MATH210 Discrete Mathematics I
Spring 2019 Section 011

Lectures Mondays, Wednesdays, Fridays 12.20pm-1.10pm Gore 308.
Instructor Dr. Sebastian Cioabă, `cioaba@udel.edu`
Office Hours Mondays and Wednesdays 10-11am, Ewing 506 or by appointment.
TA Discussion Thursdays 12.20-1.10pm Purnell Hall 324A
Teaching Assistant Tianxiao Zhao, Ewing 116, `tianxiao@udel.edu`.
TA Office Hours Tue 1-3pm, Thu 1.30-3pm or by appointment.
Textbook *Discrete Mathematics with Graph Theory* (3rd Edition, Prentice Hall 2006) by E.G. Goodaire and M.M. Parmenter.
Topics Logic, Counting, Induction, Integers, Graph Theory, Algorithms.

Course Description

Discrete Mathematics is a beautiful subject full of interesting problems and applications and closely related to many areas of mathematics, computer science, engineering and even sociology. All of you have used web-search engines, shopped online using secure credit card payment systems, or have seen nice pictures sent from far-away planets. The interesting mathematics involved in such systems includes graph theory, number theory based cryptography and coding theory which are parts of discrete mathematics.

The objectives of this course are to learn the basics of discrete mathematics: **logic, counting, induction, integers, graph theory and algorithms, to rigorously prove mathematical statements and to write correct proofs of such mathematical statements**. The lectures will present the material in a different form and even in a different order from the textbook. This means that you will have two sources of learning available to you. You should make sure you read them both as this will enhance your understanding and enable you to follow future lectures in class.

Since we have to learn to crawl before we learn to walk before we learn to run, **it is really important that you put a consistent and strong effort into learning this material**. Discrete mathematics (like all mathematics) is a constructive subject building on earlier material. You should stay on top of the subject by spending some time on it every day, and not just cram for exams. Otherwise, this course will leave you behind. To prove that you have read the syllabus, please send your instructor an email titled kittens by Wednesday, February 13. Focus on understanding the material and key concepts and not on memorizing proofs. In teaching undergraduate courses, I have met students that had troubles studying mathematics. The best way to study mathematics is with a pen and paper in a quiet and relaxing space while all the electronics and other distractions are off. In case you need study/learning tips, I have found the book *Make it stick: The Science of Successful Learning* by Peter C. Brown, Henry L. Roediger III and Mark McDaniel, very useful and interesting. I highly recommend it! When studying in this class, remember that the labor market values

the highly analytical individual that can think abstractly. [1]

[1] Page 2 *Academically Adrift: Limited Learning on College Campuses* by R. Arrum and J. Roksa.

Grading Scheme

Your final grade will be calculated based on your attendance and performance in homework assignments and exams.

Attendance You are expected to attend every lecture, but I understand if there are events that make this difficult at times. I plan to assign 5% of your grade to attendance. Attending 30 lectures will get you the maximum of 5% of your grade.

Homework I will assign homework problems every week. Depending on various things, we will have about 10 homework assignments. The assignments will contribute 30% to your final grade.

Midterms I plan to have 3 exams roughly spaced one month apart throughout the semester: on Wednesday, March 6, on Friday, April 12 and on Friday, May 17. The exams will be held during class and each will last 50 minutes. Each exam will be worth 10% of your final grade and the total contribution of the exams will be 30% to your final grade.

Final Exam The final exam will be worth 35% of your final grade. It will be a 2 hours exam scheduled on Wednesday, May 29 between 1 and 3pm in Alison Hall 206.

The correspondence between the number grade and the letter grade is the following:

A(90-100), A-(85-90), B+(80-85), B(75-80), B-(70-75), C+(65-70),C(60-65),C-(55-60), D(50-55),F(less than 50).

Resources

If you need help, please come and see me during office hours (Mondays and Wednesdays 10-11am Ewing 506) or email me to schedule an appointment at a different time.

Before coming to office hours to ask questions regarding a certain problem, **please make a serious effort** (study the class material and the textbook and work on the problem at least 15 minutes) to solve the problem on your own and write down your ideas so that we can discuss them. Based on my experience teaching this course, you will need to study two hours for each lecture.

Please do not wait until later in the semester to ask for help.

