Determining a Graph by Eigenvalues of Simplicial Complexes

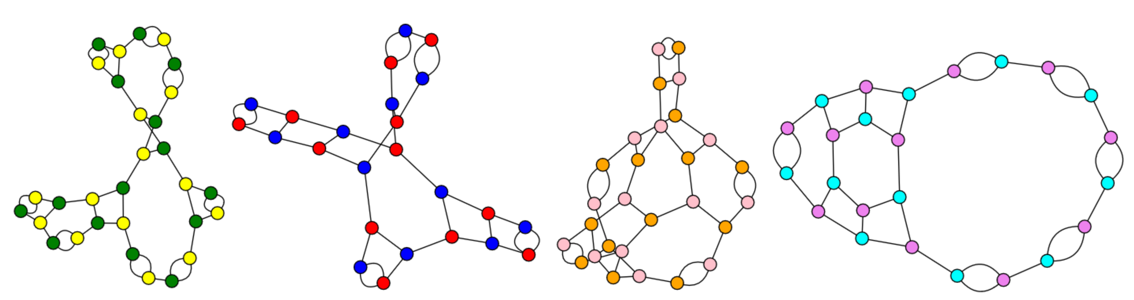
{ Chunxu Ji and Sebastian Cioabă }
University of Delaware, Department of Mathematics, Newark, DE, 19717

Abstract

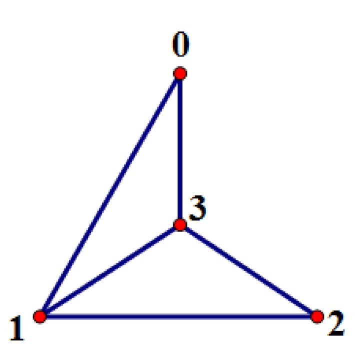
In our research, we investigate the eigenvalues of higher dimensional Laplacians associated with graphs and their use in distinguishing non-isomorphic graphs with the same ordinary Laplacian eigenvalues.

Introduction

Graph: abstract model of a network.

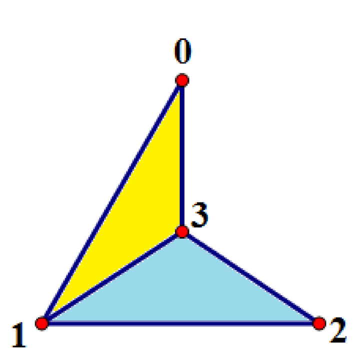


The **Laplacian matrix** of a graph:



$$\begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$

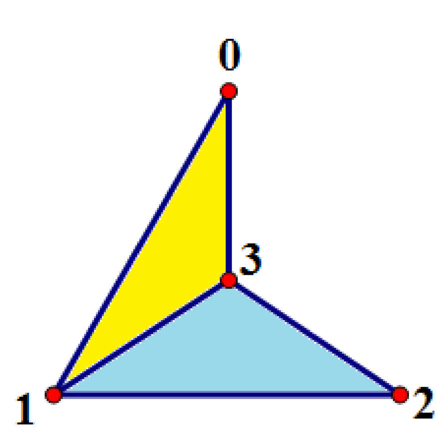
A graph as a **simplicial complex**:



$X_2 = \{013, 123\}$
$X_1 = \{01, 03, 12, 13, 23\}$
$X_0 = \{0, 1, 2, 3\}$
$X_{-1} = \{\emptyset\}$

Oriented incidence numbers:

$$[F:K] = \begin{cases} (-1)^j, & \text{if } K \subset F, \text{ and } F \setminus K = \{x_j\} \\ 0, & \text{otherwise.} \end{cases}$$



$[01:0] = -1$
$[01:1] = 1$
$[01:3] = 0$
$[013:01] = 1$
$[013:03] = -1$
$[013:13] = 1$

Definitions

$\mathcal{C}^i(X;\mathbb{R}) := \{f : X_i \to \mathbb{C}\} \simeq \mathbb{R}^{X_i}$.
Coboundary map $\delta_i : \mathcal{C}^i(X;\mathbb{R}) \to \mathcal{C}^{i+1}(X;\mathbb{R})$

