Homework 1, Math 245, Spring 2019
Due Friday, February 22, 2019

The solution of each exercise should be at most one page long. You should always show your work justifying your answer. If you can, try to write your solutions in LaTex. This will give you an extra 10%.

1. If $a > 0$, prove that $a + \frac{1}{a} \geq 2$. When does equality hold ? If $x, y, z > 0$, prove that

$$\frac{x+y}{z} + \frac{y+z}{x} + \frac{z+x}{y} \geq 6.$$

   When does equality hold ?

2. If a, b, c are three real numbers, show that

$$a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca)$$

   and

$$a^2 + b^2 + c^2 \geq ab + bc + ca$$

   . If $x, y, z > 0$, show that $\frac{x+y+z}{3} \geq \sqrt[3]{xyz} \geq \frac{3}{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}}$.

3. If a, b, x, y are real numbers, prove that $(a^2+b^2)(x^2+y^2) \geq (ax+by)^2$. When does equality happen ? Extra: do you know a geometric interpretation of this inequality ?

4. Let A, B and C be three sets. Using double inclusion, prove that

$$A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C).$$

5. If $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \frac{6x}{1+x^2}$, then show that the image of f is the interval $[-3, 3]$.

Homework 2, Math 245, Spring 2019
Due Friday, March 1st, 2019

The solution of each exercise should be at most one page long. You should always show your work justifying your answer. If you can, try to write your solutions in LaTex.

1. Let $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = 4 - 3x$ for $x \leq 2$ and $f(x) = -x$ for $x > 2$. Prove that f is bijective and find its inverse f^{-1}.

2. Let $f : A \to B$ and $g : B \to A$ be two functions. Prove that if both $g \circ f$ and $f \circ g$ are identity functions, then f is bijective.

3. Prove that the function $f : (0,1) \to \mathbb{R}, f(x) = \frac{1-2x}{x(1-x)}$ is bijective.

4. Show that intervals $(0,1)$ and $(5,12)$ have the same cardinality by constructing a bijective function from $(0,1)$ to $(5,12)$. Make sure you prove that your function is a bijection.

5. Prove that the function $h : \mathbb{N} \times \mathbb{N} \to \mathbb{N}, h(m,n) = 2^{m-1}(2n-1)$ is a bijection.

Homework 3, Math 245, Spring 2019
Due Wednesday, Wednesday, March 13, 2019

The solution of each exercise should be at most one page long. You should always show your work justifying your answer. If you can, try to write your solutions in LaTex.

1. Let n be a natural number. If the representation in base 3 of n is $a_k a_{k-1} \dots a_0$, show that n is divisible by 9 if and only if $a_1 = a_0 = 0$. If the representation in base 10 of n is $b_r b_{r-1} \dots b_0$, show that n is divisible by 9 if and only if 9 divides $b_r + \cdots + b_0$.

2. Let a, b, c be integers such that $a^2 + b^2 = c^2$. Prove that at least one of a and b is even. If c is divisible by 3, prove that both a and b are divisible by 3.

3. Let a and b be two integers such that $\mathrm{gcd(a, b)} = 1$. Show that $gcd(a+b, a-b) = 1$ or $gcd(a+b, a-b) = 2$.

4. Let a, b and n be integers such that $\gcd(a, b) = 1$, $a|n$ and $b|n$. Show that $ab|n$.

5. The least common multiple (lcm) of natural numbers a and b is the least natural number divisible by both a and b. Prove that $\mathrm{lcm}(a, b)\gcd(a, b) = ab$ for any natural numbers a and b.

Homework 4, Math 245, Spring 2019
Due Friday, March 29, 2019

The solution of each exercise should be at most one page long. You should always show your work justifying your answer. If you can, try to write your solutions in LaTex.

1. Find all integer solutions of the diophantine equation $19x + 7y = 100$.

2. Find all integers n such that $n \equiv 1 \pmod 3, n \equiv 2 \pmod 5$ and $n \equiv 3 \pmod 7$.

3. Show that if $2^n - 1$ is a prime, then n is a prime. (Primes of the form $2^n - 1$ are called Mersenne primes and only few such primes are known).

4. A natural number n is perfect if the sum of all its divisors (except n) equals n. For example, 6 is perfect since $1 + 2 + 3 = 6$ and 28 is perfect as $1 + 2 + 4 + 7 + 14 = 28$. Show that if p is a prime and $2^p - 1$ is a prime, then $2^{p-1}(2^p - 1)$ is a perfect number.