You can also ask your TA, Tianxiao Zhao, for help.

The Office of Academic Enrichment `http://ae.udel.edu/` has various resources to help students (list of tutors, supplemental instruction, study skills workshop etc.).

Homework 1, Math 210
Due Friday, February 22, 2019

The solution of each exercise should be at most one page long. You should always show your work justifying your answer. If you can, try to write your solutions in LaTex. This will give you an extra 10%.

1. Show that $[(p \to q) \wedge (q \to r)] \to (p \to r)$ is a tautology.
2. Write down *the converse* and *the contrapositive* of each of the following implications:
   (a) $ab = 0 \to (a = 0) \vee (b = 0)$.
   (b) $(a \geq b) \wedge (b \geq a) \to (a = b)$.
3. Prove that an integer is odd if and only if its cube is odd.
4. Prove that the product of any two consecutive integers is even.
5. Prove that $\frac{x+y}{2} \geq \sqrt{xy}$ for any real non-negative numbers x and y. Show that equality happens if and only if $x = y$.

Homework 2, Math 210
Due Friday, March 1st, 2019

The solution of each exercise should be at most one page long. You should always show your work justifying your answer. If you can, try to write your solutions in LaTex. This will give you an extra 10%.

1. Is it true that for any natural number n, there exists a natural number k such that $k > 5n$? Is it true that there exists a natural number k such that $k > 5n$ for any natural number n ?

2. Let $A = \{1, 3, 6, 7, 8, 10\}, B = \{2, 4, 6, 8\}, C = \{1, 2, 3, 4, 5, 6, 7\}$.

   (a) Draw the Venn diagram of the sets A, B and C.

   (b) What is $(A \cup B) \cap C$?

   (c) What is $|(A \setminus C) \cap B|$?

   (d) How many subsets $D \subseteq C$ of C are there such that $1 \in D$ and $5 \notin D$?

3. Prove that $A \cap B = A$ if and only if $A \subseteq B$. Prove that $A \cup B = B$ if and only if $A \subseteq B$.

4. Prove that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. Prove that $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$.s

5. Let A, B and C be arbitrary sets. For each of the following statements, either prove the given statement is true or exhibit a counterexample to prove it is false.

   (a) $A \times B \subseteq C \times D \to A \subseteq C$ and $B \subseteq D$.

   (b) $A \setminus (B \cup C) = (A \setminus B) \cup (A \setminus C)$.

Homework 3, Math 210
Due Friday, Wednesday, March 13, 2019

The solution of each exercise should be at most one page long. You should always show your work justifying your answer. If you can, try to write your solutions in LaTex. This will give you an extra 10%.

1. Assume that U is a set and A_1, A_2, A_3 and A_4 are subsets of U such that

   - $|U| = 100$
   - $|A_i| = 40, \forall 1 \leq i \leq 4$
   - $|A_i \cap A_j| = 20, \forall 1 \leq i < j \leq 4$
   - $|A_i \cap A_j \cap A_k| = 10, \forall 1 \leq i < j < k \leq 4$
   - $|A_1 \cap A_2 \cap A_3 \cap A_4| = 5$.

   (a) How many elements in U belong to none of the four subsets ?

   (b) How many elements belong only to A_1 ?

   (c) How many elements belong exactly to A_1 and A_2 ?

   (d) How many elements belong exactly to A_1, A_2 and A_3 ?

2. How many numbers in the range $1000 - 9999$ have exactly two 4s ? How many such numbers are even ? How many numbers in the range $1000 - 9999$ have exactly two 0s ? How many such numbers are odd ?

3. How many integers between 1 and 500 are divisible by 3 and 5, but not by 2 ?

4. Use induction on n to prove that

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

   for any natural number n.

5. Use induction on n to prove that the set $\{1, \ldots, n\}$ contains exactly 2^n subsets for any natural number n.

Homework 4, Math 210
Due Friday, March 29, 2019

You should always show your work justifying your answer. If you can, try to write your solutions in LaTex. This will give you an extra 10%.

1. Using induction or a direct proof, prove the following identities

   (a) $1 + x + x^2 + \cdots + x^{n-1} = \frac{1-x^n}{1-x}$ for real number $x \neq 1$ and any natural number n.