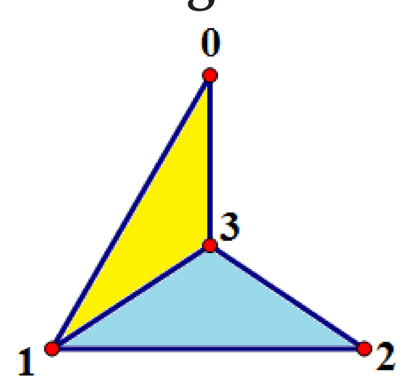
$$(\delta_i f)(H) = \sum_{F \in X_i} [H:F] f(F)$$

for any $H \in X_{i+1}$ and $f \in \mathcal{C}^i(X;\mathbb{R})$.
Boundary map $\partial_{i+1} = \delta_i^* : \mathcal{C}^{i+1}(X;\mathbb{R}) \to \mathcal{C}^i(X;\mathbb{R})$

$$(\partial_{i+1} g)(F) = \sum_{H \in X_{i+1}} [H:F] g(H)$$

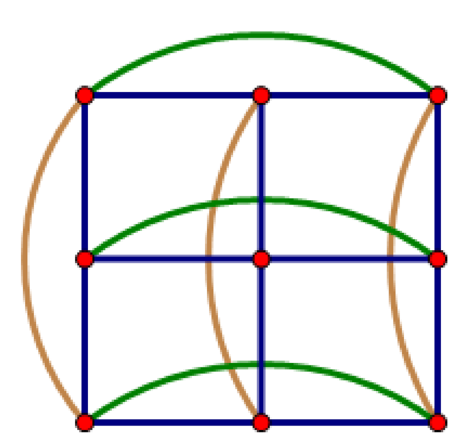
for any $F \in X_i$ and $g \in \mathcal{C}^{i+1}(X;\mathbb{R})$.

The **degree** of a face:

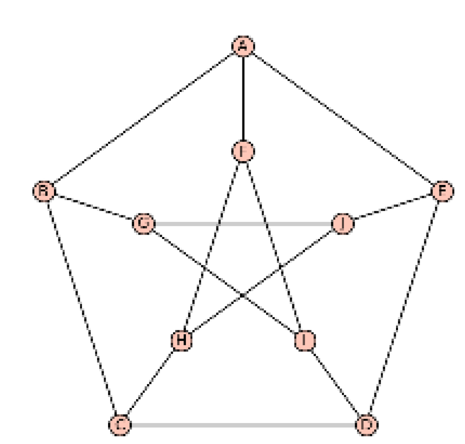


$\deg(0) = 2$
$\deg(3) = 3$
$\deg(13) = 2$
$\deg(12) = 1$
$\deg(03) = 1$

Strongly regular graphs:



Hamming graph H(2,3)
(9,4,1,2)



Petersen graph
(10,3,0,1)

Higher Order Laplacians of Graphs

Up Laplacian $L_i^{\uparrow}$ and **Down Laplacian** $L_i^{\downarrow}$.

$$\mathcal{C}^{i+1}(X;\mathbb{R}) \underset{\partial_{i+1}}{\overset{\delta_i}{\leftrightarrows}} \mathcal{C}^i(X;\mathbb{R}) \underset{\partial_i}{\overset{\delta_{i-1}}{\leftrightarrows}} \mathcal{C}^{i-1}(X;\mathbb{R})$$

$L_i^{\uparrow} := \partial_{i+1}\delta_i = \delta_i^*\delta_i$

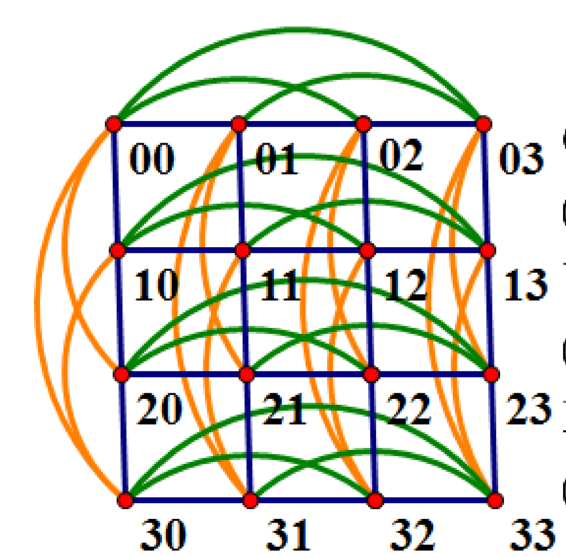
$L_i^{\downarrow} := \delta_{i-1}\partial_i = \delta_{i-1}\delta_{i-1}^*$

$$(L_i^{\uparrow})_{F,F'} = \begin{cases} \deg(F), & \text{if } F = F' \\ -[F : F\cap F'][F' : F\cap F'], & \text{if } F \cup F' \in X_{i+1} \\ 0, & \text{otherwise} \end{cases}$$

$$(L_i^{\downarrow})_{F,F'} = \begin{cases} i+1, & \text{if } F = F' \\ [F : F\cap F'][F' : F\cap F'], & \text{otherwise} \end{cases}$$

