Homework 1
Math 888 Combinatorics II
Due Friday, February 26 in class.

To get 100% on this homework, you must solve correctly all 5 questions for a total of 10 points.

1. **(2 points)** Prove that $P(X) = X^3 + X + 1$ and $Q(X) = X^3 + X^2 + 1$ are both irreducible in $\mathbb{F}_2[X]$. Write out the multiplication table for the elements in the quotient fields $\mathbb{F}_2[X]/(P(X))$ and $\mathbb{F}_2[Y]/(Q(Y))$. Find an isomorphism between these fields.

2. **(2 points)** Let O be a subset of points of a finite projective plane of order n such that no three points of O are collinear. Prove that $|O| \leq n+1$ if n is odd and $|O| \leq n+2$ if n is even. A set attaining equality in the first inequality is called an oval and a set attaining equality in the second inequality is called a hyperoval. Prove that the set $\mathcal{C} = \{\langle(t, t^2, 1)\rangle : t \in \mathbb{F}_q\} \cup \{\langle(0, 1, 0)\rangle\}$ is an oval in $PG(2, q)$.

3. **(2 points)** For $n \in \{2, 3, 4\}$, find a subset S_n of $\mathbb{Z}_{n^2+n+1}$ such that the elements of $\mathbb{Z}_{n^2+n+1}$ as points and the n^2+n+1 lines $S_n + x = \{s + x : s \in S_n\}$ form a projective plane of order n. In each case, construct an oval or a hyperoval.

4. **(2 points)** Let q be a prime power. For $a \in \mathbb{F}_q^*$, define the $\mathbb{F}_q \times \mathbb{F}_q$ matrix L_a by $L_a(x, y) = ax + y$. Prove that each L_a is a Latin square and that the collection $L_a, a \in \mathbb{F}_q^*$ is a collection of $q-1$ mutually orthogonal Latin squares.

5. **(2 points)** For $t, n \geq 2$ natural numbers, a $t \times n$ orthogonal array $OA(t, n)$ is a $t \times n^2$ matrix with entries in $[n] = \{1, \ldots, n\}$ with the property that for any two rows, the n^2 ordered pairs defined by these rows are all distinct. For $t \geq 3$, prove that an $OA(t, n)$ is equivalent to $t-2$ mutually orthogonal Latin squares. For $t \geq 2$, from an orthogonal array $OA(t, n)$ construct a graph Γ whose vertices are its n^2 columns, where two columns are adjacent if and only if they have the same entry in one coordinate position. Prove that Γ is regular of degree $t(n-1)$, any two adjacent vertices in Γ have exactly $n - 2 + (t-1)(t-2)$ common neighbors and any two non-adjacent vertices of Γ have exactly $t(t-1)$ common neighbors.

Homework 2
Math 888 Combinatorics II
Due Friday, March 12, 2021.

To get 100% on this homework, you must solve correctly all 5 questions for a total of 10 points.

1. **(2 points)** Let k and ℓ be two natural numbers and q a prime power. Let V be a $k+\ell$-dimensional vector space over $\mathbb{F}_q$. If U is a k-dimensional subspace of V, show that the number of ℓ-dimensional subspaces W of V that are skew to U (meaning that $U \cap W$ consists of only the zero vector of dimension $k+\ell$) equals $q^{k\ell}$.

2. **(2 points)**

   (a) A collection of subsets of the set $\{1,\ldots,n\}$ is called an antichain if $A \not\subset B$ for any $A \neq B \in \mathcal{A}$. Prove that if $\mathcal{A}$ is an antichain of subsets of $\{1,\ldots,n\}$, then
$$|\mathcal{A}| \leq \binom{n}{\lfloor n/2 \rfloor}.$$

   (b) A collection $\mathcal{W}$ of subspaces of F_q^n is called an antichain if $U \not\leq W$ for any $U \neq W$ subspaces of $\mathbb{F}_q^n$. Show that if $\mathcal{W}$ is an antichain of subspaces of $\mathbb{F}_q^n$, then
$$|\mathcal{W}| \leq \begin{bmatrix} n \\ \lfloor n/2 \rfloor \end{bmatrix}_q.$$

3. **(2 points)** Prove the the following identities.