5. Suppose $p > 1$ is a natural number and $(p-1)! \equiv -1 \pmod p$. Show that p is a prime.

Homework 5-6, Math 245, Spring 2019
Due Monday, April 15, 2016

The solution of each exercise should be at most one page long. You should always show your work justifying your answer. If you can, try to write your solutions in LaTex.

1. Find all integer numbers n such that $-105 \leq n \leq 105$ and
$$\begin{aligned} n &\equiv 2 \pmod 3 \\ n &\equiv 3 \pmod 5 \\ n &\equiv 1 \pmod 7. \end{aligned}$$

2. Let p be an odd prime number. If a is an integer not divisible by p, prove that $a^{\frac{p-1}{2}} \equiv 1 \pmod p$ or $a^{\frac{p-1}{2}} \equiv -1 \pmod p$.

3. What are the last 3 digits of 3^{2020} ? What are the last 3 digits of 17^{2020} ?

4. What are the last 3 digits of 8^{2020} ? What are the last 3 digits of 12^{2020} ?

5. Prove that $\sqrt{5} + \sqrt{7}$ is an irrational number. Prove that $\sqrt{2} + \sqrt{3} + \sqrt{5}$ is an irrational number.

6. Let $n \geq 2$ be a natural number such that n divides $(n-1)! + 1$. Prove that n must be a prime number.

7. Let a, b, c be natural numbers such that $ab = c^2$ and $\mathrm{gcd(a,b)} = 1$. Show that there exists natural numbers u and v such that $a = u^2$ and $b = v^2$.

8. Let a, b, c be natural numbers such that $a^2 + b^2 = c^2$ and $\mathrm{gcd(a,b)} = 1$.
   (a) Prove that c is odd.
   (b) Prove that exactly one of a and b is odd.
   (c) If a is odd, then using $b^2 = c^2 - a^2$ and the previous exercise, deduce that there exists natural numbers u and v such that $\frac{c+a}{2} = u^2$ and $\frac{c-a}{2} = v^2$.
   (d) Using previous part, show that $c = u^2 + v^2, a = u^2 - v^2$ and $b = 2uv$. This is the general form of coprime Pythagorean triples.

9. Let p be a prime.
   (a) Show that for any integer k with $1 \leq k \leq p-1$, $\binom{p}{k} \equiv 0 \pmod p$. Recall that $\binom{p}{k} = \frac{p(p-1)\ldots(p-k+1)}{k!}$ is the binomial coefficient p *choose* k counting the number of k subsets of the set $\{1, \ldots, p\}$.
   (b) Show that for any integer a, $(a+1)^p \equiv a^p + 1 \pmod p$.
   (c) Using the previous part and induction, prove that $n^p \equiv n \pmod p$ for any integer n. This is another form of Fermat's little theorem.

10. A natural number n is called *attainable* if there exists non-negative integers a and b such that $n = 5a + 8b$. Otherwise, n is called *unattainable*. Construct an 9×6 matrix whose rows are indexed by the integers between 0 and 8 and whose columns are indexed by the integers between 0 and 5 whose (x, y)-th entry equals $5x + 8y$ for any $0 \leq x \leq 8$ and $0 \leq y \leq 5$.

    (a) Mark down all the attainable numbers that are strictly less than 40 in your array.

    (b) For any $0 \leq x \leq 8, 0 \leq y \leq 5$, prove that the (x, y)-th entry and the $(8-x, 5-y)$-th entry add up to 80.

    (c) Using the previous two parts, determine the number of attainable integers that are strictly less than 40. How many numbers less than 40 are unattainable ?

    (d) Prove that 27 is the largest unattainable integer and that every integer $n \geq 28$ is attainable.

Homework 7, Math 245, Spring 2019
Due Friday, May 3, 2019

The solution of each exercise should be at most one page long. You should always show your work justifying your answer. If you can, try to write your solutions in LaTex.

1. Using the definition of a limit, determine $\lim_{n\to\infty} \frac{5n+1}{3n-2}$.

2. Let $a_n = \left(1 + \frac{1}{n}\right)^n$ for $n \geq 1$. Prove that the sequence $(a_n)_{n\geq 1}$ is bounded and monotone, and therefore convergent to a limit L_1.

