Eigenvalues and linear programming bounds for regular graphs and hypergraphs

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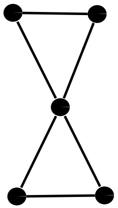
Graphs, Adjacency Matrices and Eigenvalues

- G **graph** on n vertices
- The **adjacency matrix** A is an $n \times n$ matrix where $A(x, y)$ equals the number of edges between x and y.
- The **eigenvalues** of A: $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.



$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$\frac{1+\sqrt{17}}{2}$, 1, -1, -1, $\frac{1-\sqrt{17}}{2}$

Spectral Graph Theory

Eigenvalues of Regular Graphs

1. If G is a connected k-regular graph with n vertices, then

$$k = \lambda_1 > \lambda_2 \geq \cdots \geq \lambda_n \geq -k$$

2. If G is k-regular, then G is bipartite iff $\lambda_n = -k$.

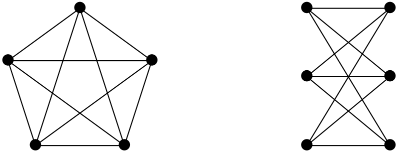
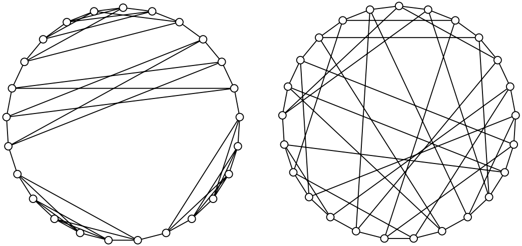


Figure: $K_5 : 4^{(1)}, -1^{(4)}$; $K_{3,3} : 3^{(1)}, 0^{(4)}, -3^{(1)}$

Expanders

Expanders are **sparse** AND

- highly connected *no bottlenecks* - **combinatorics**
- large spectral gap $k - \lambda_2$ - **algebraic**
- rapid convergence of random walk - **probabilistic**



Expanders and Ramanujan Graphs

Theorem (Alon-Milman, Cheeger, Tanner, Mohar 1980s)

If G is a k-regular graph, then

$$\sqrt{k^2 - \lambda_2^2} \geq h(G) \geq \frac{k - \lambda_2}{2}.$$

Theorem (Alon-Boppana 1986, Serre 1997)

Given $k \geq 3$ and $\theta < 2\sqrt{k-1}$, there are finitely many k-regular graphs with $\lambda_2(G) < \theta$.

Ramanujan Graph

A connected k-regular graph G is called **Ramanujan** if $|\lambda_i| \leq 2\sqrt{k-1}$ for all $\lambda_i \neq \pm k$.

Ramanujan Graphs

Lubotzky-Phillips-Sarnak, Margulis 1986, Morgenstern 1994

Explicit constructions of infinite families of k-regular Ramanujan graphs when $k-1$ is a power of a prime.

Theorem (Marcus-Spielman-Srivastava 2015)

1. *For any* $k \geq 3$, **there exists** *an infinite family of* k*-regular graphs with* $\lambda_2 \leq 2\sqrt{k-1}$.
2. *For any* $k \geq 3$, **there exists** *an infinite family of* k*-regular bipartite Ramanujan graphs.*

Maximizing the order of a regular graph with given 2nd eig

Problem

Given $k \geq 3$ and $\theta < 2\sqrt{k-1}$, what is the maximum order $v(k,\theta)$ of a k-regular graph with $\lambda_2 \leq \theta$?

Related Work

Teranishi-Yasuno 2000 $v(k,\theta)$ for bipartite graphs, design theory and drgs.
Amit-Hoory-Linial 2002 given v, k, g, what k-regular graph of order v and girth g minimizes $\lambda := \max(|\lambda_2|, |\lambda_n|)$.
Krivelevich-Sudakov 2006 pseudo-random graphs survey.
Stanić 2010, Koledin-Stanić 2013 $v(k,\theta)$ for small θ.
Hoholdt-Justesen 2012 $v(k,\theta)$ for bipartite graphs, coding theory.
Richey-Shutty-Stover 2013 $v(k,\theta)$ using Serre's proof.
Nozaki 2015 given v, k, what k-regular graph with order v minimizes λ_2.