   (a) If n is a natural number and t a complex number, then
$$\prod_{j=0}^{n-1}(1+q^j t) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}} t^k.$$

   (b) If m, n, and k are non-negative integer numbers such that $m+n \geq k$, then
$$\begin{bmatrix} m+n \\ k \end{bmatrix}_q = \sum_{j=0}^{k} q^{(m-j)(k-j)} \begin{bmatrix} m \\ j \end{bmatrix}_q \begin{bmatrix} n \\ k-j \end{bmatrix}_q.$$

4. **(2 points)** Let $n = 2t+1 \geq 3$ be an odd integer. Define $\mathcal{P} = \mathbb{Z}_n \times \mathbb{Z}_3$. Consider the design with point set $\mathcal{P}$ whose blocks are triples of the form $\{(x,0),(x,1),(x,2)\}$ with $x \in \mathbb{Z}_n$ and all triples $\{(x,j),(y,j),\left(\frac{x+y}{2}, j+1\right)\}$ with $x \neq y \in \mathbb{Z}_n$ and $j \in \mathbb{Z}_3$. Show that this block design is a Steiner triple system on $6t+3$ points.

5. **(2 points)** Let $q = 6t+1$ be a prime power and let α be a primitive element of $\mathbb{F}_q$ (meaning that α generates the multiplicative group $\mathbb{F}_q^*$). For $0 \leq j < t$ and $\beta \in \mathbb{F}_q^*$, define the triple
$$B_{j,\beta} = \{\alpha^j + \beta, \alpha^{2t+j} + \beta, \alpha^{4t+j} + \beta\}.$$
   Show that the elements of $\mathbb{F}_q$ as points and the blocks $B_{j,\beta}$ form a Steiner triple system with q points.

Homework 3
Math 888 Combinatorics II
Due Monday, April 12, 2021.

To get 100% on this homework, you must solve correctly all 5 questions for a total of 10 points.

1. Let Γ be a primitive strongly regular graph with parameters (v, k, λ, μ). Let $k^{(1)} > r^{(f)} > s^{(g)}$ be the eigenvalues of Γ.
   (a) Prove that $fg = \frac{vk(v-k-1)}{(r-s)^2}$.
   (b) If v is prime, prove that Γ is a conference graph.

2. **(2 points)** A partial geometry $pg(K, R, T)$ is an incidence structure of points and lines with the following properties:
   (a) any two distinct points are contained in at most one line,
   (b) every line contains exactly K points,
   (c) every point is on exactly R lines,
   (d) for any point p and any line L such that $p \notin L$, there are exactly T lines that contain p and intersect L.

   The point graph Γ of a partial geometry $pg(K, R, T)$ is the graph whose vertices are the points of the geometry, where $x \neq y$ are adjacent if and only if they are collinear. Prove Γ is a strongly regular graph and determine its parameters (v, k, λ, μ) as functions of K, R and T.

3. **(2 points)** A partial geometry $pg(K, R, 1)$ is called a generalized quadrangle and is usually denoted by $GQ(K-1, R-1)$. Let q be a prime power and $H = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$.
   Let V be the 4-dimensional vector space over $\mathbb{F}_q$. A subspace W of V is called totally isotropic if $u^T H v = 0$ for any $u, v \in W$.
   (a) Prove that $u^T H u = 0$ for any $u \in V$ so every 1-dimensional subspace is totally isotropic.
   (b) Prove that there are $(q^2+1)(q+1)$ 2-dimensional subspaces of V (or lines in $PG(3, q)$) that are totally isotropic.
   (c) Let $W(q)$ the incidence structure whose points and lines are the totally isotropic points and totally isotropic lines of $PG(3, q)$. Prove that $W(q)$ is a generalized quadrangle $GQ(q, q)$.

4. **(2 points)** Call two triples from a fixed set of v points adjacent if they have exactly one common point and denote by Γ the resulting graph on the triples of the set of size v. Prove that Γ is strongly regular if $v \in \{5, 7, 10\}$ and determine its parameters.

5. **(2 points)** Let L be a Latin square of order n and denote by Γ the strongly regular graph of the corresponding $OA(3, n)$.

   (a) Prove that $\alpha(\Gamma) \leq n$ and $\chi(\Gamma) \geq n$.

   (b) If L is the multiplication table of a group, show that $\alpha(\Gamma) = n$ if and only if $\chi(\Gamma) = n$.

   (c) If L is the multiplication table of the cyclic group of order $2n$, prove that $\alpha(\Gamma) < 2n$.

Homework 4
Math 888 Combinatorics II
Due Wednesday, April 28, 2021.

To get 100% on this homework, you must solve correctly all 5 questions for a total of 10 points.

