatorname{tr}(kI - A_i)^{2s} = \sum_{j=0}^{2s} \binom{2s}{j} k^j (-1)^{2s-j} \Phi_{2s-j}(X_i)$$

From the previous two relations, we get that

$$(k - \mu_l(X_i))^{2s} + \frac{4^s l k^{2s}}{|V(X_i)| - l} \geq \sum_{j=0}^{2s} \binom{2s}{j} k^j (-1)^{2s-j} \frac{\Phi_{2s-j}(X_i)}{|V(X_i)| - l}$$

Using (2.4.4) and (2.4.5), it follows that

$$\begin{aligned}
k - \limsup_{i \to +\infty} \mu_l(X_i) &\geq \left(\sum_{j=0}^{s} \binom{2s}{2j} k^{2j} \frac{(2\sqrt{k-1})^{2s-2j}}{(s-j+1)^2} \right)^{\frac{1}{2s}} \\
&> \left(\frac{1}{(s+1)^2} \sum_{j=0}^{s} \binom{2s}{2j} k^{2j} (2\sqrt{k-1})^{2s-2j} \right)^{\frac{1}{2s}} \\
&> \left(\frac{1}{2(s+1)^2} \right)^{\frac{1}{2s}} (k + 2\sqrt{k-1})
\end{aligned}$$

for any $s \geq 1$. By taking the limit as $s \to \infty$, we get

$$k - \limsup_{i \to +\infty} \mu_l(X_i) \geq k + 2\sqrt{k-1}$$

which implies the inequality stated in the theorem. □

One might be tempted to believe that if condition (2.4.1) is satisfied for sufficiently large r, then the result of the previous theorem still holds. We show that this is not the case by presenting a counterexample to this assertion. This implies that condition (2.4.1) is somewhat tight.

Consider a sequence $(X_n)_n$ of k-regular graphs with $\operatorname{girth}(X_n) \to +\infty$ as n tends to $+\infty$. Such sequences of graphs with large girth exist, see for example the LPS graphs from the first section. Now replace each vertex by a copy of K_k. The new graphs will contain some short odd cycles, will have no large odd cycles and their eigenvalues will be bounded from below by -2.

To make the description more precise, we need a few definitions and notations.

Definition 2.4.2. *Let X be a k-regular graph with vertex set $[n]$. Suppose the edges incident to each vertex of X are labeled from 1 to k in some arbitrary, but fixed way. The* rotation map $\mathrm{Rot}_X : [n] \times [k] \to [n] \times [k]$ *is defined as follows:* $\mathrm{Rot}_X(u,i) = (v,j)$ *if the i'th edge incident to u is the j'th edge incident to v.*

Note that the label of an edge may not be the same from each of its endpoints. Also, Rot_X is a permutation and $\mathrm{Rot}_X \circ \mathrm{Rot}_X$ is the identity map.

Definition 2.4.3. *If X_1 is k_1-regular graph on $[n_1]$ with rotation map Rot_{X_1} and X_2 is a k_2-regular graph on $[k_1]$ with rotation map Rot_{X_2}, then their* replacement product $X_1 \circledR X_2$ *is defined to be the (k_2+1)-regular graph on $[n_1]\times[k_1]$ whose rotation map $\mathrm{Rot}_{X_1 \circledR X_2}$ is defined as follows:*

$$\mathrm{Rot}_{X_1 \circledR X_2}((v,i),r) = \begin{cases} ((v,j),s) & \text{if } (j,s) = \mathrm{Rot}_{X_2}(i,r), \text{ when } r,s \le k_2 \\ (\mathrm{Rot}_{X_1}(v,i),r) & \text{if } r = k_2+1 \end{cases}$$

More information on the rotation maps, the replacement product and another type of product called zig-zag, can be found in [**85**] where Reingold, Vadhan and Widgerson use these products to explicitly construct (n,k,λ)-graphs with $\lambda = O(k^{\frac{2}{3}})$.

The name replacement product intuitively explains the previous definition. Roughly speaking, each vertex of X_1 is replaced by a copy of X_2 and the structure of X_1 is used in connecting these copies of X_2.

The construction of our counterexample will be based on the replacement product. From the initial sequence $(X_n)_n$ of k-regular graphs with $\mathrm{girth}(X_n) \to +\infty$ as n tends to $+\infty$, we construct a new sequence $(Y_n)_n$ of k-regular graphs where $Y_n = X_n ® K_k$. Note that Y_n does not depend on the rotation maps of X_n and K_k.

The eigenvalues of Y_n are bounded from below by -2 as shown by the following theorem.

THEOREM 2.4.4. *If X is a k-regular graph, then the eigenvalues of $X®K_k$ are at least -2.*

PROOF. From the definition of the replacement product, we notice that each vertex in $X®K_k$ is contained in two cliques, one isomorphic to K_k and one isomorphic to K_2. The cliques isomorphic to K_2 will always be labeled with k from both endpoints. All these cliques partition the edge set of $X®K_k$.

Let N be the incidence matrix whose rows are indexed by the vertices of $X®K_k$ and the columns are indexed by the cliques described earlier with $N(u, \mathcal{K}) = 1$ if vertex u is contained in clique $\mathcal{K}$ and 0 otherwise. It is not hard to see that

$$A(X®K_k) = NN^t - 2I_{nk}$$

Since NN^t has non-negative eigenvalues, it follows that the spectrum of $X®K_k$ is included in the interval $[-2, k]$. □

The following result describes the cycle structure of $X®K_k$ when X is a k-regular graph.

THEOREM 2.4.5. *For each r with $k+1 \leq r \leq 2\,\mathrm{girth}(X)-1$, we have $c_r(X®K_k) = 0$.*

PROOF. Let $r \in \{k+1, \ldots, 2\,\mathrm{girth}(X) - 1\}$ and assume there is a cycle $(v_1, y_1), (v_2, y_2), \ldots, (v_r, y_r)$ of length r in $X®K_k$.

There is at least one edge in this cycle that is labeled with k from both endpoints. Otherwise, by the definition of the replacement product, it follows that the v_j's are the same for $j \in [r]$ and that $y_1, y_2, \ldots, y_r$ forms a cycle of length r in K_k. Since $r \geq k+1$, this is impossible.

Consider the edges of the cycle that are labeled with k from both endpoints. The first coordinates of the endpoints of these edges will induce a cycle of length less than girth(X) in X, contradiction. This proves the theorem. □

The new sequence of graphs (Y_n) satisfies the condition (2.4.1) for each $2r+1 \geq k+1$, but won't satisfy the condition (2.4.1) for each $2r+1 \leq k$ and their eigenvalues will be at least -2.

2.5. Claw free graphs with small eigenvalues

A *claw free* graph X is a graph that does not contain $K_{1,3}$ as an induced subgraph. An equivalent definition is that for each vertex $x \in V(X)$, the neighbours of x induce a subgraph with independence number at most 2.

Perhaps the simplest examples of claw free graphs are the line graphs. Given a graph X, its *line graph* $L(X)$ has as vertices the edges of X with e adjacent to f if e and f share exactly one vertex in X.

As promised in the previous section, we present now a proof of the well-known result that the eigenvalues of line graphs are at least -2. Our proof is longer than the conventional proof, but is presented as a consequence of a more general result that relates the eigenvalues of a graph to partitions of its edge set into cliques. We have already used this method for proving the lower bound on the eigenvalues of $X \circledR K_k$. Recall that $\mu_1(X)$ is the smallest eigenvalue of X.

THEOREM 2.5.1. *If X is a graph on n vertices whose edge set is partitioned into cliques such that each vertex i is contained in d_i cliques, then $\mu_n(X) \geq -\max_{i\in[n]} d_i$.*

PROOF. Consider a clique partition of the edge set of X with the properties described above and let N be the vertex clique incidence matrix. Again, it is not hard to see that

$$A(X) = NN^t - \operatorname*{diag}_{i\in V(X)} (d_i)$$

Let $y \in \mathbb{R}^n$ be an unit eigenvector corresponding to μ_1. Then

$$\begin{aligned}
\mu_1 = y^t A y = y^t N N^t y - \operatorname*{diag}_{i\in V(X)} (d_i))y \\
= (N^t y)^t (N^t y) - \sum_{i=1}^{n} d_i y_i^2 \\
\geq - \max_{i\in V(X)} d_i
\end{aligned}$$

□

COROLLARY 2.5.2. *If X is a line graph, then $\mu_1(X) \geq -2$.*

PROOF. If X is a line graph of a graph G, then the cliques in X corresponding to edges in G partition the edge-set of X and each vertex of X is contained in precisely two such cliques. In this case, N^t is the vertex-edge incidence matrix and the shorter conventional proof follows from the simple observation that $A(L(G)) = N^t N - 2I$. □

Linial [**57**] asked if the property of the eigenvalues of line graphs of being bounded from below by an absolute constant is true also for claw free graphs. This is a natural question considering that the claw free graphs satisfy several of the necessary conditions for having λ_n bounded below by an absolute constant. We describe some of these conditions below.

PROPOSITION 2.5.3. *Let X be a k-regular graph with $\mu_1 = O(1)$. Then for each vertex $i \in V(X)$, $\alpha([N(i)]) = O(1)$, where by $[N(i)]$ we denote the subgraph induced by the neighbours of i.*

PROOF. Let $i \in V(X)$ and denote by S an independent set of order $t = \alpha([N(i)])$ in $N(i)$. Then $i \cup S$ induces a subgraph isomorphic to $K_{1,t}$ in X. By interlacing (see [**37**] and [**41**]), it follows that $\mu_1(X) \leq \mu_1(K_{1,t}) = -\sqrt{t}$. Thus,

$$\alpha([N(i)]) = t \leq \mu_1^2(X)$$

Since $0 > \mu_1(X) = O(1)$, the proposition follows. □

Note that each claw free graph X has the property that $\alpha([N(i)]) \leq 2$ for each vertex $i \in V(X)$.

Another necessary condition for $\lambda_n = O(1)$ is that the chromatic number is large. More precisely, the following result holds.

PROPOSITION 2.5.4. *If X is a k-regular graph with $\mu_1 = O(1)$, then $\chi(X) = \Theta(k)$.*

PROOF. From Theorem 1.3.1, it follows that

$$\chi(X) \leq 1 + \lambda_1(X) = 1 + k$$

and

$$\chi(X) \geq 1 + \frac{k}{-\mu_1}$$

The proposition follows easily from these inequalities. □

Ryjacek and Schiermeyer [**86**] proved the following bound on the independence number of a claw free graph. We include a short proof here.

THEOREM 2.5.5 (Ryjacek-Schiermeyer [**86**]). *If X is a claw free graph on n vertices having minimum degree δ, then*

$$\alpha(X) \leq \frac{2n}{\delta + 2}$$

PROOF. Let S be an independent set of size $\alpha(X)$ in X. We count the number of edges between S and its complement. Since S is independent, it follows that $e(S, V(X) \setminus S) = \sum_{u \in S} \deg(u) \geq \delta(X)\alpha(X)$. On the other hand, X claw free implies that each vertex of $V(X) \setminus S$ has at most 2 neighbours in S. Thus, $e(S, V(X) \setminus S) \leq 2(n - \alpha(X))$. These inequalities imply the desired bound on $\alpha(X)$. □

Since $\chi(X) \geq \frac{|V(X)|}{\alpha(X)}$, the previous result implies that for each k-regular claw free graph X, we have

$$\chi(X) \geq \frac{k+2}{2}.$$

Thus, a k-regular claw free graph X has chromatic number $\chi(X) = \Theta(k)$ as $k \to +\infty$.

We now prove that the answer to Linial's question is negative by describing a family of regular, claw free graphs with arbitrarily negative eigenvalues.

Let $C_{n,r}$ be the graph with vertex set $\mathbb{Z}_n$ having x adjacent to y if and only if $x - y \in S_r \pmod{n}$, where $S_r = \{\pm 1, \pm 2, \ldots, \pm r\}$. This graph is the Cayley graph of $\mathbb{Z}_n$ with generating set S_r and it is a $2r$-regular graph. It is easy to see that $C_{n,r}$ is claw free. The neighbourhood of each vertex of $C_{n,r}$ contains two disjoint cliques of order r and thus it has independence number at most 2.

We discuss Cayley graphs and their eigenvalues in Chapter 4. As a consequence of the results in Chapter 4, we obtain the following description of the eigenvalues of $C_{n,r}$.