Our results

- S.M. Cioabă, J. Koolen, H. Nozaki and J. Vermette
  Maximizing the order of a regular graph of given valency and second eigenvalue, **SIAM J. Discrete Mathematics** 2016.
- S.M. Cioabă, J. Koolen and H. Nozaki
  A spectral version of the Moore problem for bipartite regular graphs, **Algebraic Combinatorics** 2019.
- S.M. Cioabă, J. Koolen, M. Mimura, H. Nozaki and T. Okuda
  On the spectrum and linear programming bound for hypergraphs, submitted, `https://arxiv.org/abs/2009.03022`

Nozaki's LP bound

Singleton 1966

Let $F_i = F_i^{(k)}$ be the orthogonal polynomials defined by
$F_0(x) = 1, F_1(x) = x, F_2(x) = x^2 - k$ and
$F_i(x) = xF_{i-1}(x) - (k-1)F_{i-2}(x), i \geq 3$.
$F_i(A)_{a,b}$ equals the number of non-backtracking a, b-walks of length i.

Theorem (Nozaki 2015)

Let G be a connected k-regular graph with v vertices and distinct eigenvalues $\theta_1 = k > \theta_2 > \ldots > \theta_d$. If there exists a polynomial $f(x) = \sum_{i \geq 0} f_i F_i(x)$ such that $f(k) > 0$, $f(\theta_j) \leq 0$ for any $j \geq 2$, $f_0 > 0$, and $f_\ell \geq 0$ for any $\ell \geq 1$, then

$$v \leq \frac{f(k)}{f_0}.$$

Our Results

Theorem (Cioabă-Koolen-Nozaki-Vermette 2016)

Let $T(k,t,c)$ be the $t \times t$ tridiagonal matrix:

$$T(k,t,c) = \begin{bmatrix} 0 & k & & & & & \\ 1 & 0 & k-1 & & & & \\ & 1 & 0 & k-1 & & & \\ & & . & . & . & & \\ & & & . & . & . & \\ & & & & 1 & 0 & k-1 \\ & & & & & c & k-c \end{bmatrix}$$

If $\theta = \theta_2(T(k,t,c))$, then

$$v(k,\theta) \leq 1 + \sum_{i=0}^{t-3} k(k-1)^i + \frac{k(k-1)^{t-2}}{c}.$$

Our Results

Theorem (Cioabă-Koolen-Nozaki-Vermette 2016)

If $\theta = \theta_2(T(k,t,c))$, *then*

$$v(k,\theta) \leq 1 + \sum_{i=0}^{t-3} k(k-1)^i + \frac{k(k-1)^{t-2}}{c}.$$

Proof Idea

1. The eigenvalues of T are the roots of $(x-k)(G_{t-1} + (c-1)G_{t-2})$, where $G_t = \sum_{i=0}^{t} F_i$.
2. Apply Nozaki's LP bound to $f(x) = (x-\theta_2)\prod_{j\geq 3}(x-\theta_j)^2$. $f(\theta_i) \leq 0$ for $i \geq 2$.
3. $v(k,\theta) \leq \frac{f(k)}{f_0} = \sum_{i=0}^{t-2} F_i(k) + F_{t-1}(k)/c = 1 + \sum_{i=0}^{t-3} k(k-1)^i + \frac{k(k-1)^{t-2}}{c}$.

Our Results

Theorem (Cioabă-Koolen-Nozaki-Vermette 2016)

If $\theta = \theta_2(T(k,t,c))$, then

$$v(k,\theta) \le 1 + \sum_{i=0}^{t-3} k(k-1)^i + \frac{k(k-1)^{t-2}}{c} =: N(k,t,c).$$

If G is a k-regular graph with second eigenvalue θ and $N(k,t,c)$ vertices, then G is distance-regular with quotient matrix $T(k,t,c)$.

Behavior of $\theta = \theta_2(T(k,t,c))$

1. θ is the largest root of $(c-1)G_{t-2} + G_{t-1}$, where $G_t = \sum_{i=0}^{t} F_i$.
2. θ can take any value between -1 and $2\sqrt{k-1}$.

A Consequence of Our Results

Theorem (Alon-Boppana 1986, Serre 1997)

Given $k \geq 3$ and $\theta < 2\sqrt{k-1}$, there are finitely many k-regular graphs with $\lambda_2(G) < \theta$.

Proof.

$\lambda_2(t) := \theta_2(T(k,t,1))$ is the largest root of G_{t-1}.
$\lambda_2(t) = 2\sqrt{k-1}\cos\tau$, where $\tau \leq \pi/(t-1)$ (Bannai-Ito book).
As $\theta < 2\sqrt{k-1}$, there is a τ such that $\theta \leq \lambda_2(\tau)$.
Thus $v(k,\theta) \leq v(k,\lambda_2(\tau)) \leq 1 + \sum_{i=0}^{t-2} k(k-1)^i$. □

Other proofs of Alon-Boppana/Serre's theorem.