1. **(2 points)** Let n and d be two natural numbers such that $d \leq n/2$. The Johnson graph $J(n,d)$ has as vertices all the d-subsets of the set $[n] = \{1,\dots,n\}$ and adjacency is defined as follows:
$$A \sim B \Leftrightarrow |A \cap B| = d-1.$$
   (a) Let $0 \leq j \leq d$ be an integer. If A and B are two d-subsets of $[n]$, prove that the distance in $J(n,d)$ is j if and only if $|A \cap B| = d - j$.
   (b) Show that $J(n,d)$ is a distance-regular graph of diameter d and with intersection array parameters
$$b_j = (d-j)(n-d-j),\ c_j = j^2, 0 \leq j \leq d.$$

2. **(2 points)** Let n and d be two natural numbers such that $d \leq n/2$ and let q be a prime power. Let V be an n-dimensional vector space over $\mathbb{F}_q$. The Grassmann graph $J_q(n,d)$ has as its vertices the d-dimensional subspaces of V, where two d-subspaces are adjacent if and only if their intersection has dimension $d-1$.
   (a) Let $0 \leq j \leq d$ be an integer. If A and B are two d-subspaces of V, prove that the distance in $J_q(n,d)$ is j if and only if $\dim(A \cap B) = d - j$.
   (b) Prove that $J_q(n,d)$ is a distance-regular graph of diameter d with intersection array parameters
$$b_j = q^{2j+1} \begin{bmatrix} d-j \\ 1 \end{bmatrix}_q \begin{bmatrix} n-d-j \\ 1 \end{bmatrix}_q,\ c_j = \begin{bmatrix} j \\ 1 \end{bmatrix}_q^2, 0 \leq j \leq d.$$

3. **(2 points)** Let $k \geq 2$ be an integer. The Odd graph O_k has as vertices all the $(k-1)$-subsets of the set $[2k-1] = \{1,\dots,2k-1\}$ and adjacency is defined as follows:
$$A \sim B \Leftrightarrow A \cap B = \emptyset.$$
   (a) Show that O_k is k-regular and has girth 3 for $k=2$, girth 5 for $k=3$ and girth 6 when $k \geq 4$.
   (b) Show that O_k is a distance-regular graph with diameter $k-1$ and intersection array $\{b_0, b_1, \dots, b_{k-2}; c_1, c_2, \dots, c_{k-1}\}$, where
$$b_j = k - \lfloor \frac{j+1}{2} \rfloor, c_j = \lfloor \frac{j+1}{2} \rfloor$$

4. **(2 points)** Let $\mathcal{C}$ be a linear code over $\mathbb{F}_2$ with parity-check matrix

$$H = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

Determine the parameters n, k, d of $\mathcal{C}$ and a generator matrix of $\mathcal{C}$.

5. **(2 points)**

(a) Let q be a prime power, k a natural number and $a_1, \ldots, a_k \in \mathbb{F}_q$, not all zero. Consider the linear map $f : \mathbb{F}_q^k \to \mathbb{F}_q, f(x_1, \ldots, x_k) = a_1x_1 + \ldots + a_kx_k$. Show that every element of $\mathbb{F}_q$ is the image under f of exactly q^{k-1} vectors in $\mathbb{F}_q^k$.

(b) Let $\mathcal{C}$ be a linear $[n, k, d]$ code over $\mathbb{F}_q$ and let T be a $q^k \times n$ matrix whose rows are the codewords of $\mathcal{C}$. Show that each element of $\mathbb{F}_q$ appears in every non-zero column of T exactly q^{k-1} times.

(c) Show that every linear $[n, k, d]$ code over $\mathbb{F}_q$ satisfies the inequality:

$$d \leq \frac{n(q^k - q^{k-1})}{q^k - 1}.$$

(d) Let q be a prime power and m a natural number. Denote $n = \frac{q^m-1}{q-1}$. Let $\mathcal{C}$ be the dual of the $[n, n-m]$ Hamming code over $\mathbb{F}_q$. Show that the Hamming weight of each non-zero codeword in $\mathcal{C}$ is q^{m-1} and that the code $\mathcal{C}$ attains equality in the previous bound.

Homework 5
Math 888 Combinatorics II
Due Friday, May 14, 2021.

To get 100% on this homework, you must solve correctly all 5 questions for a total of 10 points.

1. **(2 points)** Prove that if there exists a $p \in (0,1)$ such that

$$\binom{n}{k} p^{\binom{k}{2}} + \binom{n}{\ell}(1-p)^{\binom{\ell}{2}} < 1,$$

then $R(k, \ell) > n$.