PROPOSITION 2.5.6. *The nontrivial eigenvalues of $C_{n,r}$ are*

$$-1+\frac{\sin\left((2r+1)\frac{\pi l}{n}\right)}{\sin\frac{\pi l}{n}}$$

for $l \in [n-1]$.

PROOF. The next equality follows from Lemma 4.2.1. For $l \neq 0, \epsilon_l = e^{\frac{2\pi i l}{n}}$, an eigenvalue of $C_{n,r}$ is

$$\begin{aligned}
&\sum_{j=1}^{r}\epsilon_l^j+\sum_{j=1}^{r}\epsilon_l^{-j}\\
&=\frac{1-\epsilon_l^{r+1}}{1-\epsilon_l}-1+\frac{1-\epsilon_l^{-(r+1)}}{1-\epsilon_l^{-1}}-1\\
&=-2+\frac{1-\epsilon_l^{r+1}-\epsilon_l(1-\epsilon_l^{-r-1})}{1-\epsilon_l}\\
&=-1+\frac{\epsilon_l^{-r}-\epsilon_l^{r+1}}{1-\epsilon_l}=-1+\frac{\epsilon_l^{r+\frac{1}{2}}-\epsilon_l^{r-\frac{1}{2}}}{\epsilon_l^{\frac{1}{2}}-\epsilon_l^{-\frac{1}{2}}}\\
&=-1+\frac{\sin\left((2r+1)\frac{\pi l}{n}\right)}{\sin\frac{\pi l}{n}}.
\end{aligned}$$

□

If we choose n and r such that $l=\frac{3n}{2(2r+1)}$ is an integer, then the previous proposition implies

$$\mu_1(C_{n,r})\leq -1-\frac{1}{\sin\frac{3\pi}{2(2r+1)}}\sim -1-\frac{2}{3\pi}-\frac{2}{3\pi}2r$$

as r tends to $+\infty$. Hence, the eigenvalues of the claw free graphs $C(n,r)$ can be arbitrarily negative.

It would be interesting to determine how close $\mu_1(X)$ can get to $-k$ when X is a k-regular claw free graph on n vertices.

CHAPTER 3

Eigenvalues of irregular graphs

3.1. Preliminaries

Serre's Theorem 2.2.1 is related to a result obtained by Greenberg in [**39**] whose proof has not appeared to our knowledge in any journal as yet. Greenberg's result is stated in many places, [**63**] and [**64**] (Theorem 2.3) for example. In this chapter, a simplified version of Greenberg's proof is presented. We also present a slight improvement of Greenberg's result as well as a theorem regarding the smallest eigenvalues of irregular graphs.

If X is a connected graph (not necessarily finite) such that the maximum degree of X is finite, then $V(X)$ is countable and we let $l^2(X)$ denote the space of functions $f : V(X) \to \mathbb{R}$ with $\sum_{x\in V(X)} |f(x)|^2 < \infty$. Let $\delta : l^2(X) \to l^2(X)$ be the *adjacency operator* of X, i.e., $(\delta f)(x) = \sum_{y\in V(X)} a_{x,y} f(y)$, where $a_{x,y}$ is the number of edges with endpoints x and y. Note that if X is finite, we have been writing functions f as vectors x and the adjacency operator as left multiplication by a matrix A, that is $x \to Ax$.

Recall that if $T : H \to H$ is a *linear operator* on a Hilbert space H, then the *norm* of T is defined to be $||T|| = \sup_{f\neq 0} \dfrac{||Tf||}{||f||}$. We say T is *bounded* if its norm is finite.

Let $I : H \to H$ be the identity operator. The *resolvent set* of T is the set of all complex λ for which $T - \lambda I$ has a bounded inverse. The *spectrum* of T is the complement of the resolvent set of T.

Denote by $\rho(X)$ the *spectral radius* of X:

$$\rho(X) = \max\{|\lambda| : \lambda \in \text{spectrum of } \delta\}$$

Recall that we denote by $t_s(u)$ the number of closed walks of length s that start at u in a graph X.

LEMMA 3.1.1. *Let X be a connected graph. Then $\limsup_{s\to+\infty} \sqrt[s]{t_s(u)}$ is independent of the vertex $u \in V(X)$.*

PROOF. Since X is connected, it is enough to prove that

$$\limsup_{s\to+\infty} \sqrt[s]{t_s(u)} = \limsup_{s\to+\infty} \sqrt[s]{t_v(y)}$$

for any adjacent vertices u and v. The previous assertion follows easily from the fact that

$$t_{s+2}(u) \geq t_s(v) \geq t_{s-2}(u)$$

for each $s \geq 2$. □

It is well known (cf. [**62**] and [**64**]) that

$$\rho(X) = \limsup_{s\to\infty} \sqrt[s]{t_s(u)}, \tag{3.1.1}$$

for each u in X.

We now show that actually the sequence $(\sqrt[2s]{t_{2s}(u)})_s$ converges for each u. Our argument is based on the following simple observation. A closed walk of length $2r$ starting at a vertex u of X together with a closed walk of length $2s$ starting at u form a closed walk of length $2r + 2s$. Thus, for each u in X and each r and s nonnegative integers, we have

$$t_{2r+2s}(u) \geq t_{2r}(u)t_{2s}(u) \tag{3.1.2}$$

The next result is known as Fekete's Lemma [**28**]. For an English version of the lemma and a proof, see van Lint and Wilson [**59**].

LEMMA 3.1.2 (Fekete). *Let* $h : \mathbb{N} \to \mathbb{N}$ *be a function such that*

$$h(m+n) \geq h(m)h(n) \tag{3.1.3}$$

for each $m, n \in \mathbb{N}$. *Then* $\lim_{n\to+\infty} \sqrt[n]{h(n)}$ *exists (possibly infinite).*

Hence, we obtain that if X is an infinite tree with finite maximum degree

$$\rho(X) = \lim_{s\to+\infty} \sqrt[2s]{t_{2s}(u)}, \tag{3.1.4}$$

for each $u \in V(X)$.

Given two graphs X_1 and X_2 (not necessarily finite), a *homomorphism* from X_1 to X_2 is a function $g : V(X_1) \to V(X_2)$ such that $xy \in E(X_1)$ implies $g(x)g(y) \in E(X_2)$ for each $x, y \in V(X_1)$. We call g an *isomorphism* from X_1 to X_2 if g is bijective and $xy \in E(X_1)$ if and only if $g(x)g(y) \in E(X_2)$. An isomorphism from a graph X to itself is called an *automorphism* of X. The automorphisms of X form a group, called the *automorphism group* of X that we denote by $\mathrm{Aut}(X)$. If x is a vertex of X, then the *automorphism orbit* of x is $\mathrm{Orb}(x) = \{y \in V(X) : y = g(x) \text{ for some } g \in Aut(X)\}$.

If X_1 and X_2 are two graphs, a homomorphism $g : V(X_1) \to V(X_2)$ is called a *cover map* if it is surjective and for each $x \in V(X_1)$, g induces an isomorphism from $N_{X_1}(x)$ to $N_{X_2}(g(x))$. Since g maps closed walks to closed walks, it follows that if $g : V(X_1) \to V(X_2)$ is a cover map, then $\rho(X_1) \leq \rho(X_2)$. If X_1 and X_2 are both finite, then it is straightforward to show that each eigenvalue of X_2 is an eigenvalue of X_1, and so $\rho(X_1) = \rho(X_2)$. Denote by $\mathcal{C}(X)$ the family of finite graphs covered by X. Using a result of Leighton [**52**], the next theorem is proved by Greenberg [**39**].

THEOREM 3.1.3 (Greenberg [**39**]). *Let X be a connected graph with finite maximum degree. Then for each X_1 and X_2 in $\mathcal{C}(X), \rho(X_1) = \rho(X_2)$. This common value is denoted by $s(X)$.*

For a finite graph Z, its *universal cover* $\tilde{Z}$ is the graph with the property that for any graph Y with a cover map $g : V(Y) \to V(Z)$, there exists a cover map $g' : V(\tilde{Z}) \to V(Y)$. The universal cover of any finite graph is a tree. A precise description can be found in Leighton's paper [**52**]. For example, the universal cover of any k-regular graph is the infinite k-regular tree. However, not every infinite tree X covers a finite graph. It is easy to see that a necessary condition for covering a finite graph is that $\mathrm{Aut}(X)$ have finitely many orbits. A survey paper of Lubotzky [**63**] contains more details on the universal covers of finite graphs.

3.2. A theorem of Greenberg

If X is a connected, infinite graph with finite maximum degree, denote by $\mathcal{C}(X)$ the family of finite and connected graphs covered by X. In his Ph.D. thesis ([**39**]), Greenberg proved the following result.

THEOREM 3.2.1 (Greenberg). *Let X be a connected infinite graph with finite maximum degree. Given $\epsilon > 0$, there exists $c = c(X, \epsilon) > 0$, such that for every $Y \in \mathcal{C}(X)$,*

$$|\{\lambda_i \in \textit{spectrum of } Y : |\lambda_i| \geq \rho(X) - \epsilon\}| \geq c|Y|.$$

PROOF. [1]

Let $\epsilon > 0$ and $Y \in \mathcal{C}(X)$. Let c be the proportion of eigenvalues of Y that have absolute value $\geq \rho(X) - \epsilon$.

[1] I thank Shlomo Hoory for translating and adapting Greenberg's argument.

Obviously, if x and y are in the same orbit of $\mathrm{Aut}(X)$, then $t_r(x) = t_r(y)$ for any nonnegative integer r. Hence, the fact that X has finitely many automorphism orbits and (3.1.4) imply that there exists a nonnegative integer $r_0 = r(X, \epsilon)$ such that

$$t_{2r}(y) \geq \left(\rho(X) - \frac{\epsilon}{2}\right)^{2r}$$

for each vertex $y \in V(Y)$ and $r \geq r_0$.

Using the previous inequality, we obtain

$$\left(\rho(X) - \frac{\epsilon}{2}\right)^{2r} \leq \min_{y \in V(Y)} t_{2r}(y) \leq \frac{\mathrm{tr}(A^{2r}(Y))}{|V(Y)|} \leq c s^{2r}(X) + (1-c)\left(\rho(X) - \epsilon\right)^{2r}$$

for each $r \geq r_0$. This implies

$$c \geq \frac{\left(\rho(X) - \frac{\epsilon}{2}\right)^{2r} - \left(\rho(X) - \epsilon\right)^{2r}}{s^{2r}(X) - \left(\rho(X) - \frac{\epsilon}{2}\right)^{2r}}$$

for each $r \geq r_0$. Letting $r = r_0$, this proves the theorem. □

The previous theorem is also cited in [**64**] (Theorem 2.3), but it seems that no proof of it exists in the literature other than in Greenberg's thesis. Note that Theorem 3.2.1 implies a weaker form of Serre's theorem. This is because if X is the infinite k-regular tree, then $\rho(X) = 2\sqrt{k-1}$. Thus, if Y is a finite k-regular graph we obtain that for each $\epsilon > 0$, a positive proportion (that depends only on ϵ and k) of the eigenvalues of Y have *absolute value* at least $(2-\epsilon)\sqrt{k-1}$. This is slightly weaker than Theorem 2.2.1.

3.3. An improvement of Greenberg's theorem

We prove here a stronger form of Greenberg's theorem. The proof given below follows the ideas of our proof of Serre's theorem from Chapter 2.

THEOREM 3.3.1. *Let X be a connected infinite graph with finite maximum degree. Given $\epsilon > 0$, there exists $c = c(X, \epsilon) > 0$, such that for every $Y \in \mathcal{C}(X)$, the number of eigenvalues λ_i of Y such that $\lambda_i \geq \rho(X) - \epsilon$ is at least $c|Y|$.*

PROOF. Let $Y \in \mathcal{C}(X)$ with eigenvalues $\lambda_1(Y) > \lambda_2(Y) \geq \cdots \geq \lambda_n(Y)$ and cover map $g : V(X) \to V(Y)$. Given $\epsilon > 0$, let $m = |\{i : \lambda_i \geq \rho(X) - \epsilon\}|$. From (3.1.4) we know that $\rho(X) = \lim_{s \to +\infty} \sqrt[2s]{t_{2s}(x)}$, for any vertex $x \in V(X)$. Since $\rho(X) > 0$, it follows that there exists a positive integer $N = N(X, \epsilon)$ such that $\rho(X) > \frac{\epsilon}{N}$. Obviously, if x and y are in the same orbit of $\mathrm{Aut}(X)$, then $t_r(x) = t_r(y)$ for any nonnegative integer r. Hence, the fact that X has finitely many automorphism orbits implies that there exists a nonnegative integer $s_0 = s(X, \epsilon)$ such that $t_{2s}(x) \geq \left(\rho(X) - \frac{\epsilon}{N}\right)^{2s}$, for each $s \geq s_0$ and any $x \in V(X)$.