   (b) $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1})$ for any real numbers a and b and any natural number n.

   (c) $1 - x + x^2 + \cdots - x^{2k-1} + x^{2k} = \frac{1+x^{2k+1}}{1+x}$ for any real number $x \neq -1$ and any natural number k.

   (d) $a^{2k+1} + b^{2k+1} = (a+b)(a^{2k} - a^{2k-1}b + a^{2k-2}b^2 - \cdots - ab^{2k-1} + b^{2k})$ for any real numbers a and b and any natural number k.

2. Suppose that the sum of the first n terms of an arithmetic progression is given by the formula $S_n = 4n^2 - 3n$ for every $n \geq 1$. Find the first three terms of the arithmetic progression and its difference.

3. A sequence $(a_n)_{n\geq 0}$ is defined recursively by $a_0 = 2, a_1 = 3$ and $a_n = 3a_{n-1} - 2a_{n-2}$ for $n \geq 2$.

   (a) Find the first five terms of this sequence.

   (b) Guess a formula or a_n.

   (c) Use strong induction to prove that your formula is correct.

   (d) Find a formula for a_n that involves only one preceding term.

4. Let $(a_n)_{n\geq 1}$ be an arithmetic progression with non-zero common difference. Let $(b_n)_{n\geq 1}$ be a geometric progression with a positive common ratio. Prove that there exists numbers x and y such that
$$b_n = xy^{a_n} \text{ for all } n \geq 1.$$
   [This exercise shows that arithmetic and geometric progressions are related!]

5. The following construction leads to a *fractal-like* object called the Koch snowflake. Start with an equilateral triangle K_1 have sides of length 1. Divide each side into 3 segments of equal length and build an equilateral triangle on the middle segment of each side as the base in the exterior of the original triangle. Then delete these 3 bases. We obtain a polygon with 12 sides, each of length 1/3. We continue by dividing each of the 12 sides of K_2 into 3 congruent segments, and building an equilateral triangle on the middle segment as the base in the exterior of K_2. Then we delete each of these 12 bases obtaining a polygon K_3 with 48 sides. Continue this procedure for K_3 to get K_4 and so on. The picture below shows K_1, K_2, K_3 and K_4.

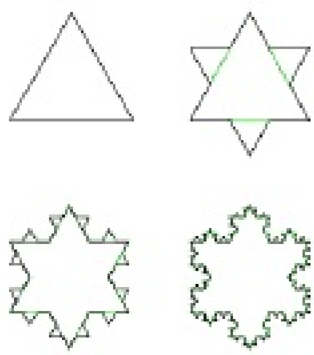


Figure 1: $K_i, i = 1, 2, 3, 4.$

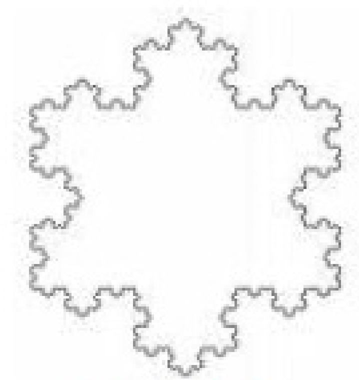


Figure 2: K_4

Figure 1:

For $n \geq 1$, compute the following:

(a) the number of sides of K_n.

(b) the perimeter p_n of K_n.

(c) the area a_n of K_n.

(d) What do the results above imply about p_n and a_n as n gets larger ?

Homework 5&6, Math 210
Due Monday, April 15, 2019

You should always show your work justifying your answer. If you can, try to write your solutions in LaTex. This will give you an extra 10%.

For the first 6 questions please include the answer in terms of binomial coefficients, exponentials or factorials as well as the numerical value, you may use your calculator to do that.

1. In how many ways can 12 people be divided in two teams of 6 for a game of hockey ? If there are two brothers among the 12 people, in how many ways can we divide the 12 people in two teams such that the brothers are on the same team ?

2. How many permutations of the lettters a, b, c, d, e, f, g are there ? How many such permutations do not contain the pattern abc (as consecutive letters) ? How many permutations do not contain the patterns abc nor def ? How many permutations do not contain the patterns abc nor cde ?