2. **(2 points)** For $r \geq 2$, an r-uniform hypergraph H is a pair (V, E) consisting of a finite set of vertices V and a finite set E of edges that are r-subsets of V. A 2-uniform is a graph. A Steiner triple system is an example of a 3-uniform hypergraph. Consider an r-uniform hypergraph with at most $4^{r-1}/3^r$ edges. Show that there is a coloring of the vertices of H by 4 colors such that in every edge all 4 colors are represented.

3. **(2 points)** Let $A_1, \ldots, A_m$ be k-subsets and $B_1, \ldots, B_m$ be ℓ-subsets of a finite set X such that

   (a) $A_i \cap B_i = \emptyset, \forall i \in [m]$,

   (b) $A_i \cap B_j \neq \emptyset, \forall i \neq j \in [m]$.

   Show that $m \leq \binom{k+\ell}{k}$.

4. **(2 points)** Let $\mathcal{C}$ be a finite collection of binary strings of finite lengths (at most some n) such that no member of $\mathcal{F}$ is a prefix of another one. For $j \geq 1$, let f_j denote the number of words of length j in $\mathcal{F}$. Show that

$$\sum_{j=1}^{n} \frac{f_j}{2^j} \leq 1.$$

5. **(2 points)** For $k \geq 1$, a tournament is called k-unrankable if for every set of k players, there is one player that beats them all. Show that for any k, there exists a tournament that is k-unrankable.

Exam 1
Math 888 Graduate Combinatorics II
Due Sunday, March 21, 2021.

1. **(2 points)** Let $n > k \geq 1$ be two natural numbers and q a prime power. Show that

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{\ell=0}^{k(n-k)} a_\ell q^\ell,$$

where a_ℓ is the number of partitions of ℓ whose Ferrers diagrams fit in a $k \times (n-k)$ rectangle.

2. **(2 points)** Let $n > k \geq 1$ be two natural numbers and q a prime power. Let W be a vector subspace of dimension k of a vector space V of dimension n over $\mathbb{F}_q$. Let $\ell \geq j \geq 1$ be two natural numbers. Show that the number of $\ell-$dimensional subspaces U of V such that $\dim(W \cap U) = j$ equals

$$q^{(k-j)(\ell-j)} \begin{bmatrix} n-k \\ \ell-j \end{bmatrix}_q \begin{bmatrix} k \\ j \end{bmatrix}_q.$$

3. **(2 points)** Let $r \geq 1$ be a natural number. Denote by N the $2r \times 2r$ block diagonal matrix with r diagonal blocks of the form $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Define the graph $Sp(2r)$ whose vertex set is the set of all non-zero vectors in $\mathbb{F}_2^{2r}$, with adjacency defined by $x \sim y$ if and only if $x^T N y = 1$. Show that $Sp(2r)$ is a strongly regular graph with parameters $(2^{2r}-1, 2^{2r-1}, 2^{2r-2}, 2^{2r-2})$.

4. **(2 points)** Let $v > k \geq 2$ be two natural numbers. Let $\mathcal{D}$ be a $2-(v,k,1)$ design. Define a graph Γ whose vertices are the blocks of $\mathcal{D}$ with adjacency defined as follows $B \sim B'$ if and only if $B \cap B' \neq \emptyset$. Prove that Γ is a strongly regular graph, determine its parameters (v,k,λ,μ) and its eigenvalues.

5. **(2 points)** Consider the graph on the set of flags (ordered pairs (p,L) of incident point-line pairs) of the projective plane $PG(2,4)$, where $(p,L) \sim (q,M)$ when $p \neq q, L \neq M$ and either $p \in M$ or $q \in L$. Show that this graph is strongly regular with parameters $(v,k,\lambda,\mu) = (105, 32, 4, 12)$.

Exam 2
Math 888 Graduate Combinatorics II
Due Wednesday, May 5, 2021.

1. **(2 points)** Let q be a prime power, m a natural number and $n = q^m$. The first order Reed-Muller code over $\mathbb{F}_q$ is defined as the $[n, m+1]$ linear code $\mathcal{C}$ with an $(m+1) \times n$ generator matrix whose columns range over all the vectors in $\mathbb{F}_q^{m+1}$ with a first entry equaling 1. Show that the minimum distance of $\mathcal{C}$ equals $q^m - q^{m-1}$ and that this number is the weight of $q^{m+1} - q$ codewords in $\mathcal{C}$. What are the weights of the remaining q codewords of $\mathcal{C}$? Show that no linear $[n, m+1]$ code over $\mathbb{F}_q$ can have minimum distance greater than $q^m - q^{m-1}$.