Since every closed walk of length r in X starting at $x \in V(X)$ covers a closed walk of length r in Y starting at $g(x)$, it follows that $\Phi_r(Y) \geq \sum_{\substack{y \in V(Y) \\ y = \pi(x)}} t_r(x)$ for each non-negative integer r. From the previous two relations it follows that $\Phi_{2s}(Y) \geq n\left(\rho(X) - \frac{\epsilon}{N}\right)^{2s}$ for $s \geq s_0$.

Let K be a positive constant that does not depend on Y and is larger than $\lambda_1(Y)$. We can take $K = \Delta(X)$, the maximum degree of X. From the previous inequality we deduce

$$\begin{aligned}
\mathrm{tr}(K \cdot I + A(Y))^{2l} &= \sum_{i=0}^{2l} \binom{2l}{i} K^{2l-i} \Phi_i(Y) \geq \sum_{j=s_0}^{l} \binom{2l}{2j} K^{2l-2j} \Phi_{2j}(Y) \\
&\geq n \sum_{j=s_0}^{l} \binom{2l}{2j} K^{2l-2j} \left(\rho(X) - \frac{\epsilon}{N}\right)^{2j} \\
&\geq n \sum_{j=0}^{l} \binom{2l}{2j} K^{2l-2j} \left(\rho(X) - \frac{\epsilon}{N}\right)^{2j} - n \sum_{j=0}^{s_0-1} \binom{2l}{2j} K^{2l-2j} \left(\rho(X) - \frac{\epsilon}{N}\right)^{2j} \\
&\geq \frac{n}{2} \left(K + \rho(X) - \frac{\epsilon}{N}\right)^{2l} - n s_0 \binom{2l}{2s_0} K^{2l}
\end{aligned}$$

for each $l \geq 2s_0$. It follows that

$$\operatorname{tr}(K \cdot I + A)^{2l} = \sum_{i=1}^{n} (K + \lambda_i(Y))^{2l}$$

$$\leq (n-m)(K + \rho(X) - \epsilon)^{2l} + m(2K)^{2l}.$$

Hence, we obtain

$$\text{(3.3.1)} \qquad \frac{m}{n} \geq \frac{\frac{\left(K+\rho(X)-\frac{\epsilon}{N}\right)^{2l}}{2} - (K + \rho(X) - \epsilon)^{2l} - s_0\binom{2l}{2s_0}K^{2l}}{(2K)^{2l} - (K + \rho(X) - \epsilon)^{2l}}$$

for each $l \geq 2s_0$.

Now

$$\lim_{l\to\infty} \sqrt[2l]{\frac{\left(K + \rho(X) - \frac{\epsilon}{N}\right)^{2l}}{2}} = K + \rho(X) - \frac{\epsilon}{N}$$

and

$$\lim_{l\to\infty} \sqrt[2l]{2\left(K + \rho(X) - \epsilon\right)^{2l} + s_0\binom{2l}{2s_0}K^{2l}} = \max(K + \rho(X) - \epsilon, K) < K + \rho(X) - \frac{\epsilon}{N}$$

imply that there exists $l_0 = l(X, \epsilon)$ such that

$$\frac{\left(K + \rho(X) - \frac{\epsilon}{N}\right)^{2l}}{2} - (K + \rho(X) - \epsilon)^{2l} - s_0\binom{2l}{2s_0}K^{2l} > (K + \rho(X) - \epsilon)^{2l},$$

for each $l \geq l_0$. Hence,

$$\frac{m}{n} > \frac{(K + \rho(X) - \epsilon)^{2l_0}}{(2K)^{2l_0} - (K + \rho(X) - \epsilon)^{2l_0}} = c(X, \epsilon) > 0$$

□

By using a similar argument as before, we can prove the following.

THEOREM 3.3.2. *Let X be a connected, infinite graph with finite maximum degree. Given $\epsilon > 0$, there exist a non-negative integer $g = g(X, \epsilon)$ and $c = c(X, \epsilon) > 0$, such that for every $Y \in \mathcal{C}(X)$ with* $\operatorname{oddg}(Y) \geq g$,

$$|\{\mu_i \in \textit{spectrum of } Y : \mu_i \leq -(\rho(X) - \epsilon)\}| \geq c|Y|$$

PROOF. Let $Y \in \mathcal{C}(X)$ with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and cover map $g : V(X) \to V(Y)$. Given $\epsilon > 0$, let $m = |\{i : \lambda_i \leq -(\rho(X) - \epsilon)\}|$.

As in the proof of Theorem 3.3.1, we deduce that there exist $N = N(X, \epsilon) > 1$, $s_0 = s(X, \epsilon)$ and $l_0 = l(X, \epsilon)$ with $l_0 \geq 2s_0$ such that

$$\frac{\left(K + \rho(X) - \frac{\epsilon}{N}\right)^{2l_0}}{2} - (K + \rho(X) - \epsilon)^{2l_0} - s_0 \binom{2l_0}{2s_0} K^{2l_0} > (K + \rho(X) - \epsilon)^{2l_0}$$

Consider

$$\operatorname{tr}(K \cdot I - A(Y))^{2l} = \sum_{i=1}^{n} (K - \lambda_i)^{2l} \leq (n - m)(K + \rho(X) - \epsilon)^{2l} + m(2K)^{2l}$$

Let $g(X, \epsilon) = 2l_0$. If Y has no odd cycles of length less than $2l_0$, then

$$\begin{aligned}
\operatorname{tr}(K \cdot I - A(Y))^{2l_0} &= \sum_{j=0}^{l_0} \binom{2l_0}{2j} K^{2l_0 - 2j} \Phi_{2j}(Y) \\
&\geq \sum_{j=s_0+1}^{l_0} \binom{2l_0}{2j} K^{2l_0 - 2j} \Phi_{2j}(Y) \\
&\geq n \sum_{j=0}^{2l_0} \binom{2l_0}{2j} K^{2l_0 - 2j} \left(\rho(X) - \frac{\epsilon}{N}\right)^{2j} - s_0 \binom{2l_0}{2s_0} K^{2l_0} \\
&\geq \frac{n}{2} \left(K + \rho(X) - \frac{\epsilon}{N}\right)^{2l_0} - s_0 \binom{2l_0}{2s_0} K^{2l_0}
\end{aligned}$$

From the previous two inequalities, we deduce

$$\begin{aligned}
\frac{m}{n} &\geq \frac{\frac{\left(K + \rho(X) - \frac{\epsilon}{N}\right)^{2l_0}}{2} - (K + \rho(X) - \epsilon)^{2l_0} - s_0 \binom{2l_0}{2s_0} K^{2l_0}}{(2K)^{2l_0} - (K + \rho(X) - \epsilon)^{2l_0}} \\
&> \frac{(K + \rho(X) - \epsilon)^{2l_0}}{(2K)^{2l_0} - (K + \rho(X) - \epsilon)^{2l_0}}
\end{aligned}$$

This proves the theorem. □

In his Ph.D. thesis, Greenberg introduced the notion of Ramanujan graph for general finite graphs (not necessarily regular). A finite graph Y is called *Ramanujan* if for any nontrivial eigenvalue λ of Y, the inequality $|\lambda| \leq \rho(\tilde{Y})$ holds, where $\tilde{Y}$ is the universal cover of Y. If Y is regular, then we obtain the definition given by Lubotzky, Phillips and Sarnak in [**65**] (see Definition 2.1.3).

Recently, Hoory [**44**] proved that if Y is a finite graph with average degree d, then $\rho(\tilde{Y}) \geq 2\sqrt{d-1}$. Hoory used this result to prove a generalization of the asymptotic Alon-Boppana theorem. Denote by $B_r(v)$ the ball of radius r around v. A graph Y has an r-robust average-degree d if for every vertex v the graph induced on $V(Y) \setminus B_r(v)$ has average degree at least d. Hoory's generalization is the following result.

THEOREM 3.3.3. *Let Y_i be a sequence of graphs such that Y_i has an r_i-robust average degree $d \geq 2$, where $\lim_{i \to +\infty} r_i = +\infty$. Then*

$$\liminf_{i \to +\infty} \lambda(Y_i) \geq 2\sqrt{d-1}$$

CHAPTER 4

Abelian Cayley graphs

4.1. Preliminaries

In this chapter, we prove that Abelian Cayley graphs have a large number of closed walks of even length. We use this fact to prove that for any Abelian group G and any symmetric k-subset S of G, the number of eigenvalues λ_i of the Cayley graph $X(G, S)$ such that $\lambda_i \geq k - \epsilon$ is at least $c \cdot |G|$, where c is a positive constant that depends only on ϵ and k. This proves that constant-degree Abelian Cayley graphs have many large non-trivial eigenvalues and thus, they are bad expanders. We discuss the chromatic and independence numbers of some finite analogues of Euclidean graphs.

Cayley graphs are defined as follows.

Definition 4.1.1. *Let G be a finite multiplicative group, with identity 1 and suppose $S = \{x_1, x_2, \ldots, x_s, x_1^{-1}, x_2^{-1}, \ldots, x_s^{-1}, y_1, \ldots, y_t\}$ is a subset of G such that $1 \notin S$, $x_i^2 \neq 1$ for all i and $y_j^2 = 1$ for all j. The (s,t)-Cayley graph $X = X(G, S)$ is the simple graph with vertex set G and with $x, y \in G$ adjacent if $xy^{-1} \in S$.*

Notice that adjacency is well-defined since S is symmetric, i.e. $a \in S$ if and only if $a^{-1} \in S$. Also, G is regular with valency $k = |S|$ and it contains no loops since $1 \notin S$. It is easy to see that X is connected if and only if S generates G.

A similar way of constructing regular graphs uses the product instead of the ratio in the previous definition. Given an Abelian group G and a k-subset S of G, the *product graph*[1] $Y(G,S)$ has as vertices the elements of G with x adjacent to y if $x \cdot y \in S$. The fact that G is Abelian implies that $Y(G,S)$ is an undirected k-regular graph. Note also that S does not have to be a symmetric set for the product graphs.

4.2. Spectra of Abelian Cayley graphs

It is well known (see Lovász [**61**]) that the eigenvalues of Abelian Cayley graphs $X = X(G,S)$ can be expressed in terms of the irreducible characters of the group G. We have already used this fact in Chapter 2.

PROPOSITION 4.2.1. *If G is an Abelian group and S is a subset of k elements of G, then the eigenvalues of $X(G,S)$ are*

$$\lambda_\chi = \sum_{s \in S} \chi(s)$$

where χ ranges over all the irreducible characters of G.

PROOF. Let χ be an irreducible character of G and define the vector $v_\chi = (\chi(g))_{g \in G}$. Then

$$(A(X)v_\chi)(x) = \sum_{s \in S} \chi(xs) = \chi(x) \sum_{s \in S} \chi(s) = \lambda_\chi v_\chi(x)$$

which implies that v_χ is an eigenvector of A with eigenvalue λ_χ. Since distinct characters are orthogonal, this completely determines the spectrum of X. $\square$

The eigenvalues of product graphs can be also expressed as sums of the irreducible characters of the group G.

PROPOSITION 4.2.2. *Let G be an Abelian group and χ an irreducible character of G. If $\lambda_\chi = 0$, then v_χ and $v_{\chi^{-1}}$ are both eigenvectors of $Y(G,S)$ with eigenvalue*

[1] In additive notation, these are usually *sum graphs*.

zero. If χ is nontrivial and $\lambda_\chi \neq 0$, then $|\lambda_\chi| v_\chi \pm \lambda_\chi v_{\chi^{-1}}$ are two eigenvectors with eigenvalues $\pm|\lambda_\chi|$.