3. A coin is tossed 12 times. How many different sequences of heads and tails are possible ? In how many of these sequences are 4 heads ? In how many of these sequences are 4 tails ? In how many of these sequences we get at least 4 tails ?

4. An urn contains 7 red balls and 7 blue balls. A sample of 4 balls is extracted. How many different samples are possible ? How many samples contain exactly one red ball ? How many samples contain at least one red ball ? How many samples contain 2 red balls ?

5. What is the coefficient of x^{14} in the binomial expansion of $(x - 2x^{-2})^{20}$? What is the coefficient of x^{-15} is the same expansion ?

6. How many possible five-card hands can be dealt from a standard deck of 52 cards ? How many five-card hands are all of the same suit ? How many five-card hands have at least one pair ? How many five-card hands have two pairs ?

7. Let n be natural number. Give a combinatorial proof that $\binom{2n}{2} = 2\binom{n}{2} + n^2$.

8. Prove that $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ for any natural numbers $n \geq k$. Show that
$$\binom{n}{k} = \binom{n-1}{k-1} + \cdots + \binom{k}{k-1} + \binom{k-1}{k-1}.$$

9. Let $a \geq b \geq k$ be natural numbers. Prove that
$$\binom{a+b}{k} = \binom{a}{0}\binom{b}{k} + \binom{a}{1}\binom{b}{k-1} + \cdots + \binom{a}{k-1}\binom{b}{1} + \binom{a}{0}\binom{b}{k}.$$
Hint: Think about choosing a team of k people from a girls and b boys.

10. Prove that for any natural number $\binom{2n}{n} \geq \frac{2^{2n}}{n+1}$ for any natural number n. You may use induction on n.

Homework 7, Math 210
Due Friday, May 3, 2019

You should always show your work justifying your answer. If you can, try to write your solutions in LaTex. This will give you an extra 10%.

1. Let a and b be two integers and n be a natural number. Show that n divides $a - b$ if and only a and b have the same remainder when divided by n.

2. Given two natural numbers a and b, let $lcm(a, b)$ denote the smallest natural number that is divisible by both a and b. For example, $lcm(4, 6) = 12$. Show that $lcm(a, b) = \frac{ab}{gcd(a,b)}$.

3. Let p be a prime and $a_1, \ldots, a_n$ be natural numbers, where $n \geq 1$. Show that if p divides $a_1 \ldots a_n$, then p divides at least one of the numbers $a_1, \ldots, a_n$.

4. Let $n \geq 2$ be a natural number. Show that if $2^n - 1$ is a prime, then n must be a prime.

5. Let $n \geq 2$ be a natural number. Show that if $2^n + 1$ is a prime, then n must be a power of 2.

Homework 8, Math 210
Due Wednesday, May 15, 2019

You should always show your work justifying your answer. If you can, try to write your solutions in LaTex. This will give you an extra 10%.

1. In each of the following cases, find the greatest common divisor of a and b and express it in the form $ma + nb$ for suitable integers m and n.
   (a) $a = -93, b = 119$
   (b) $a = 1575, b = 231$

2. Find all integers x satisfying the following congruence:
$$155x \equiv 1185 \pmod{1404}$$

3. Find all integers $0 \le x, y < 15$ that satisfy each of the following pairs of congruences. If no x, y exists, explain why not.
$$7x + 2y \equiv 3 \pmod{15}$$
$$9x + 4y \equiv 6 \pmod{15}$$

4. Suppose the natural number N is $(a_{n-1} \dots a_0)_b$ (in base b). Prove that $n - 1 = \lfloor \log_b N \rfloor$ and hence that N has $1 + \lfloor \log_b N \rfloor$ digits in base b. How many digits does 7^{254} have in its base 10 representation ? Hint: $0.845 < \log_{10} 7 < 0.8451$.

5. Find all integers x such that
$$x \equiv 2 \pmod{3}$$
$$x \equiv 3 \pmod{4}$$
$$x \equiv 4 \pmod{5}.$$

Homework 9, Math 210
Due Wednesday, May 22, 2019.

You should always show your work justifying your answer. If you can, try to write your solutions in LaTex. This will give you an extra 10%.