2. **(2 points)** For $m > 1$, let $\mathcal{C}$ be the $[n, n-m, 3]$ Hamming code over $\mathbb{F}_2$ where $n = 2^m - 1$. Show that the number of codewords of weight 3 in $\mathcal{C}$ is $n(n-1)/6$. How many codewords are there in $\mathcal{C}$ of weight $n-1$? $n-2$? and $n-3$?

3. **(2 points)** Let $\mathcal{C}$ be an (n, M, d) block code over the alphabet $\mathbb{F}$. Show that $M \leq q^{n-d+1}$. If there exist two orthogonal Latin squares L_1 and L_2 of order n over an alphabet $\mathbb{F}$ of order n, construct the code $\mathcal{C}$ whose codewords are the n^2 words of length 4 of the form:

$$(i, j, L_1(i,j), L_2(i,j))$$

for any $1 \leq i, j \leq n$. Show that this code has minimum distance 3 and attains equality in the previous bound.

4. **(2 points)** Let $\mathcal{C}$ be a linear $[n, k, d]$ code over $\mathbb{F}$. For $1 \leq j \leq n$, denote by $\mathcal{C}_j$ the code

$$\mathcal{C}_j = \{(c_1, c_2, \ldots, c_{j-1}, c_{j+1}, \ldots, c_n) : (c_1, c_2, \ldots, c_{j-1}, c_j, c_{j+1}, \ldots, c_n) \in \mathcal{C}\}.$$

The code $\mathcal{C}_j$ is said to be obtained by *puncturing* $\mathcal{C}$ at the j-th coordinate. Show that $\mathcal{C}_j$ is a $[n-1, k_j, d_j]$ linear code over $\mathbb{F}$ where $k_j \geq k-1$ and $d_j \geq d-1$. Prove that there are at least $n-k$ indices j for which $k_j = k$.

5. **(2 points)** Let G_{12} be the ternary code whose generator matrix is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 2 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 2 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 1 & 0 \end{bmatrix}$$

Show that G_{12} is a $[12, 6, 6]$ linear code. Prove that it may be punctured to get a $[11, 6, 5]$ linear code that is perfect. This is the ternary Golay code.

Exam 3
Math 888 Graduate Combinatorics II
Due Wednesday, May 26, 2021.

1. **(2 points)** Let A be a $m \times n$ matrix and B another matrix. The matrix

$$\begin{bmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B & \dots & a_{2n}B \\ \vdots & \vdots & & \vdots \\ a_{m1}B & a_{m2}B & \dots & a_{mn}B \end{bmatrix}$$

consisting of mn blocks of the size of B is called the Kronecker or tensor product $A \otimes B$ of A and B.

   (a) Prove that $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$.

   (b) Prove that $(A \otimes B)^T = A^T \otimes B^T$.

2. **(2 points)** Let $n \geq 2$ be an integer. A Hadamard matrix of order n is an $n \times n$ matrix H with entries $+1$ or -1 is called if $HH^T = nI_n$. A Hadamard matrix is called normalized if its first row is formed by all 1s.

   (a) Show if H is a Hadamard matrix of order n then $n \in \{1, 2\}$ or $n \equiv 0 \pmod 4$.

   (b) Show that if H is Hadamard matrix of order n and K is a Hadamard matrix of order m, then $H \otimes K$ is a Hadamard matrix of order nm.

   (c) Deduce that for any $k \geq 1$, there exists a Hadamard matrix of order 2^k.

3. **(2 points)** Determine a signed adjacency matrix whose largest eigenvalue is minimum for the complete graph K_4, complete bipartite graphs $K_{3,3}, K_{4,4}$ and the Petersen graph. You can use the computer for your computations.

4. **(2 points)** Determine the expansion/isoperimetric constant for the complete graph K_n, the complete bipartite graph $K_{n,n}$, the Petersen graph and the 3-dimensional cube Q_3.

5. **(2 points)** Let Γ be a connected bipartite d-regular graph with partite sets V_1 and V_2, each of size n. Denote by $d = \lambda_1 > \lambda_2 \geq \ldots \geq \lambda_{2n-1} > \lambda_{2n} = -d$ the eigenvalues of the adjacency matrix of Γ.

   (a) Determine an eigenvector of length 1 corresponding the eigenvalue $-d$.

   (b) If $n \geq 2$, show that $\lambda_2 \geq 0$.

   (c) Show that for any $X \subset V_1, Y \subset V_2$, with $|X| = x, |Y| = y$,

$$|e(X, Y) - \frac{dxy}{n}| \leq \lambda_2 \sqrt{xy(1 - x/n)(1 - y/n)}.$$