3. For $n \geq 1$, let $b_n = \sum_{k=0}^{n} \frac{1}{k!}$. Show that the sequence $(b_n)_{n\geq 1}$ is monotone and bound, and therefore convergent to a limit L_2. Prove that $L_1 = L_2$.

4. Let $(x_n)_{n\geq 1}$ be a sequence defined recursively as $x_1 = 2$ and $x_{n+1} = \frac{x_n + 2/x_n}{2}$ for $n \geq 1$. Show that the sequence $(x_n)_{n\geq 1}$ is bounded and monotone, and therefore convergent. Determine $\lim_{n\to\infty} x_n$.

5. Let $y_n = \frac{1}{n+1} + \cdots + \frac{1}{n+n}$ for $n \geq 1$. Show that $(y_n)_{n\geq 1}$ is bounded and monotone, and therefore convergent.

Homework 8, Math 245, Spring 2019
Due Wednesday, May 15, 2019

The solution of each exercise should be at most one page long. You should always show your work justifying your answer. If you can, try to write your solutions in LaTex.

1. Prove that there is a rational number between any two irrational numbers and an irrational number between any two rational numbers.

2. If $\lim_{n\to\infty} x_n = L_1$ and $\lim_{n\to\infty} y_n = L_2$, by using the definition of a limit, show that $\lim_{n\to\infty}(3x_n - 4y_n + 5) = 3L_1 - 4L_2 + 5$.

3. Determine $\lim_{n\to\infty}(\sqrt{n+1} - \sqrt{n})$ and $\lim_{n\to\infty}(\sqrt{n^2+n} - n)$.

4. Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ be two sequences of real numbers such that $(a_n)_{n\geq 1}$ is bounded and $\lim_{n\to\infty} b_n = 0$. Show that $\lim_{n\to\infty} a_n b_n = 0$.

5. Let $[a_n, b_n]_{n\geq 1}$ be a sequence of closed intervals such that $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$ for every $n \geq 1$ and $\lim_{n\to\infty}(b_n - a_n) = 0$. Show that there exists a unique real number L that is contained in all intervals $[a_n, b_n]$ for any $n \geq 1$; in other words, the intersection $\cap_{n\geq 1}[a_n, b_n]$ of all the intervals equals $\{L\}$.

Homework 9, Math 245, Spring 2019
Due Wednesday, May 22, 2019

The solution of each exercise should be at most one page long. You should always show your work justifying your answer. If you can, try to write your solutions in LaTex.

1. Prove that the addition and multiplication of complex numbers are associative and commutative and satisfy the distributive law.

2. If z and w are complex numbers, prove that $|zw| = |z||w|$ and that $|z+w|^2 = |z|^2 + |w|^2 + 2Re(z\overline{w})$.

3. If z and w are complex numbers, show that $\overline{zw} = \overline{z} \cdot \overline{w}$, $\overline{z+w} = \overline{z} + \overline{w}$ and $|z| = |\overline{z}|$.

4. Determine all the solutions to $x^2 + y^2 = 0$ with x and y real. Determine all solutions to $z^2 + w^2 = 0$ with z and w complex.

5. Solve the equation $z^{20} = -1024(1+i)$.

Name:

Midterm 1, Math 245 - Spring 2019

Duration: 50 minutes
Please turn off your cellphones and laptops, close your books and notebooks.
To get full credit you should explain your answers for questions 2-4.
Each question is worth 5 points so 20 points=100%.

1. [5 points] (TRUE/FALSE) For each of the following questions, determine whether it is true or false. You do not need to justify your answers. Each question is worth 1 point.

a) The square of any real number is non-negative.

b) For any finite sets A and B, $|A \setminus B| + |B \setminus A| = |A \cup B|$.

c) For any real numbers x and y, $|x + y| = |x| + |y|$.

d) If $x_1, x_2, x_3, x_4, x_5 > 0$ and $x_1 x_2 x_3 x_4 x_5 = 1$, then $x_1 + x_2 + x_3 + x_4 + x_5 \geq 5$.

e) For any sets A, B and C, $(A \cup B) \setminus C \subset [A \setminus (B \cup C)] \cup [B \setminus (A \cap C)]$.