Friedman 1997, Alon 2004, Cioabă 2006, Mohar 2010.

Our Results

Determined $v(k,1)$ for every k. $v(k,1) = 2k+2$ for $k \geq 11$.

(k,θ)	$v(k,\theta)$	Graph meeting bound
$(q+1, \sqrt{q})$	$2(q^2+q+1)$	incidence graph of $PG(2,q)$
$(q+1, \sqrt{2q})$	$2(q+1)(q^2+1)$	incidence graph of $GQ(q,q)$
$(q+1, \sqrt{3q})$	$2(q+1)(q^4+q^2+1)$	incidence graph of $GH(q,q)$
$(3,1)$	10	Petersen graph
$(4,2)$	35	Odd graph O_4
$(7,2)$	50	Hoffman–Singleton graph
$(5,1)$	16	Clebsch graph
$(10,2)$	56	Gewirtz graph
$(16,2)$	77	M_{22} graph
$(22,2)$	100	Higman–Sims graph

Richey-Shutty-Stover Results/Conjectures

$v(3,2) \leq 105$, conj. $v(3,2) = 30$. **TRUE**. Tutte-Coxeter graph or Tutte 8-cage

$v(4,2) \leq 77$, conj. $v(4,2) = 24$. **FALSE**. $v(4,2) = 35$. Odd graph O_4

conj. $v(4,3) = 27$. **FALSE**. $v(4,3) = 728$. $GH(3,3)$

Degree-Diameter Problem and Eigenvalues

Theorem (Hoffman-Singleton 1960)

If G is a k-regular graph with diameter 2 and k^2+1 vertices, then $k \in \{2, 3, 7, 57\}$.

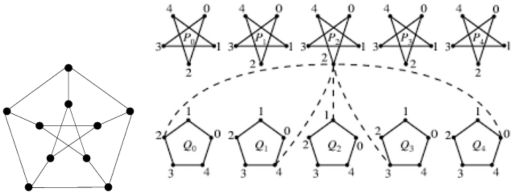


Figure: The Petersen graph and the Hoffman-Singleton graph

Open Problem

$k = 57$?

Degree-Diameter Problem and Eigenvalues

The Moore bound

If G is a k-regular graph of diameter d, then its number of vertices is at most

$$1 + k + k(k-1) + \cdots + k(k-1)^{d-1}.$$

Theorem (Bannai-Ito 1973; Damerell 1973)

No equality in the Moore bound when $k, d \geq 3$.

Connections with the Moore problem

Dinitz, Schapiro and Shahaf 2020

If G is connected k-regular of diameter d and $\lambda = \max(|\lambda_2|, |\lambda_n|)$, then

$$n \leq G_d(k) - G_d(\lambda) = 1 + \sum_{i=0}^{d-1} k(k-1)^i - G_d(\lambda).$$

If 2nd eigenvalue of G is *large*, then improvement of Moore bound.

Our results

If $\lambda_2(G)$ is *small*, then

$$n \leq 1 + \sum_{i=0}^{d-2} k(k-1)^i + \frac{k(k-1)^{d-1}}{c}$$

where $c > 1$.

Connections with the Moore problem

(k, d)	known	defect	Lower	Moore	Upper
(8,2)	57	8	2.09503	2.19258	3.40512
(9,2)	74	8	2.29956	2.37228	3.53113
(10,2)	91	10	2.46923	2.54138	3.88748
(4,3)	41	12	2.11232	2.25342	2.88396
(5,3)	72	34	2.42905	2.62620	3.77862
(4,4)	98	63	2.53756	2.69963	3.44307
(5,4)	212	214	2.91829	3.12941	4.41922
(3,5)	70	24	2.32340	2.39309	2.64401
(4,5)	364	121	2.89153	2.93996	3.42069
(3,6)	132	58	2.45777	2.51283	2.75001
(4,6)	740	717	3.00233	3.08314	3.73149

If the order of a graph is at least “known”, then the second eigenvalue is between “Lower” and “Upper”. “Moore” is the second eigenvalue of a Moore graph.

Our bipartite results

Problem

Given $k \geq 3$ and $\theta < 2\sqrt{k-1}$, what is the maximum order $v(k,\theta)$ of a k-regular graph with $\lambda_2 \leq \theta$?

Bipartite Problem

Given $k \geq 3$ and $\theta < 2\sqrt{k-1}$, what is the maximum order $b(k,\theta)$ of a **bipartite** k-regular graph with $\lambda_2 \leq \theta$?

$b(k,\theta)$ is not the same as $v(k,\theta)$

$b(3,1) = 8 < 10 = v(3,1)$.