Proof. Let χ be an irreducible character. Then

$$(A(Y)v_\chi)(x) = \sum_{s \in S} \chi(sx^{-1}) = \chi(x^{-1}) \sum_{s \in S} \chi(s) = \lambda_\chi v_{\chi^{-1}}(x)$$

If $\lambda_\chi = 0$, then it is easy to see that both v_χ and $v_{\chi^{-1}}$ are eigenvectors of Y with eigenvalue 0. The second part of the proposition also follows from the previous equation. $\square$

Thus, the absolute values of the eigenvalues of $X(G, S)$ and $Y(G, S)$ are the same when S is symmetric. They are $|\sum_{s \in S} \chi(s)|$ as χ runs through all the irreducible characters of G.

4.3. Some Ramanujan Abelian Cayley graphs

In this section, we present examples of Abelian Cayley graphs that are Ramanujan in some cases. Often, the proof that they are Ramanujan is based on nontrivial character sum estimates from number theory.

Paley graphs

Let q be a prime power such that $q \equiv 1 \pmod 4$. The Paley graph $P(q)$ is the Cayley graph of the group $(\mathbb{F}_q, +)$ with respect to the set of non-zero squares in $\mathbb{F}_q$. Note that the condition $q \equiv 1 \pmod 4$ ensures that the set of non-zero squares is symmetric about 0.

For example, if $q = 5$, then the set of non-zero square in $\mathbb{F}_5$ is $\{1, -1\}$. Thus, the Paley graph $P(5)$ is the cycle on 5 vertices.

The Paley graph $P(q)$ is strongly regular with parameters $\left(q, \frac{q-1}{2}, \frac{q-5}{4}, \frac{q-1}{4}\right)$. Thus, $P(q)$ is $\frac{q-1}{2}$-regular and its nontrivial eigenvalues are $\frac{-1\pm\sqrt{q}}{2}$, both with multiplicity $\frac{q-1}{2}$. It is easy to see that $P(q)$ is Ramanujan for each prime power $q \equiv 1 \pmod 4$.

Finite analogues of the Euclidean plane

Consider the infinite graph whose vertex set is the Euclidean plane $\mathbb{R}^2$ and two vertices are adjacent if and only if the distance between them is 1. The graph we just described is usually called the *unit distance graph* or the *Euclidean graph*.

Finding the chromatic number of the unit distance graph is one of the most outstanding open problems in graph theory. Its origin is due to Hadwiger and Nelson in 1944. From the works of Hadwiger [**40**] and Moser [**75**], we know that the chromatic number of this graph lies between 4 and 7. Ron Graham has offered $1000 for a solution to this problem.

More details on the unit distance graphs can be found in a recent paper of Szekély [**90**]. Medrano, Myers, Stark and Terras have considered finite analogues of the Euclidean graph.

In [**69**], Medrano, Myers, Stark and Terras study the finite analogue of this problem over finite fields.

Let $\mathbb{F}_q$ be a finite field with $q = p^r$ elements, where p is an odd prime. For $a \in \mathbb{F}_q$, let $S_q(n, a) = \{x \in \mathbb{F}_q^n : x^t x = a\}$. Then the *Euclidean graph over* $\mathbb{F}_q$, $E_q(n, a)$ is the Cayley graph of $(\mathbb{F}_q^n, +)$ with respect to $S_q(n, a)$.

The quadratic character χ is defined as usual

$$\chi(a) = \begin{cases} 1 & \text{if } a \neq 0, a = u^2, u \in \mathbb{F}_q, \\ -1 & \text{if } a \neq 0, a \neq u^2, u \in \mathbb{F}_q, \\ 0 & \text{if } a = 0. \end{cases}$$

The following result is proved by Medrano, Myers, Stark and Terras [**69**].

THEOREM 4.3.1 (Medrano, Myers, Stark, Terras [**69**]). *The Euclidean graph $E_q(n,a)$, where q is a power of an odd prime, is a regular graph with q^n vertices of degree given by $|S_q(n,a)|$. If $a \neq 0$, then*

$$|S_q(n,a)| = \begin{cases} q^{n-1} + \chi((-1)^{\frac{n-1}{2}}a)q^{\frac{n-1}{2}} & \textit{for } n \textit{ odd,} \\ q^{n-1} - \chi((-1)^{\frac{n}{2}})q^{\frac{n-1}{2}} & \textit{for } n \textit{ even.} \end{cases}$$

If $a = 0$, then

$$|S_q(n,0)| = \begin{cases} q^{n-1} & \textit{for } n \textit{ odd} \\ q^{n-1} + \chi((-1)^{\frac{n}{2}})(q-1)q^{\frac{n}{2}} & \textit{for } n \textit{ even.} \end{cases}$$

Note that $|S_q(n,a)| > 1$ if $n \geq 3$. When $n = 2$, $|S_q(2,a)| > 1$ if $a \neq 0$ or if $a = 0$ and $\chi(-1) = 1$. The graphs are connected unless $(q,n,a) = (q,2,0)$ with $\chi(-1) = -1$. In the latter case, the graph is just a set of loops on each point in $\mathbb{F}_q^n$.

The nontrivial eigenvalues of $E_q(n,a)$ are also studied in [**69**]. By expressing the eigenvalues of $E_q(n,a)$ as Kloosterman sums, Medrano, Myers, Stark and Terras prove the following result.

THEOREM 4.3.2 (Medrano, Myers, Stark, Terras [**69**]). *All the non-trivial eigenvalues λ_i of $E_q(n,a)$ satisfy the inequality*

$$|\lambda_i| \leq 2q^{\frac{n-1}{2}} \tag{4.3.1}$$

Using the previous two results, it is determined in [**69**] when the graphs $E_q(n, a)$ are Ramanujan. The problem of finding the chromatic number of these graphs is not discussed in [**69**]. The graphs $E_q(n, a)$ may have interesting extremal properties. For example, it was proved by Brown [**13**] that when q is a prime, $E_q(3, 1)$ has asymptotically the largest number of edges among the graphs on q^3 vertices that do not contain $K_{3,3}$.

For q a prime power (not necessarily odd), $n = 2$ and $a = 1$, the chromatic numbers of the graphs $E_q(n, a)$ have been studied by Moorhouse [**73**] who proves the next result.

THEOREM 4.3.3 (Moorhouse [**73**]). *If q is even ($q = 2^r$), then $\chi(E_q(2, 1)) = 2$.*

PROOF. There exists an $\mathbb{F}_2$-linear map $\theta : \mathbb{F}_q \to \mathbb{F}_2$ such that $\theta(1) = 1$. One checks that the map $\mathbb{F}_q^2 \to \mathbb{F}_2$ given by $(x, y) \to \theta(x + y)$ is a proper 2-coloring. □

By computer search, Moorhouse [**73**] computes the exact chromatic number of $\chi(E_q(2, 1))$ for small values of q; for $q = 3, 5, 7, 9, 11$, the chromatic numbers are $2, 3, 4, 3, 5$.

By computer search we determined the following bounds on the chromatic and the independence number of $E_q(2, 1)$ for q an odd prime.

q	$\chi(E_q(2,1))$	$\alpha(E_q(2,1))$
7	4	14
11	5	≥ 28
13	≤ 6	≥ 39
17	≤ 6	≥ 66
19	≤ 8	≥ 75
23	≤ 9	≥ 79

The lower bound of 14 for $\alpha(E_7(2,1))$ is by computer search. David Gregory has proved 14 is an upper bound. As seen earlier, the graphs $E_q(2,1)$ with q an odd prime, are regular of degree $|S_q(2,1)| = q - \chi(-1)$ and the nontrivial eigenvalues of $E_q(2,1)$ are in absolute value at most $2\sqrt{q}$. By using the Hoffman bound from Chapter 1, we get

Proposition 4.3.4. *If q is an odd prime, then*

$$\chi(E_q(2,1)) \geq 1 + \frac{q - \chi(-1)}{2\sqrt{q}} > \frac{\sqrt{q}}{2}$$

Perhaps it is not surprising that, similarly to the unit distance graph, we obtain a gap in estimating the chromatic number of the finite Euclidean graphs over fields.

$$\frac{\sqrt{q}}{2} < \chi(E_q(2,1)) \leq q + 1$$

Also, from Hoffman's bound on the independence number of a graph, we obtain the following result.

Proposition 4.3.5. *If q is an odd prime, then*

$$\alpha(E_q(2,1)) \leq \frac{q^2 \cdot 2\sqrt{q}}{q - \chi(-1) + 2\sqrt{q}} \sim 2q^{\frac{3}{2}}$$

It would be interesting to determine the exact order of magnitude for $\chi(E_q(2,1))$ and $\alpha(E_q(2,1))$.

In [**70**], Medrano, Myers, Stark and Terras study the analogue of the Euclidean graph over finite rings.

Let $\mathbb{Z}_q$ be the ring $\mathbb{Z}/q\mathbb{Z}$, where $q = p^r$ and p is a prime. Define the distance between $x, y \in \mathbb{Z}_q^n$ by $d(x,y) = (x-y)\cdot(x-y)$. For $a \in \mathbb{Z}_q$, define the *Euclidean graph over $\mathbb{Z}_q$*, $X_q(n,a)$ as follows: the vertices are the vectors in $\mathbb{Z}_q^n$ and $x, y \in \mathbb{Z}_q^n$

are adjacent if $d(x,y) = a$. Let

$$T_q(n,a) = \{x \in \mathbb{Z}_q^n : d(x,0) = a\}.$$

Then $X_q(n,a)$ is the Cayley graph of $\mathbb{Z}_q^n$ with respect to $T_q(n,a)$. The next result is proved in [**70**].

THEOREM 4.3.6. *If $a \in \mathbb{Z}_q^*$, the degree of $X_{p^r}(n,a)$ is given by*

$$|T_{p^r}(n,a)| = p^{(n-1)(r-1)}|T_p(n,a)|$$

By reducing the elements of $\mathbb{Z}_{p^{r+1}}$ modulo p^r we obtain a homomorphism ϕ from $X_{p^{r+1}}(n,a)$ to $X_{p^r}(n,a)$. The next result is a simple consequence of this fact.

THEOREM 4.3.7. *For each $r \geq 1$,*

$$\chi(X_{p^{r+1}}(n,a)) \leq \chi(X_{p^r}(n,a)) \ , \ \alpha(X_{p^{r+1}}(n,a)) \geq p^n \alpha(X_{p^r}(n,a))$$

I believe that equality holds in both of the previous results.

There are other number theoretic constructions of Abelian Cayley graphs that are Ramanujan. They appear in the works of Friedman [**31**], Winnie Li [**53**] and Murty [**76**].

All these examples show that it is possible to construct Ramanujan graphs using Abelian groups. However, we should notice that for each choice of a degree of regularity, all these constructions produce a *finite* number of graphs that are Ramanujan. We will show in the next sections that it is actually impossible to construct infinite sequences of constant-degree Ramanujan graphs using Abelian groups.

4.4. Codes and Abelian Cayley graphs over $\mathbb{F}_2^n$

A *code* of *length* n is subset C of $\mathbb{F}_2^n$. C is called *linear* if it is a subspace of the vector space $\mathbb{F}_2^n$. For $x, y \in \mathbb{F}_2^n$, the *Hamming distance* $d(x, y)$ between x and y is the number of coordinates in which x and y differ. The *weight* of a codeword x is just $d(x, 0)$. The *minimum distance* of a code C is

$$d_{min}(C) = \min\{d(x, y) : x \neq y \in C\}$$

If C is linear, then $d_{min}(C) = \min_{x \in C \setminus \{0\}} w(x)$.

An Abelian Cayley graph over $\mathbb{F}_2$ can be associated with a linear code such that the eigenvalues of the graph are in relationship with the weights of the codewords. This can be done as follows.

Let $X = X(\mathbb{F}_2^n, S)$, where $S = \{c_1, c_2, \ldots, c_k\}$. Denote by M the n by k matrix whose columns are the elements of S. The rows of M generate a subspace C of $\mathbb{F}_2^k$. For a vector u in C, denote by $e(u)$ the difference between the number of 0's and the number of 1's of u. This is the same as $e(u) = k - 2w(u)$. The connection between the eigenvalues of X and the linear code C is given by the next result.

THEOREM 4.4.1. *For $u \in C$, $e(u) = k - 2w(u)$ is an eigenvalue of X.*

For a proof and more details, see [**25, 34**]. The previous theorem was used by Alon and Roichman [**4**] to obtain lower bounds on the nontrivial eigenvalues of Cayley graphs over $\mathbb{F}_2^n$. We discuss some of their results in the next section.