1. What is the Prüfer code of the tree below ?

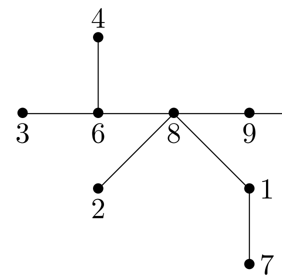
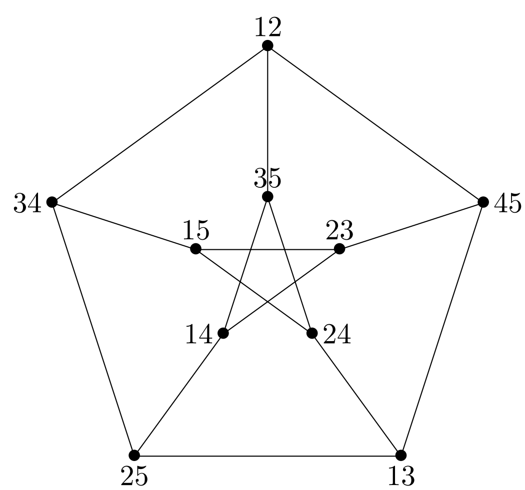


Figure 1: A tree on 9 vertices

2. The graph below is famous in graph theory and is called the Petersen graph. The vertex set is formed by all subsets of 2 elements (or 2-subsets) from the set $\{1, 2, 3, 4, 5\}$ and two 2-subsets are adjacent if and only if they are disjoint. To simplify things, I will write 12 to represent $\{1, 2\}$ so 12 is adjacent to 34 since $\{1, 2\} \cap \{3, 4\} = \emptyset$, but 12 is not adjacent to 14 since $\{1, 2\} \cap \{1, 4\} \neq \emptyset$. How many vertices and edges does the Petersen graph have ? Does it have any cycles of length 5 ? Does it have any cycles of length 6 ? What is the smallest number of colors that you can assign to its vertices such that any two adjacent vertices get different colors ?



3. Find the tree whose Prüfer code is $(1, 2, 3, 2, 2, 5)$?

4. Is there a graph with degree sequence $5, 4, 3, 2, 1, 1$? Is there a graph with degree sequence $3, 3, 3, 3, 2, 1, 1$? If yes, draw the graph. If no, prove there is no such graph.

5. Use induction on n to prove that any tree on $n \geq 2$ vertices has at least two vertices of degree 1 (a vertex of degree 1 is called a leaf).

Name:

Exam 1, Math 210 - Spring 2019

Duration: 50 minutes
Please turn off your cellphones and laptops, close your books and notebooks.
To get full credit you should explain your answers for questions 2-4.
Each question is worth 5 points so 20 points=100%.

1. [5 points] (TRUE/FALSE) For each of the following questions, determine whether it is true or false. You do not need to justify your answers. Each question is worth 1 point.

a) If 2019 is even, then 2020 is odd.

b) For any finite sets A and B, $|A \setminus B| = |A| - |B|$.

c) There exists a real number y such that $y > x$ for every real number x.

d) There are exactly 36 6-letter license plates formed only with the letters U and D.

e) For every real number x, there exists a real number y such that $y > x$.

#	Score
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Total	

2. [5 points] Let $A = \{1, 2, 3, 4, 5\}, B = \{2, 3, 5, 7, 9, 11\}, C = \{4, 8\}$.

1. Draw the Venn diagram for these 3 sets.
2. List all the subsets D of A such that $\{1, 2\} \subseteq D$ and $3 \notin D$.
3. List all subsets E of $A \cap B$ such that $3 \in E$.
4. Determine $|(A \cup C) \setminus B|$.
5. Determine $|A \times B \times C|$.

3. a) [3 points] How many integers between 100 and 999 are divisible by 2 or 3 or 5 ? How many integers between 100 and 999 are divisible by 5, but not by 2 nor by 3 ? How many integers between 100 and 999 are divisible by 2 and 5, but not by 3 ?

b) [2 points] In how many different ways can we arrange 5 people on a line ? In how many different ways can we arrange 5 people at a circular table ?

4. a) [3 points] How many numbers in the range $2000 - 8999$ ddo not have any repeated digits? How many of them are odd ? How many of them are even and do not contain the digit 4 ?

b) [2 points] Let $n \geq 2$ be a natural number. How many subsets does $\{1, \ldots, n\}$ have ? How many such subsets A have the property that $1 \in A$, but $2 \notin A$?