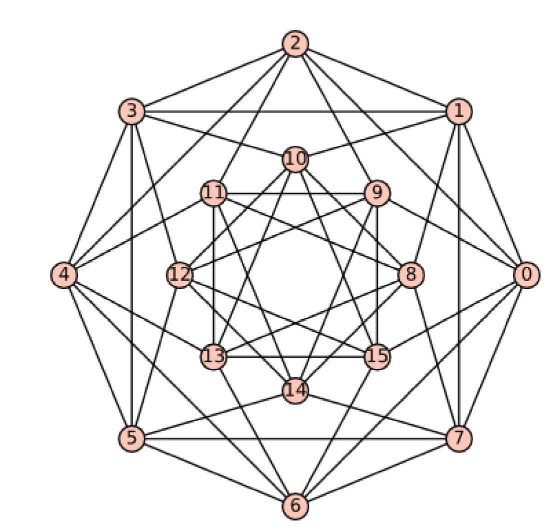
Two strongly regular graphs with same parameters $(16, 6, 2, 2)$:

Hamming Graph $H(2,4)$



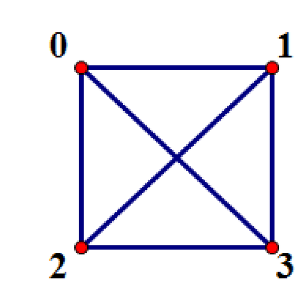
Ordinary Laplacian eigenvalues:
$0^{(1)}4^{(6)}8^{(9)}$
Up Laplacian:
$0^{(24)}4^{(24)}$
Down Laplacian:
$0^{(33)}4^{(6)}8^{(9)}$

Shrikhande Graph

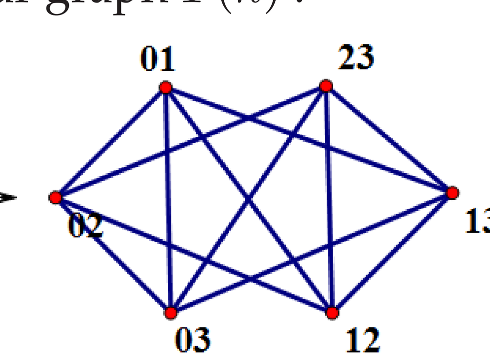


Ordinary Laplacian eigenvalues:
$0^{(1)}4^{(6)}8^{(9)}$
Up Laplacian:
$0^{(17)}(3-\sqrt{5})^{(6)}2^{(9)}4^{(9)}(3+\sqrt{5})^{(6)}6^{(1)}$
Down Laplacian:
$0^{(33)}4^{(6)}8^{(9)}$

Triangular graph $T(n)$:



K4



T(4)

Chang graphs and Triangular graph $T(8)$.

There are 4 non-isomorphic strongly regular graphs with parameters $(28, 12, 6, 4)$ which have the same ordinary Laplacian eigenvalues. Three of them are called Chang graphs, and the other one is the triangular graph $T(8)$. Our computation shows that these four graphs have the same eigenvalues of down Laplacian, but different eigenvalues of up Laplacian.

Computation Results

1.Hamming Graph $H(2,3)$
Up Laplacian: $0^{(12)}3^{(6)}$
Down Laplacian: $0^{(10)}3^{(4)}6^{(4)}$
2. Hamming Graph $H(2,5)$
Up Laplacian: $0^{(40)}5^{(60)}$
Down Laplacian: $0^{(76)}5^{(8)}10^{(16)}$

3. Hamming Graph $H(2,6)$
Up Laplacian: $0^{(60)}6^{(120)}$
Down Laplacian: $0^{(145)}6^{(10)}12^{(15)}$
4.Triangular Graph $T(6)$.
Up Laplacian:$0^{(14)}2^{(10)}5^{(16)}6^{(10)}8^{(10)}$
Down Laplacian: $0^{(46)}6^{(5)}10^{(9)}$

5. Triangular Graph $T(7)$.
Up Laplacian:$0^{(20)}2^{(15)}6^{(35)}7^{(15)}9^{(20)}$
Down Laplacian: $0^{(85)}7^{(6)}12^{(14)}$
6. Triangular Graph $T(8)$
Up Laplacian: $0^{(27)}2^{(21)}7^{(64)}8^{(21)}10^{(35)}$
Down Laplacian: $0^{(141)}8^{(7)}14^{(20)}$

Current Work

1. Determine the spectrum of Hamming graphs and Triangular graphs.
2. Prove that the eigenvalues of the down Laplacian are determined by the parameters of the strongly regular graph.

Conclusion and Future Work

The eigenvalues of higher order Laplacians seem to be able to distinguish between non-isomorphic graphs that have the same adjacency matrix eigenvalues.

We will test this hypothesis on other strongly regular graphs with the same parameters and further investigate this problem in the near future.

References

[1] C. Bachoc, A. Gundert & A.Passuello *The Theta Number of Simplicial Complexes*. arXiv: 1704.01836v1 [math.CO] (2017)