4.5. Bounding the eigenvalues of Abelian Cayley graphs

For any k-regular graph X (not necessarily Abelian Cayley graph), we have (see also Chapter 1 for more details),

$$\lambda_2 = \max_{\substack{1^t x = 0 \\ x \neq 0}} \frac{x^t A x}{x^t x}$$

This fact and Lemma 4.2.1 were used in [**33**] where Friedman, Murty and Tillich prove that the second largest eigenvalue of a k-regular Abelian Cayley graph with n vertices is at least $k - O(kn^{-\frac{4}{k}})$ as n tends to infinity. This implies that constant-degree Abelian Cayley graphs are bad expanders. Friedman, Murty and Tillich [**33**] also show that their bound is sharp.

Using Proposition 1.1.1, we obtain

$$k^{2r} + \lambda^{2r}(X) \leq \Phi_{2r}(X)$$

for each $r \geq 1$.

The previous inequality was used by Alon and Roichman [**4**] to show that random Cayley graphs are expanders. More precisely, they proved the following theorem.

Theorem 4.5.1. *For each $\epsilon \in (0, 1)$, there is a $c = c(\epsilon) > 0$ such that for every group G of order n, and for a set S of $c \log n$ random elements in the group, the expected value of the second largest eigenvalue of the Cayley graph $X(G, S)$ is at most $(1 - \epsilon)|S|$.*

The Abelian Cayley graphs show that this theorem is tight, since they are not expanders for $o(\log |G|)$ generators (we will prove this result in the next sections). The value of $c(\epsilon)$ was later improved by Landau and Russell in [**51**] and by Loh and Schulman in [**60**].

Note that $\lambda_2^l(X) = \max_{1 \perp x} \frac{x^t A^l(X) x}{x^t x}$ for each odd l. Also, it is easy to see that $\lambda^l(X) = \max_{1 \perp x} \frac{x^t A^l(X) x}{x^t x}$ for each even l. This equation can be used to obtain lower bounds on $\lambda(X)$.

For a lower bound on $\lambda_2(X)$, one can also use the following equation

$$(k + \lambda_2(X))^l = \max_{1 \perp x} \frac{x^t (kI + A)^l x}{x^t x} \tag{4.5.1}$$

For a vertex $u \in V(X)$, denote by χ_u its characteristic vector. Let u and v two vertices of X such that the distance from u to v is greater than $2r$. Then $\chi_u^t A^l \chi_v = 0$ for each $l \leq 2r$. Also, $\chi_u^t A^l \chi_u = t_l(u)$ and $\chi_v A^l \chi_v = t_l(v)$ for each l.

If $x = \chi_u - \chi_v$, then obviously $1 \perp x$. It follows from the previous paragraph that

$$x^t A^l x = t_l(u) + t_l(v)$$

for each $l \leq 2r$. By equation (4.5.1)

$$(k + \lambda_2(X))^{2r} \geq \frac{x^t (kI + A)^{2r} x}{x^t x}$$

Since $x^t x = 2$, it follows that

$$\begin{aligned} \frac{x^t (kI + A)^{2r} x}{x^t x} &= \frac{1}{2} \sum_{i=0}^{2r} \binom{2r}{i} k^i x^t A^{2r-i} x \\ &\geq \frac{1}{2} \sum_{j=0}^{r} \binom{2r}{2j} k^{2j} x^t A^{2r-2j} x \\ &\geq \frac{1}{2} \sum_{j=0}^{r} \binom{2r}{2j} k^{2j} (t_{2r-2j}(u) + t_{2r-2j}(v)). \end{aligned}$$

Hence, whenever the distance between u and v is greater than $2r$, we obtain

$$k + \lambda_2(X) \geq \sqrt[2r]{\frac{1}{2} \sum_{j=0}^{r} \binom{2r}{2j} k^{2j} (t_{2r-2j}(u) + t_{2r-2j}(v))} \tag{4.5.2}$$

We will use this inequality in the following sections to obtain lower bounds on $\lambda_2(X)$ when X is an Abelian Cayley graph.

4.6. Closed walks of even length in Abelian Cayley graphs

Let G be a finite Abelian group. There is a simple bijective correspondence between the closed walks of length r starting at a vertex u of a Cayley graph $X(G,S)$ and the r-tuples $(a_1,\ldots,a_r) \in S^r$ with $\prod_{i=1}^{r} a_i = 1$. To each closed walk $u = u_0, u_1, \ldots, u_{r-1}, u_r = u$, we associate the r-tuple $(u_0u_1^{-1}, u_1u_2^{-1}, \ldots, u_{r-2}u_{r-1}^{-1}, u_{r-1}u_r^{-1}) \in S^r$.

Suppose that

$$S = \{x_1, x_2, \ldots, x_s, x_1^{-1}, x_2^{-1}, \ldots, x_s^{-1}, y_1, \ldots, y_t\}, \tag{4.6.1}$$

where each x_i has order greater than 2 for $1 \le i \le s$ and each y_j has order 2 for $1 \le j \le t$. Let $W_{2r}(X)$ be the number of $2r$-tuples from S^{2r} in which the number of appearances of x_i is the same as the number of appearances of x_i^{-1} for all $1 \le i \le s$ and the number of appearances of y_j is even for all $1 \le j \le t$. More precisely, $W_{2r}(X)$ counts $2r$-tuples from S^{2r} in which p positions are occupied by x_i's, p positions are occupied by x_i^{-1}'s and the remaining $2r-2p$ positions are occupied by y_j's (each of them appearing an even number of times), where $0 \le p \le r$. These choices imply that the product of the $2r$ elements in this type of $2r$-tuple is 1.

Thus, $t_{2r}(u) \ge W_{2r}(X)$ for each $u \in V(X)$. This implies

$$\Phi_{2r}(X) = \sum_{u \in V(X)} t_{2r}(u) \ge nW_{2r}(X), \tag{4.6.2}$$

for each $r \ge 1$.

We evaluate $W_{2r}(X)$ by choosing first the $2p$ positions for the x_i's and their inverses. This can be done in $\binom{2r}{2p}$ ways. Then we choose p positions for the x_i's. This is done in $\binom{2p}{p}$ ways and the rest are left for x_i^{-1}'s. Since this happens for all $0 \leq p \leq r$, we get the following expression for $W_{2r}(X)$

$$(4.6.3) \qquad \sum_{p=0}^{r} \binom{2r}{2p}\binom{2p}{p} \sum_{i_1+\cdots+i_s=p} \binom{p}{i_1,\ldots,i_s}^2 \sum_{2j_1+\cdots+2j_t=2r-2p} \binom{2r-2p}{2j_1,\ldots,2j_t}$$

It remains to estimate the sums

$$c(p,s) = \sum_{i_1+\cdots+i_s=p} \binom{p}{i_1,\ldots,i_s}^2$$

and

$$d(r-p,t) = \sum_{2j_1+\cdots+2j_t=2r-2p} \binom{2r-2p}{2j_1,\ldots,2j_t}$$

Obtaining a closed formula for either of these two sums seems to be an interesting and difficult combinatorial problem in itself. In the next section, we will find lower bounds for these two expressions that will help us obtain lower bounds on the second largest eigenvalue of an Abelian Cayley graph.

4.7. Estimating two combinatorial sums

If $l = 2$, then we can find simple formulas for both $c(m,2)$ and $d(m,2)$. In this case, we have

$$c(m,2) = \sum_{i_1+i_2=m} \binom{m}{i_1,i_2} = \sum_{i=0}^{m} \binom{m}{i} = \binom{2m}{m}$$

and

$$d(m,2) = \sum_{2j_1+2j_2=2m} \binom{2m}{2j_1,2j_2} = \sum_{j=0}^{m} \binom{2m}{2j} = 2^{2m-1}$$

The following argument was suggested by David Wehlau. Using the multinomial formula to expand $(x_1+\cdots+x_l)^m(x_1^{-1}+\cdots+x_l^{-1})^m$, we notice that the constant term is $\sum_{i_1+\cdots+i_l=m} \binom{m}{i_1,\ldots,i_l}^2 = c(m,l)$. On the other hand, if $y = x_1 + \cdots + x_{l-1}$ and $z = x_1^{-1} + \cdots + x_{l-1}^{-1}$, then

$$(x_1+\cdots+x_{l-1}+x_l)^m(x_1^{-1}+\cdots+x_{l-1}^{-1}+x_l^{-1})^m = \left(\sum_{i=0}^{m}\binom{m}{i}y^i x_l^{m-i}\right)\left(\sum_{j=0}^{m}\binom{m}{j}z^j x_l^{j-m}\right).$$

It follows that the constant term in this expansion is $\sum_{j=0}^{m}\binom{m}{j}^2 c(j,l-1)$. Hence, we deduce the following recurrence relation

$$c(m,l) = \sum_{j=0}^{m}\binom{m}{j}^2 c(j,l-1). \tag{4.7.1}$$

I have not been able to find a nice formula for $c(m,l)$ from this recurrence relation though.

Let $\mathcal{S}_b^{(m)}$ be the elementary symmetric polynomial of degree b in m indeterminates: $\mathcal{S}_0^{(m)} = 1$ and for $1 \le b \le m$

$$\mathcal{S}_b^{(m)}(X_1, X_2, \ldots, X_m) = \sum_{1\le i_1<i_2<\cdots<i_b\le m} X_{i_1}X_{i_2}\ldots X_{i_b} \tag{4.7.2}$$

For $0 \le a \le m$, let $\mathcal{S}^{(m)}(a,b) = \mathcal{S}_b^{(m)}(-1,\ldots,-1,1,\ldots,1)$, where the number of -1's is exactly a. Note that $\mathcal{S}^{(m)}(0,b) = \binom{m}{b}$.

In [**29**], Fixman gave a formula for the sum of multinomial coefficients

$$C(l,q,a) = \sum \binom{q}{h_1\ldots h_l},$$

where the summation is over the nonnegative integers $h_1,\ldots,h_l$ satisfying

(1) $h_1 + \cdots + h_l = q$;

(2) $h_1, h_2, \ldots, h_a$ are odd; and

(3) $h_{a+1}, \ldots, h_n$ are even

in terms of the $\mathcal{S}^{(l)}(a,b)$'s. Fixman's result is the following formula.

THEOREM 4.7.1.

$$C(l,q,a) = 2^{-l}\sum_{j=0}^{l}(l-2j)^q \mathcal{S}^{(l)}(a,j)$$

Note that $C(l,2m,0) = d(m,l)$. Theorem 4.7.1 implies that

$$d(m,l) = 2^{-l}\sum_{j=0}^{l}\binom{l}{j}(l-2j)^{2m} \tag{4.7.3}$$

This implies $d(m,l) > 2^{1-l}l^{2m}$.

We use the Cauchy-Schwarz inequality to obtain a lower bound on $c(m,l)$.

$$\begin{aligned} c(m,l) = \sum_{i_1+\cdots+i_l=m}\binom{m}{i_1,\ldots,i_l}^2 &\geq \frac{\left(\sum_{i_1+\cdots+i_l=m}\binom{m}{i_1,\ldots,i_l}\right)^2}{\binom{m+l-1}{l-1}} \\ &= \frac{l^{2m}}{\binom{m+l-1}{l-1}}. \end{aligned}$$

Hence, we have proved that

$$c(m,l) > \frac{l^{2m}}{\binom{m+l-1}{l-1}}\ ,\ d(m,l) > \frac{l^{2m}}{2^{l-1}} \tag{4.7.4}$$

4.8. A Serre-type theorem for Abelian Cayley graphs

The following result states that the constant-degree Abelian Cayley graphs have many large non-trivial eigenvalues. This theorem can be also regarded as an analogue of Serre's Theorem 2.2.1 for Abelian Cayley graphs.