#	Score
1	
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Total	

2. [5 points] Consider the following two functions

$$f : \mathbb{R} \to [6, +\infty), f(x) = x^2 - 2x + 7$$

and

$$g : [6, +\infty) \to [0, +\infty), g(x) = \sqrt{x - 4}.$$

1. Prove that f is not injective.
2. Prove that f is surjective.
3. Prove that g is injective.
4. Prove that g is not surjective.
5. Find the domain, co-domain of $g \circ f$. For x in the domain of $g \circ f$, what is $(g \circ f)(x)$?

3. a) [3 points] Let a be a positive real number. Show that $|x| \leq a$ if and only if $x \in [-a, a]$. Prove that if $|y - 1| < 1$, then $|y^2 - 4y + 3| < 3$.

b) [2 points] Let $f : A \to B$ and $g : B \to C$ be two functions. Prove that if $g \circ f$ is injective, then f is injective. Show that if $g \circ f$ is surjective, then g is surjective.

4. a) [3 points] Show that the intervals $[0, 1]$ and $[5, 10]$ have the same cardinality. Prove that $[0, 1]$ and $(0, 1]$ have the same cardinality.

b) [2 points] Let $\lfloor x \rfloor$ denote the largest integer that is smaller than or equal to x. Prove that for any real numbers x and y,

$$\lfloor x + y \rfloor \geq \lfloor x \rfloor + \lfloor y \rfloor.$$

Give an example of two numbers x and y where the inequality above is strict.

Name:

Exam 2, Math 245 - Spring 2019

Duration: 50 minutes
Please turn off your cellphones and laptops, close your books and notebooks.
To get full credit you should explain your answers for questions 2-4.
Each question is worth 5 points so 20 points=100%.

1. [5 points] (TRUE/FALSE) For each of the following questions, determine whether it is true or false. You do not need to justify your answers. Each question is worth 1 point.

a) 2020 divides $2^{2021} - 1$.

b) The product of any two irrational numbers is an irrational number.

c) If a, b, c are natural numbers and $ab = c^2$, then a and b are squares of integers.

d) If $2^n + 1$ is prime, then n is a power of two.

e) If p is prime, then $2^p - 1$ is prime.

#	Score
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Total	

2. [3 points] Find all integers x and y such that $4x + 7y = 13$.

b) [2 points] Find all integers n such that $2n \equiv 1 \pmod{3}$ and $3n \equiv 4 \pmod{5}$.

3. a) [3 points] Let a, b and d be integers such that d divides ab and $\gcd(d, a) = 1$. Prove that d divides b.

b) [2 points] Find the last two digits of 3^{2003} and 4^{2003}.

4. a) [3 points] Let $p \neq q$ be two primes numbers. Show that $\sqrt{p}$ is irrational and that $\sqrt{p} - \sqrt{q}$ is irrational.

b) [2 points] Let a and b be two integers such that $\gcd(a, b) = 3$. Show that $\gcd(a + b, a - b)$ equals 3 or 6.

Name:

Exam 3, Math 245 - Spring 2019

Duration: 50 minutes
Please turn off your cellphones and laptops, close your books and notebooks.
To get full credit you should explain your answers for questions 2-4.
Each question is worth 5 points so 20 points=100%.

1. [5 points] (TRUE/FALSE) For each of the following questions, determine whether it is true or false. You do not need to justify your answers. Each question is worth 1 point.

a) The sum of any two irrational numbers is always an irrational number.

b) There is a convergent sequence of real numbers that is not monotone.

c) If a is an upper bound for a set S, then a^2 is also an upper bound for S.

d) If b is a lower bound for a set T, then $b/2$ is also a lower bound for T.

e) A complex number z is real if and only if $Im(z) = 0$.

#	Score
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Total	

2. [3 points] For any natural number n, define $x_n = \frac{3n}{2n+1}$. Determine whether the sequence $(x_n)_{n \geq 1}$ is monotone, whether it is bounded, and whether it converges. If it converges, find its limit. Justify your work.

b) [2 points] State the Completeness Axiom for Real Numbers. Prove that the supremum of the interval $(3, 7)$ equals 7.

3. a) [3 points] Prove that a convergent sequence is bounded. Give an example of a bounded sequence that is not convergent.

b) [2 points] If $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ are two sequences such that $\lim_{n\to\infty} a_n = L_1$ and $\lim_{n\to\infty} b_n = L_2$, then prove that

$$\lim_{n\to\infty} (3a_n - 2b_n + 4) = 3L_1 - 2L_2 + 4.$$

4. a) [3 points] Calculate and simplify each of the following 3 expressions:

$$(4+3i)(2-i)+3-5i$$
$$\frac{4+3i}{2-i}-3+5i$$
$$(1+i)^{20}.$$

b) [2 points] Find all complex numbers z such that $z^6 = 8(1-i)$.