Our bipartite results

Theorem (Cioabă-Koolen-Nozaki 2019)

Let $B(k,t,c)$ be the $t \times t$ tridiagonal matrix:

$$B(k,t,c) = \begin{bmatrix} 0 & k & & & & & \\ 1 & 0 & k-1 & & & & \\ & 1 & 0 & k-1 & & & \\ & & . & . & . & & \\ & & & . & . & . & \\ & & & & c & 0 & k-c \\ & & & & & k & 0 \end{bmatrix}$$

If $\theta = \theta_2(B(k,t,c))$, then

$$b(k,\theta) \leq 2\left(1 + \sum_{i=0}^{t-4}(k-1)^i + \frac{(k-1)^{t-3}}{c} + \frac{(k-1)^{t-2}}{c}\right).$$

Our bipartite results

Theorem (Cioabă-Koolen-Nozaki 2019)

If $\theta = \theta_2(B(k,t,c))$, *then*

$$b(k,\theta) \leq 2\left(1 + \sum_{i=0}^{t-4}(k-1)^i + \frac{(k-1)^{t-3}}{c} + \frac{(k-1)^{t-2}}{c}\right) := M(k,t,c).$$

Equality happens if and only if there exists a bipartite distance-regular graph whose quotient matrix of the distance partition from a vertex is $B(k,t,c)$ *for* $1 \leq c < k$ *or* $B(k,t-1,1)$ *for* $c = k$.

Bipartite bound is always less than or equal than the general bound.

Our bipartite results

Table: Known bipartite graphs meeting the bound $M(k, d+1, c)$

k	θ	$b(k,\theta)$	d	c	Name
2	$2\cos(2\pi/n)$	n	$n/2$	1	C_n
k	0	$2k$	2	1	$K_{k,k}$
k	$\sqrt{k-\tau}$	$2(1+k(k-1)/\tau)$	3	τ	Symmetric (v,k,τ)-design
r^2-r+1	r	$2(r^2+1)\times$ (r^2-r+1)	4	$(r-1)^2$	$pg(r^2-r+1, r^2-r+1, (r-1)^2)$
q	$\sqrt{q}$	$2q^2$	4	$q-1$	$AG(2,q)$ minus a parallel class
$q+1$	$\sqrt{2q}$	$2\sum_{i=0}^{3} q^i$	4	1	$GQ(q,q)$
$q+1$	$\sqrt{3q}$	$2\sum_{i=0}^{5} q^i$	6	1	$GH(q,q)$
6	2	162	4	2	$pg(6,6,2)$

Hypergraphs

Regular Graphs

Given $k \geq 3$ and $\theta < 2\sqrt{k-1}$, what is the maximum order $v(k,\theta)$ of a k-regular graph with $\lambda_2 \leq \theta$?

Bipartite Regular Graphs

Given $k \geq 3$ and $\theta < 2\sqrt{k-1}$, what is the maximum order $b(k,\theta)$ of a **bipartite** k-regular graph with $\lambda_2 \leq \theta$?

Regular Uniform Hypergraphs

Given $r, u \geq 2$ and $\theta < u - 2 + 2\sqrt{(r-1)(u-1)}$, what is the maximum order $h(r,u,\theta)$ of an u-**uniform** r-regular hypergraph with $\lambda_2 \leq \theta$?

Hypergraphs

Hypergraphs

Given $r, u \geq 2$ and $\theta < u - 2 + 2\sqrt{(r-1)(u-1)}$, what is the maximum order $h(r, u, \theta)$ of an u-**uniform** r-regular hypergraph with $\lambda_2 \leq \theta$?

$h(4, 3, 1) = 15 < b(8, 1) = 18 < v(8, 1) = 21.$

Our results

Proved an LP bound generalizing Nozaki's bound for graphs.
General upper bound for $h(r, u, \theta)$.
Our results imply the Alon-Boppana bound for hypergraphs by K. Feng and Winnie Li (1996).

Open problems

Problems

1. $v(k, \sqrt{2}) =?$ for $k \geq 4$.
   $v(3, \sqrt{2}) = 14$ (Heawood graph).
2. $v(k, \sqrt{k}) =?$ for $k \geq 5$.
   $v(3, \sqrt{3}) = 18$ (Pappus graph)
   $v(4, 2) = 35$ (Odd graph O_4).
3. $v(6, 2) =?$
   $v(6, 2) \geq 42$ (The 2nd subconsitutent of Hoffman-Singleton graph).
   $v(6, 2) \leq 47$ (LP bound).