THEOREM 4.8.1. *Given $k \geq 3$, for each $\epsilon > 0$, there exists a positive constant $C = C(\epsilon,k)$ such that for any Abelian group G and for any symmetric set S of elements of G with $|S| = k$ and $1 \notin S$, the number of the eigenvalues λ_i of the Cayley graph $X = X(G,S)$ such that $\lambda_i \geq k-\epsilon$ is at least $C\cdot|G|$.*

PROOF. Let $\epsilon > 0$. Consider an Abelian group G and S a subset of G of size k. Denote by n the order of G and by m the number of eigenvalues λ_i of $X = X(G, S)$ such that $\lambda_i \geq k - \epsilon$. Then there are exactly $n - m$ eigenvalues of X that are less than $k - \epsilon$. It follows that

$$\operatorname{tr}(kI + A)^{2l} = \sum_{i=1}^{n} (k + \lambda_i)^{2l} < (n - m)(2k - \epsilon)^{2l} + m(2k)^{2l}, \tag{4.8.1}$$

for each $l \geq 1$. Recall now that X is a (s, t)-Cayley graph (see (4.6.1)). Using inequalities (4.7.4) in (4.6.3), for each $r \geq 1$ we get

$$\begin{aligned}
W_{2r}(X) &= \sum_{p=0}^{r} \binom{2r}{2p} \cdot \binom{2p}{p} \cdot \sum_{i_1+\cdots+i_s=p} \binom{p}{i_1, \ldots, i_s}^2 \cdot \sum_{2j_1+\cdots+2j_t=2r-2p} \binom{2r-2p}{2j_1, \ldots, 2j_t} \\
&= \sum_{p=0}^{r} \binom{2r}{2p} \cdot \binom{2p}{p} \cdot c(p, s) \cdot d(r - p, t) \\
&> \sum_{p=0}^{r} \binom{2r}{2p} \cdot \frac{2^{2p}}{2p+1} \cdot \frac{s^{2p}}{\binom{s+p-1}{s-1}} \cdot \frac{t^{2r-2p}}{2^{t-1}} \\
&> \frac{1}{2^{t-1}(2r+1)\binom{s+r-1}{r-1}} \sum_{p=0}^{r} \binom{2r}{2p} (2s)^{2p} t^{2r-2p} \\
&= \frac{(2s+t)^{2r} + (2s-t)^{2r}}{2^t(2r+1)\binom{s+r-1}{r-1}} \\
&> \frac{k^{2r}}{2^t(2r+1)\binom{s+r-1}{s-1}}
\end{aligned}$$

Using the previous inequality and (4.6.2), we obtain the following

$$
\begin{aligned}
\operatorname{tr}(kI+A)^{2l} &= \sum_{i=0}^{2l} \binom{2l}{i} k^i \Phi_{2l-i}(X) \\
&\geq \sum_{j=0}^{l} \binom{2l}{2j} k^{2j} \Phi_{2l-2j}(X) \\
&\geq \sum_{j=0}^{l} \binom{2l}{2j} k^{2j} n W_{2(l-j)}(X) \\
&> \sum_{j=0}^{l} \binom{2l}{2j} k^{2j} \frac{k^{2l-2j}}{2^t(2(l-j)+1)\binom{s+l-j-1}{s-1}} \\
&> \frac{k^{2l}}{2^t(2l+1)\binom{s+l-1}{s-1}} \sum_{j=0}^{l} \binom{2l}{2j} \\
&= \frac{(2k)^{2l}}{2^{t+1}(2l+1)\binom{s+l-1}{s-1}}
\end{aligned}
$$

for each $l \geq 1$. Combining this inequality with (4.8.1), it follows that

$$\frac{m}{n} > \frac{\frac{1}{2^{t+1}(2l+1)\binom{s+l-1}{s-1}}(2k)^{2l} - (2k-\epsilon)^{2l}}{(2k)^{2l} - (2k-\epsilon)^{2l}}, \tag{4.8.2}$$

for each $l \geq 1$. Now

$$\lim_{l\to\infty} \sqrt[2l]{\frac{1}{2^{t+1}(2l+1)\binom{s+l-1}{s-1}}(2k)^{2l}} = 2k$$

and

$$\lim_{l\to\infty} \sqrt[2l]{2(2k-\epsilon)^{2l}} = 2k-\epsilon.$$

This implies that there exists $l_0 = l(\epsilon, k)$ such that

$$\frac{1}{2^{t+1}(2r+1)\binom{s+r-1}{s-1}}(2k)^{2l} - (2k-\epsilon)^{2l} > (2k-\epsilon)^{2l}, \tag{4.8.3}$$

for each $l \geq l_0$. Letting

$$C(\epsilon, k) = \frac{(2k-\epsilon)^{2l_0}}{(2k)^{2l_0} - (2k-\epsilon)^{2l_0}} > 0$$

it follows that

$$\frac{m}{n} > C(\epsilon, k)$$

□

The fact that the Abelian Cayley graphs have large numbers of closed walks can be also used to prove the following theorem. This is weaker than the result of Friedman, Murty and Tillich [**33**].

THEOREM 4.8.2. *Let X be a k-regular Cayley graph with n vertices on an Abelian group G. Then*

$$\lambda_2(X) \geq k\left(1 - \mathrm{O}\left(\frac{\log n}{n^{\frac{1}{k}}}\right)\right) \tag{4.8.4}$$

as n tends to infinity.

PROOF. If $D = \mathrm{diam}(X)$, then every element of X is a product of the form $z_1^{a_1} \cdot z_2^{a_2} \ldots z_k^{a_k}$, where $S = \{z_1, z_2, \ldots, z_k\}$, each a_i is a nonnegative integer and $\sum_{i=1}^{k} a_i \leq D$. The total number of products is at most $\binom{D+k}{k}$. Hence,

$$n \leq \binom{D+k}{k} < (D+1)^k$$

This implies that

$$\mathrm{diam}(X) > n^{\frac{1}{k}} - 1 \tag{4.8.5}$$

From the proof of Theorem 4.8.1, we know that

$$W_{2r}(X) > \frac{k^{2r}}{2^t(2r+1)\binom{s+r-1}{s-1}},$$

for each $r \geq 1$.

Let u and v two vertices in X such that $d(u,v) = D$ and let $r = \lfloor \frac{D}{2} \rfloor$. Using (4.5.2), we obtain

$$\begin{aligned} k + \lambda_2 &\geq \sqrt[2r]{\frac{1}{2}\sum_{j=0}^{r}\binom{2r}{2j}k^{2j}(t^{2r-2j}(u) + t_{2r-2j}(v))} \\ &\geq \sqrt[2r]{\sum_{j=0}^{r}\binom{2r}{2j}k^{2j}W_{2r-2j}(X)} \\ &> \sqrt[2r]{\sum_{j=0}^{r}\binom{2r}{2j}k^{2j}\frac{k^{2r-2j}}{2^t(2r-2j+1)\binom{s+r-j+1}{s-1}}} \\ &> \sqrt[2r]{\frac{(2k)^{2r}}{2^t(2r+1)\binom{s+r-1}{s-1}}} = 2k \cdot \left(2^t(2r+1)\binom{s+r-1}{s-1}\right)^{-\frac{1}{2r}}. \end{aligned}$$

Notice that $\binom{s+r-1}{s-1} > \frac{r^{s-1}}{(s-1)!}$. Then

$$\left(2^t(2r+1)\binom{s+r-1}{s-1}\right)^{-\frac{1}{2r}} > \left(\frac{(s-1)!}{2^t}\right)^{\frac{1}{2r}} [(2r+1)r^{s-1}]^{-\frac{1}{2r}}$$

Now for r large we have

$$\left(\frac{(s-1)!}{2^t}\right)^{\frac{1}{2r}} = e^{\frac{\ln(s-1)!-t\ln 2}{2r}} \sim 1 + \frac{\ln(s-1)! - t\ln 2}{2r}$$

and

$$[(2r+1)r^{s-1}]^{-\frac{1}{2r}} = e^{-\frac{\ln[(2r+1)r^{s-1}]}{2r}} \sim 1 - \frac{\ln[(2r+1)r^{s-1}]}{2r}$$

Since $r = \lfloor \frac{D}{2} \rfloor$, we obtain

$$\lambda_2(X) \geq k\left(1 - \mathrm{O}\left(\frac{\log D}{D}\right)\right)$$

as n tends to infinity. Using (4.8.5), this proves the theorem. □

This shows that the result of Alon and Roichman is best possible. Abelian Cayley graphs with $\mathrm{o}(\log n)$ generators will have large nontrivial eigenvalues and hence, they won't be expanders.

4.9. A short proof

We now present a very short proof of the fact that

$$\lambda(X) \geq \frac{k}{2e} - k\left(\frac{n-2}{2}\right)^{-\frac{1}{k}}$$

for any k-regular Abelian Cayley graph X on n vertices. This is a weaker statement than the previous lower bound on λ_2 for Abelian Cayley graphs, but the result still shows that constant-degree Abelian Cayley graphs have large nontrivial eigenvalues and the proof is very simple.

Suppose S contains no elements of order 2. Thus, k is even and $S = \{x_1, x_1^{-1}, x_2, x_2^{-1}, \ldots, x_{\frac{k}{2}}, x_{\frac{k}{2}}^{-1}\}$. There are at least $r!\frac{k}{2}(\frac{k}{2}-1)\ldots(\frac{k}{2}-r+1)$ closed walks of length $2r$ in X for each $1 \leq r \leq \frac{k}{2}$. Place r of the x_i's on the first r positions and permute their inverses on the last r positions. Using the inequality (4.6.2), we get $\Phi_{2r}(X) > nr!\frac{k}{2}(\frac{k}{2}-1)\ldots(\frac{k}{2}-r+1)$ for each $1 \leq r \leq \frac{k}{2}$. Since $(n-1)\lambda^{2r} + k^{2r} > \Phi_{2r}(X)$, it follows that

$$\lambda \geq \left(\binom{\frac{k}{2}}{r}(r!)^2\right)^{\frac{1}{2r}} - k\left(\frac{n-1}{2}\right)^{-\frac{1}{2r}}$$

for all $1 \leq r \leq \frac{k}{2}$. Choose $r = \frac{k}{2}$. Then

$$\begin{aligned}\lambda(X) &\geq \left(\left(\frac{k}{2}\right)!\right)^{\frac{2}{k}} - k\left(\frac{n-1}{2}\right)^{-\frac{1}{k}} \\ &> \frac{k}{2e} - k\left(\frac{n-1}{2}\right)^{-\frac{1}{k}}.\end{aligned}$$

When S contains some elements of order 2, the proof is similar.

CHAPTER 5

Gaps between primes and new expanders

5.1. Perfect matchings and eigenvalues

A *matching* P in a graph X is a set of mutually disjoint edges. The vertices incident to the edges in P are *saturated* by P. We call P a *perfect matching* if all the vertices of G are saturated by P, i.e. P is a 1-regular graph with vertex set $V(X)$. A *factor* of X is a spanning subgraph of X. A *t-factor* is a spanning t-regular subgraph. Thus, a perfect matching is the same as a 1-factor.

For each $S \subseteq V(X)$, denote by $N(S)$ the set of vertices adjacent to at least one vertex in S. The following theorem is well-known (see West [**93**], page 110).

THEOREM 5.1.1 (P.Hall, 1935). *A bipartite graph X with partite sets A and B has a matching that saturates A if and only if $|N(S)| \geq |S|$, for each $S \subseteq A$.*

The following corollary to Hall's theorem is also known as the Marriage Theorem. It was originally proved by Frobenius in 1917.

COROLLARY 5.1.2. *For $k > 0$, every k-regular bipartite graph has a perfect matching.*

For the existence of perfect matchings in general (not necessarily bipartite) graphs, the following characterization was found by Tutte in 1947. An *odd component* of a graph is a component of odd order. The number of odd components of a graph G will be denoted by $\mathrm{odd}(G)$.

THEOREM 5.1.3 (Tutte, 1947). *A graph G contains a perfect matching if and only if*

$$\mathrm{odd}(G \setminus S) \leq |S|,$$

for each $S \subseteq V(G)$.

In [**12**], Brouwer and Haemers used Tutte's theorem to prove the following eigenvalue condition that is sufficient for the existence of a perfect matching in a regular graph.

THEOREM 5.1.4 (Brouwer-Haemers [**12**]). *If G is a connected, k-regular graph on n vertices (n even) and*

$$\lambda_3(G) \leq \begin{cases} k - 1 + \frac{3}{k+1}, & \text{for } k \text{ even} \\ k - 1 + \frac{3}{k+2}, & \text{for } k \text{ odd,} \end{cases}$$

then G has a perfect matching.