Name:

Final Exam, Math 245 - Spring 2019

Duration: 2 hours
Please turn off your cellphones and laptops, close your books and notebooks.
To get full credit you should explain your answers for questions 2-8.
Each question is worth 5 points so 40 points=100%.

1. [5 points] (TRUE/FALSE) For each of the following questions, determine whether it is true or false. You do not need to justify your answers. Each question is worth 1 point.

a) A function $f : A \to B$ is surjective if for any $x \in A$, there is $y \in B$ such that $y = f(x)$.

b) A function $f : A \to B$ is injective if $f(a) \neq f(b)$ implies that $a \neq b$.

c) If A, B and C are sets such that $A \cap B = A \cap C$ and $A \cup B = A \cup C$, then $B = C$.

d) If n is an integer, then n^2 cannot equal $3 \pmod 7$.

e) If a, b, c are natural numbers and $ab = c^3$, then a and b are cubes of integers.

#	Score
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8	
Total	

2. a) [3 points] Show that the intervals $[0, 2)$ and $(4, 10]$ have the same cardinality. Prove that $[0, 1]$ and $[0, 1)$ have the same cardinality.

b) [2 points] Let $\lceil x \rceil$ denote the smallest integer that is greater than or equal to x. Prove that for any real numbers x and y,

$$\lceil x + y \rceil \leq \lceil x \rceil + \lceil y \rceil.$$

Give an example of two numbers x and y where the inequality above is strict.

3. [5 points] Consider the following two functions

$$f : [0, +\infty) \rightarrow (-\infty, 4], f(x) = -x^2 + 2x + 3$$

and

$$g : (-\infty, 4] \rightarrow (-\infty, 3], g(x) = 2 - \sqrt{4 - x}.$$

1. Prove that f is not injective.
2. Prove that f is surjective.
3. Prove that g is injective.
4. Prove that g is not surjective.
5. Find the domain, co-domain of $g \circ f$. For x in the domain of $g \circ f$, what is $(g \circ f)(x)$?

4. a) [3 points] Prove by mathematical induction on n that

$$\frac{1}{1^2} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}$$

for any natural number $n \geq 1$.

b) [2 points] Let $a_n = \sum_{k=1}^{n} \frac{1}{k^2}$ for $n \geq 1$. Prove that the sequence $(a_n)_{n\geq 1}$ is monotone and bounded.

5. a) [3 points] Let $f : A \to B$ and $g : B \to C$ be two functions.

If f is injective and g is injective, then prove that $g \circ f : A \to C$ is injective.

If f is surjective and g is surjective, then prove that $g \circ f : A \to C$ is surjective.

If $g \circ f$ is injective and f is surjective, then prove that g is injective.

b) [2 points] Prove that the function $f : (0, +\infty) \to (0, 1), f(x) = \frac{1}{\sqrt{1+x^2}}$ is invertible and determine its inverse f^{-1} (including its domain and co-domain).

6. a) [3 points] For any natural number n, define $x_n = \frac{5n-3}{2n-1}$. Determine whether the sequence $(x_n)_{n\geq 1}$ is monotone and whether it is bounded. Using the definition of a limit, prove that the sequence converges and find its limit. Justify your work.

b) [2 points] State the Completeness Axiom for Real Numbers. Prove that the infimum of the interval $(3, 7)$ equals 3.

7. a) [3 points] Write the definition of a convergent sequence. Write the definition of a Cauchy sequence. Prove that any convergent sequence must be Cauchy.

b) [2 points] If $A \subset B \subset \mathbb{R}$ are two sets of real numbers, show that $\inf(A) \leq \inf(B)$ and $\sup(A) \geq \sup(B)$.

8. a) [3 points] If $z = 1 - i$ and $w = 2 + 3i$, calculate

$$\frac{2\overline{z} + w}{z - \overline{w}},$$

$$|z^3 \overline{z} w^2|,$$

and

$$z^{30}.$$

b) [2 points] Solve the equation $z^6 = -8$ over complex numbers.