PROOF. The proof is by contradiction. Assume that G satisfies the inequality stated above and G has no perfect matching. Using Tutte's theorem, it follows that there exists a set S of s vertices such that $G \setminus S$ has $q > s$ odd components. Since n is even, we deduce that $q + s$ must be even. Thus, $q \geq s + 2$. Let $G_1, G_2, \ldots, G_q$ be the odd components of $G \setminus S$. Denote by n_i the order of G_i and by t_i the number of edges between G_i and S for each i with $1 \leq i \leq q$. Then $\sum_{i=1}^{q} t_i \leq e(S, G \setminus S) \leq k|S| = ks$. Because G is connected, it follows that $t_i \geq 1$ for each i with $1 \leq i \leq q$. Combining this with $\sum\limits_{i=1}^{q} t_i < ks$ and $q \geq s + 2$, it follows that for at least three values of i, say $1, 2$ and 3, $t_i < k$ and $n_i > 1$. Denote by ν_i the largest eigenvalue of G_i and suppose $\nu_1 \geq \nu_2 \geq \nu_3$. The union of G_1, G_2 and G_3 is an induced subgraph of G and by interlacing of eigenvalues, we obtain that $\lambda_i \geq \nu_i$ for $i = 1, 2, 3$.

Since $t_3 = 2e_3 - kn_3$, the average degree d_3 of G_3 equals $\frac{2e_3}{n_3} = k - \frac{t_3}{n_3}$. Since $n_3(n_3-1) \geq 2e_3 = kn_3 - t_3 > kn_3 - k$, it follows that $n_3 > k$. If k is even, then t_3 is even. Because $t_3 < k$, it follows that $t_3 \leq k-2$. If k is odd, then $k < n_3$ implies $k \leq n_3 - 2$ (n_3 is also odd).

We deduce that

$$d_3 \geq \begin{cases} k - \frac{k-2}{k+1}, & \text{for } k \text{ even,} \\ k - \frac{k-1}{k+2}, & \text{for } k \text{ odd.} \end{cases}$$

From $\lambda_3 \geq \nu_3 > d_3$ and the previous inequality , we obtain a contradiction. The second inequality is true since G_3 is not regular ($kn_3 > 2e_3 = kn_3 - t_3 > (k-1)n_3$). This completes the proof of the theorem. □

In the same paper [**12**], Brouwer and Haemers construct examples of k-regular graphs with no perfect matchings and having $\lambda_3 = k - 1 + \frac{3}{k+1} + \mathrm{O}(k^{-2})$, for any $k \geq 3$. We should note here that the converse of Brouwer and Haemers theorem is not true, i.e., the existence of a perfect matching in G does not imply $k - \lambda_3(G) > \epsilon > 0$. The Cayley graph of $\mathbb{Z}_{2n}$ with generating set $S = \{\pm 1, n\}$ is a 3-regular graph that contains 3 disjoint perfect matchings and satisfies $\lambda_3(G) = 2\cos\frac{4\pi}{n} + 1$. The difference $k - \lambda_3 = 2 - 2\cos\frac{4\pi}{n}$ tends to 0 as n gets large.

To a permutation $\sigma \in S_n$, we can associate an n by n matrix P_σ defined as follows:

$$P_\sigma(i,j) = \begin{cases} 1 & \text{if } \sigma(i) = j \\ 0 & \text{otherwise} \end{cases}$$

An n by n matrix P is called a *permutation matrix* if there is $\sigma \in S_n$ such that $P = P_\sigma$. The adjacency matrix of a perfect matching on $2n$ vertices is a permutation matrix. Its eigenvalues are 1 and -1, each with multiplicity n. Notice that the vertex set may be indexed so that $\mathcal{E}(A(P), 1) = \left\{ \begin{pmatrix} x \\ x \end{pmatrix} : x \in \mathbb{R}^n \right\}$ and $\mathcal{E}(A(P), -1) = \left\{ \begin{pmatrix} -x \\ x \end{pmatrix} : x \in \mathbb{R}^n \right\}$

If A is the adjacency matrix of a k-regular graph G and B is the adjacency matrix of a perfect matching P on $V(G)$, then from Theorem 1.6.3 we obtain that

$$|\lambda_i(G \cup P) - \lambda_i(G)| \leq 1, \tag{5.1.1}$$

for each integer i with $1 \leq i \leq n$. Note that $G \cup P$ might have multiple edges. Using (5.1.1), we immediately obtain the following lemma.

Lemma 5.1.5. *Let G be a k-regular graph on n vertices (assume n even) and P be a perfect matching on $V(G)$ such that $E(G) \cap E(P) = \emptyset$. If G is an (n, k, λ)-graph, then $G \cup P$ is an $(n, k+1, \lambda+1)$-graph.*

Of course, if we extend the definition of (n, k, λ)-graphs to (n, k, λ)-multigraphs, then the previous theorem is true without the assumption that $E(P) \cap E(G) = \emptyset$.

Using Theorem 1.6.3 and Theorem 5.1.4 we can prove the following lemma.

Lemma 5.1.6. *Let G be an (n, k, λ)-graph such that n is even and $k - \lambda > 2$. Then G contains at least one perfect matching and for each perfect matching P of G, $G \setminus P$ is an $(n, k-1, \lambda+1)$-graph.*

Proof. If $k - \lambda > 2$, then $k - \lambda_3(G) > 2$. By Theorem 5.1.4, we deduce that G has a perfect matching P. Then by Theorem 1.6.3, we obtain

$$|\lambda_i(G \setminus P) - \lambda_i(G)| \leq 1,$$

for each $i \neq 1$. It follows that $\lambda(G \setminus P) \leq \lambda(G) + 1$. Since $\lambda(G) < k - 2$, the previous inequality implies that $\lambda(G \setminus P) < k - 1$. Since $G \setminus P$ is $(k-1)$-regular and $\lambda(G \setminus P) < k-1$, we deduce that $G \setminus P$ is connected. Hence, $G \setminus P$ is an $(n, k-1, \lambda+1)$-graph. □

Again, notice that this theorem is true if (n, k, λ)-graph is replaced by (n, k, λ)-multigraph throughout.

LEMMA 5.1.7. *If G is an (n, k, λ)-graph and $n > k + \lambda$, then its complement $\overline{G}$ is an $(n, n-k-1, \lambda+1)$-graph.*

PROOF. If $P_G(x) = \det(xI - A(G))$ is the characteristic polynomial of G, then it follows (see Biggs [**8**], page 20) that

$$(x + k + 1)P_{\overline{G}}(x) = (-1)^n(x - n + k + 1)P_G(-x - 1)$$

Thus, if $k = \lambda_1 > \lambda_2 \geq \cdots \geq \lambda_n$ are the eigenvalues of G, then the eigenvalues of $\overline{G}$ are $n-k-1, -1-\lambda_2, \ldots, -1-\lambda_n$. Since $n-k > \lambda$, it follows that $n-k-1 > -1-\lambda_i$, for each i with $2 \leq i \leq n$. Hence, $\overline{G}$ is $(n-k-1)$-regular and the multiplicity of the eigenvalue $n-k-1$ is 1. This implies that $\overline{G}$ is a $(n, n-k-1, \lambda+1)$-graph. □

The following result is an easy consequence of the previous results and of Theorem 5.1.4.

COROLLARY 5.1.8. *If n is even, G is an (n, k, λ)-graph and $n \geq k + \lambda + 3$, then the complement of G contains at least one perfect matching.*

5.2. Gaps between primes

Denote by p_m the m-th largest prime number and let $\Delta_m = p_{m+1} - p_m$. Let $\pi(x)$ be the number of primes that are less than x. The Prime Number Theorem (see Ribenboim [**83**] for example) states that $\pi(x) \sim \frac{x}{\log x}$ or equivalently, $p_m \sim m \log m$, as $m \to \infty$. This implies that

$$\frac{\Delta_1 + \Delta_2 + \cdots + \Delta_m}{m} = \frac{p_{m+1} - 2}{m} \sim \log m \sim \log p_m,$$

as $m \to \infty$. Thus, the average order of the difference $p_{m+1} - p_m$ is $\log p_m$.

The study of gaps between primes has a large history. Recently, a major advance was made in the theory by Goldston, Pintz and Yildirim [**38**]. If p_n denotes the n-th prime, they proved that

$$\liminf_{n\to\infty} \frac{p_{n+1} - p_n}{\log p_n} = 0 \tag{5.2.1}$$

Crámer proved in [**19**] (see also [**18**]) the following result concerning the maximum order of Δ_m.

Theorem 5.2.1. *If the Riemann hypothesis is true, then there is a positive constant c such that*

$$\pi(x + c\sqrt{x}\log x) - \pi(x) > \sqrt{x},$$

for each $x \geq 2$. Thus,

$$\Delta_m = \mathrm{O}(\sqrt{p_m}\log p_m).$$

as $m \to +\infty$.

Based on probability arguments, Crámer conjectured in 1937 that $\Delta_m = \mathrm{O}((\log p_m)^2)$.

In 1943, Selberg proved the following.

Theorem 5.2.2. *Let $\Phi(x)$ be a positive and increasing function such that $\frac{\Phi(x)}{x}$ is decreasing for $x > 0$. Assume that $\frac{\Phi(x)}{x} \to 0$ and $\frac{\Phi(x)}{\log x} \to \infty$ as $x \to \infty$. Then assuming the Riemann hypothesis is true, we have for almost all $x > 0$*

$$\pi(x + \Phi(x)) - \pi(x) \sim \frac{\Phi(x)}{\log x}$$

This implies that for almost all $x > 0$, there is a prime between $[x - k(x)\log^2 x, x]$, where $k(x)$ is a function tending arbitrarily slowly to infinity.

For our purposes, it suffices to show that, given $\epsilon > 0$, then almost always $\Delta_m \leq \epsilon\sqrt{p_m}$. The following argument was suggested by Ram Murty. Let $B(x)$ denote the set of primes $p_m \leq x$ such that the interval $(p_m, p_m + \epsilon\sqrt{p_m})$ contains no primes and let $b(x) = |B(x)|$. Consider the following function

$$S(x) = \sum_{p_{m+1} \leq x} \Delta_m$$

Obviously,

$$S(x) = \sum_{p_{m+1} \leq x} (p_{m+1} - p_m) = p_{n+1} - 2 < Ax \log x,$$

where p_{n+1} is the largest prime less than or equal to x and A is some positive constant. On the other hand,

$$\begin{aligned} S(x) &\geq \sum_{\substack{\frac{x}{2} \leq p_m < p_{m+1} \leq x \\ p_m \in B(x)}} \Delta_m \\ &\geq \epsilon \sum_{\substack{\frac{x}{2} \leq p_m < p_{m+1} \leq x \\ p_m \in B(x)}} \sqrt{p_m} \\ &> \epsilon \left(b(x) - b\left(\frac{x}{2}\right)\right) \sqrt{C\frac{x}{2}\log\frac{x}{2}} \\ &> \epsilon \left(b(x) - b\left(\frac{x}{2}\right)\right) C'\sqrt{x \log x}, \end{aligned}$$

where C and C' are some positive constants. From the previous two relations, we obtain that for each positive x

$$b(x) - b\left(\frac{x}{2}\right) \leq A_\epsilon \sqrt{x \log x},$$

where A_ϵ is some positive constant. By iteration, this implies

$$b(x) \leq A'_\epsilon \sqrt{x} \log^{\frac{3}{2}} x. \tag{5.2.2}$$

Hence, for each positive x

$$\frac{b(x)}{\pi(x)} \le D_\epsilon \frac{\log^{\frac{5}{2}} x}{\sqrt{x}}, \tag{5.2.3}$$

where D_ϵ is a positive constant.

This inequality states that $\Delta_m \le \epsilon\sqrt{p_m}$ for almost all m.

Using more complicated arguments, Crámer proved the following stronger result in [**20**].

THEOREM 5.2.3 (Crámer [**20**]). *Let $h(x)$ be the number of primes $p_n \le x$ such that $p_{n+1} - p_n > p_n^k$, where $k \in (0, \frac{1}{2}]$ is a constant. Then*

$$h(x) = \mathrm{O}(x^{1-\frac{3}{2}k+\epsilon}),$$

for each $\epsilon > 0$.

Hence, $b(x) = O(x^{\frac{1}{4}+\epsilon})$ for each $\epsilon > 0$ which is better than the bound (5.2.2).

In 2001, Baker, Harman and Pintz [**6**] obtained the best unconditional result on the maximum value of Δ_m. In [**6**], they proved that

$$\Delta_m \le p_m^{0.525}, \tag{5.2.4}$$

if p_m is large enough. Ribenboim [**83**] states that the hope is to prove unconditionally that $\Delta_m = \mathrm{O}(p_m^{\frac{1}{2}+\epsilon})$ for each $\epsilon > 0$.

5.3. New expanders from old

Given $k \ge 3$, it is of great interest to construct infinite families of (n, k, λ)-graphs with $\lambda < k$ as small as possible. These graphs are called expanders. By the Alon and Boppana theorem [**1**], it is easy to see $\lambda = 2\sqrt{k-1}$ is best possible.

Using standard probabilistic arguments, one can prove the existence of infinite families of k-regular expanders. This was done by Pinsker for $k = 3$ in [**80**] and it is a folklore (and messy) result for $k \geq 4$.

The first explicit construction of an infinite family of expanders was given by Margulis in [**66**]. In 2002, Reingold, Vadhan and Widgerson [**85**] constructed expanders by using the zig-zag product and the replacement product. They obtain infinite families of (n, k, λ)-graphs with $\lambda = \mathrm{O}(k^{\frac{2}{3}})$ as k tends to infinity. Bilu and Linial [**9**] constructed expanders using random lifts. They construct infinite families of (n, k, λ)-graphs with $\lambda = \mathrm{O}(\sqrt{k}\log^{\frac{3}{2}} k)$. For an account of known expander constructions, see [**85**].

In this section, we would like to address the question of constructing Ramanujan graphs when $k - 1$ is not a prime power. In this context, no explicit constructions of infinite families is known, though there is the important non-constructive work of Friedman [**32**].

Our goal is to begin with the infinite families of Ramanujan graphs described above and perturb them in an explicit way to obtain what we call almost Ramanujan graphs. Thus, when $k - 1$ is not a prime power, the question of how close it is to a prime power becomes important. It turns out that when gaps between consecutive primes are small, the almost Ramanujan graphs are easily constructed.

The best expanders are the Ramanujan graphs. Infinite families of k-regular Ramanujan graphs have been constructed (see Chapter 2) when $k - 1$ is a power of a prime (see [**65, 67, 74**]). We will show how one can use these graphs to construct families of d-regular graphs that have a large spectral gap (and good expanding properties) when $d - 1$ is not a power of a prime.

Recently, Friedman [**32**] proved that for any integer $k \geq 3$ and any $\epsilon > 0$, the probability that a random k-regular graph G satisfies $\lambda(G) \leq 2\sqrt{k-1} + \epsilon$ tends to 1 as the number of vertices of G tends to infinity. Roughly speaking, this means that *almost* all k-regular graphs are *almost Ramanujan.*

Our idea to construct expanders is to slightly modify known explicit expanders by adding (removing) perfect matchings to (from) their edge set. This simple idea was pursued in a slightly different direction by Bollobás and Chung in [**10**]. We describe their approach in the next three paragraphs.

As seen in Chapter 1, a very important problem in graph theory with connections to network optimization is constructing k-regular graphs on n vertices with small diameter. The random k-regular graph has diameter $\log_{k-1} n + \mathrm{o}(\log_{k-1} n)$ which is very close to the optimum value (see [**11**]). In our search for k-regular graphs with small diameter, we look first at the k-regular graphs G with small $\lambda(G)$. This is because of the results connecting the diameter and the eigenvalues of a k-regular graph (see Chapter 1).

The best possible upper bound on the diameter of a k-regular graph that these theorems can provide, is $2\log_{k-1} n + \mathrm{O}(1)$. This follows from the Alon-Boppana theorem $\lambda(G) \geq 2\sqrt{k-1} + \mathrm{o}(1)$ (see Theorem 2.1.2).

Bollobás and Chung proved that the diameter of a k-regular expander on n vertices plus a random perfect matching is almost surely less than $\log_{k-1} n + \log_{k-1}\log(n) + \mathrm{O}(1)$ as n goes to infinity. This shows that we can achieve the best possible asymptotic diameter by a very small random perturbation of an explicit expander graph. The Ramanujan graphs have diameter less than $2\log_{k-1} n + O(1)$. The result of Bollobás and Chung implies that the Ramanujan graphs plus a random perfect matching have diameter $\log_{k-1} n + \mathrm{o}(\log_{k-1} n)$ almost surely as $n \to +\infty$.

By repeatedly applying Lemma 5.1.5, the following result is immediate.

THEOREM 5.3.1. *Let $d > k \geq 2$ be two integers. Let X be a k-regular graph on n vertices (assume n even) and $P_1, \ldots, P_{d-k}$ be a family of perfect matchings on $V(X)$ such that $E(X) \cap \left(\cap_{i=1}^{d-k} E(P_i)\right) = \emptyset$. If X is an (n, k, λ)-graph, then $X \cup \left(\cup_{i=1}^{d-k} P_i\right)$ is an $(n, d, d-k+\lambda)$-graph.*

Let $d > k \geq 3$ be two integers. Suppose we can construct an explicit family (X_n) of $(|V(X_n)|, k, \lambda)$-graphs, with $\lambda < k$ and $|V(X_n)|$ even for each n. Add $d-k$ perfect matchings to X_n for each n. By the previous theorem, we obtain a family (X'_n) of $(n, d, \lambda + d - k)$-graphs. The spectral gap of the new family (X'_n) is at least $d - (\lambda + d - k) = k - \lambda$ which is the spectral gap of the X_n's. Thus, if there is an absolute constant c such that $k < d < k + c\lambda$, then X'_n is an $(|V(X'_n)|, d, (c+1)\lambda)$-graph for each n.

The intuition behind this procedure of adding perfect matchings to expander graphs is that the expansion properties of the new graphs will inherit the good expansion properties of the old graphs. This idea also appears in the work of Mohar [**71**]. If we require that the new graphs have no repeated edges, then when adding perfect matchings, we need to make sure the edge set of the perfect matching and the edge set of our current graph are disjoint. This can be done easily by applying Lemma 5.1.7 and its corollary.

It seems very natural to apply this procedure of adding perfect matchings to the families of best possible expanders, namely the Ramanujan graphs.

If X is a k-regular Ramanujan graph, then $\lambda(X) \leq 2\sqrt{k-1}$ and by Lemma 5.1.5, we obtain $\lambda(X \cup P) \leq 2\sqrt{k-1} + 1$. This observation was made by De la Harpe and Musitelli in [**42**] where they construct 7-regular graphs with spectral graph

$7-\lambda_2 \geq 6-2\sqrt{5}=1.52$. This falls short of the desired spectral gap for 7-regular Ramanujan graphs which is $7-2\sqrt{6}=2.10$.

Let $d \geq 3$ be an integer such that $d-1$ is not a prime power. Let $m \geq 1$ be the integer such that $p_m < d-1 < p_{m+1}$. The results of Lubotzky, Phillips and Sarnak [**65**] provide us with an infinite family of graphs $(X_{p_m,n})$ such that $X_{p_m,n}$ is a $(|V(X_{p_m,n})|, p_m+1, 2\sqrt{p_m})$-graph for each n. Each $X_{p_m,n}$ has $n(n^2-1)$ or $\frac{n(n^2-1)}{2}$ vertices, depending on whether or not n is a square in $\mathbb{F}_{p_m}$.

By adding any $d-p_m-1$ perfect matchings to each $X_{p_m,n}$, we obtain a new family of (multi)graphs (X'_n). Using the results in the previous section, we obtain the following theorem.

THEOREM 5.3.2. *If $d-1$ is not prime, then X'_n is a $(|V(X_{p_m,n})|, d, p_{m+1}-p_m-1+2\sqrt{p_m})$-graph.*

Assuming the Riemann hypothesis, it follows from Theorem 5.2.1, that $2\sqrt{p_m}+p_{m+1}-p_m = \mathrm{O}(\sqrt{p_m}\log p_m)$. This implies the following theorem.

THEOREM 5.3.3. *If $d-1$ is not prime and the Riemann hypothesis is true, then X'_n is a $(|V(X_{p_m,n})|, d, \lambda)$-graph with $\lambda = \mathrm{O}(\sqrt{d}\log d)$.*

Note that the X'_n's are actually multigraphs, they do not contain loops, but might have multiple edges. If we want to make sure that the X_n's are simple, then the perfect matching added at each step, must be chosen from the complement of the current graph.

In general, if X is a k-regular graph, then we can find perfect matchings in its complement using the following procedure. If X is a k-regular bipartite graph with partite sets A and B of equal size, then consider the bipartite graph X^c with partite sets A and B with $xy \in E(X^c)$ if and only if $xy \notin E(G)$. Then X^c is $(n-k)$-regular. By Hall's theorem, it follows that X^c contains a matching P that saturates A, i.e., a perfect matching. Actually, X^c contains $(n-k)!$ perfect matchings. This implies that $X \cup P$ is a bipartite $(k+1)$-regular graph with partite sets A and B. The best known algorithm for finding such a matching P in X^c is due to Rizzi and it has complexity $O(n(\log n)^2)$ (see [**84**]).

If we start with a non-bipartite k-regular graph X, then we can use Lemma 5.1.7 to check whether or not we can find a perfect matching in the complement of G. The best known algorithm for finding a maximum matching in a k-regular graph on n vertices is due to Micali and Vazirani and has complexity $O(kn^{\frac{3}{2}})$ (see [**77**]).

By removing perfect matchings from Ramanujan graphs, we can also obtain new families of graphs with small non-trivial eigenvalues. We suspect these families have worse expanding properties than the ones obtained by adding perfect matchings to Ramanujan graphs. Let d be a nonnegative integer with $d-1$ not a power of a prime. Let p_{m+1} be the smallest prime that is larger than $d-1$. We can construct an infinite family of d-regular graphs Y_n' with large spectral gap by removing perfect matchings from the LPS graphs $(X_{p_{m+1},n})$ (see [**65**]) that are $(p_{m+1}+1)$-regular and have an even number of vertices $n(n^2-1)$ or $\frac{n(n^2-1)}{2}$.

THEOREM 5.3.4. *If* $d-1$ *is not prime, then* Y_n' *is a* $(|V(X_{p_m,n})|, d, p_{m+1}-p_m+1+2\sqrt{p_{m+1}})$*-graph.*

Unconditionally, if d is sufficiently large, then we know from the previous section that $p_{m+1}-d < d^{0.525}$.

Hence, we can construct infinite families of $(f(n), d, \lambda)$-graphs, such that $f(n) \to \infty$ as $n \to \infty$ and

- $\lambda = O(d^{0.525})$ if d is large enough and $d-1$ not a prime power.
- $\lambda = O(\sqrt{d}\log d)$ if we assume the Riemann hypothesis and $d-1$ is not a prime power.

In a previous section, we proved that, given $\epsilon > 0$, then for almost all primes p_m, we have $\Delta_m \leq \epsilon\sqrt{p_m}$. Together with the previous arguments, this implies the next result.

THEOREM 5.3.5. *Let $\epsilon > 0$. Then for almost all d, one can explicitly construct infinite families of $(n, d, (2+\epsilon)\sqrt{d})$-graphs.*

CHAPTER 6

Some Open Problems

In this thesis, we study the eigenvalues of regular graphs and their connections with expansion and gaps between the primes.

There are some problems that arise as natural continuations of the results discussed in this thesis. We state them in this section and we hope to study them in the future.

- How small is λ_n for a k-regular claw free graph on n vertices ? More precisely, we conjecture there is a non-vanishing gap between λ_n and $-k$ for k-regular claw free graphs.
- The chromatic number of the Euclidean graph is between 4 and 7. It would be interesting to determine the order of magnitude of the chromatic and the independence numbers of the analogues of the Euclidean graphs over finite fields and rings.
- We have seen that by adding a perfect matching to a k-regular graph, the eigenvalues of the new graph can increase by at most 1 in absolute value.

  Conjecture 6.0.6. *Let X be a k-regular Ramanujan graph with an even number of vertices. Then there exists a perfect matching P with $V(P) = V(X)$ such that $X \cup P$ is Ramanujan.*

  This would imply that infinite families of Ramanujan graphs exist for each degree. It would also interesting to see how adding perfect matchings affects other parameters of a graph.

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