Name:

Exam 2, Math 210 - Spring 2019

Duration: 50 minutes
Please turn off your cellphones and laptops, close your books and notebooks.
To get full credit you should explain your answers for questions 2-4.
Each question is worth 5 points so 20 points=100%.

1. [5 points] (TRUE/FALSE) For each of the following questions, determine whether it is true or false. You do not need to justify your answers. Each question is worth 1 point.

a) $1 + 2 + \cdots + 39 + 40 = 780$.

b) $\binom{11}{1} + \binom{11}{3} + \binom{11}{5} + \binom{11}{7} + \binom{11}{9} + \binom{11}{11} = 1024$.

c) There are exactly 36 ways to choose 2 people from a group of 6.

d) A function $f : A \to B$ is injective if $a = b$ implies $f(a) = f(b)$ for any $a, b \in A$.

e) A function $g : A \to B$ is not surjective if there exists $y \in B$ such that for any $x \in A, y \neq f(x)$.

#	Score
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2. a) [3 points] Use induction on n to prove that

$$\sum_{k=1}^{n} k2^{k-1} = (n-1)2^n + 1$$

for any $n \geq 1$.

b) [2 points] Use strong induction on n to prove that any integer $n \geq 28$ can be written as

$$n = 8a + 5b$$

for some non-negative integers a and b (that depend on n, of course).

3. a) [3 points] Consider the following sequence $(a_n)_{n\geq 1}$ defined recursively as follows: $a_0 = 0, a_1 = 1$ and

$$a_{k+1} = 3a_k - 2a_{k-1}$$

for any $k \geq 1$. Use the characteristic equation of the recurrence and find the value of a_k for any $k \geq 0$.

b) [2 points] Prove that $\binom{2019}{1001} = \binom{2018}{1001} + \binom{2018}{1000}$. Determine the coefficient of x^{14} in the expansion of $\left(\frac{3}{x} - x^2\right)^{10}$.

4. a) [3 points] An urn contains 6 red balls and 4 blue balls. A sample of 4 balls is selected. How many different samples are possible ? How many samples contain exactly 3 red balls ? How many samples contain at least 3 red balls ?

b) [2 points] Consider a circular pizza and for $n \geq 1$, denote by $P(n)$ the maximum number of pieces of pizza we can make with n straight cuts ? For example, $P(0) = 1, P(1) = 2$ and $P(2) = 4$. Determine $P(3)$ and $P(4)$ (you can a drawing of a pizza with 3 or 4 cuts for that). Can you find a formula for $P(n)$ for general n ? Hint: Consider the difference $P(n) - P(n-1)$ for $n = 1, 2, 3, ...$

Name:

Exam 3, Math 210 - Spring 2019

Duration: 50 minutes
Please turn off your cellphones and laptops, close your books and notebooks.
To get full credit you should explain your answers for questions 2-4.
Each question is worth 5 points so 20 points=100%.

1. [5 points] (TRUE/FALSE) For each of the following questions, determine whether it is true or false. You do not need to justify your answers. Each question is worth 1 point.

a) There are infinitely many integers a, b, c such that $a^2 + b^2 = c^2$.

b) There exists an integer n such that $n^2 \equiv 2 \pmod 4$.

c) A function $f : A \to B$ is not surjective if there exists $y \in B$ such that $f(x) \neq y$ for any $x \in A$.

d) There exists a graph (without loops nor multiple edges) with 10 vertices and 46 edges.

e) A function $f : A \to B$ is injective if for any $a, b \in A$, $a \neq b$ implies that $f(a) \neq f(b)$.

#	Score
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2. a) [3 points] Consider the function $f : [5, +\infty) \to (-\infty, 3]$, $f(x) = 3 - \sqrt{x - 5}$. Prove that f is injective. Prove that f is surjective. Determine the inverse function f^{-1} of f. What are the domain and co-domain of f^{-1} ?

b) [2 points] Let $g : \mathbb{Z} \to \mathbb{N}$, $g(n) = |n| + 1$ and $h : \mathbb{N} \to \mathbb{Z}$, $h(n) = n(-1)^n + 4$. Determine $g \circ h$ and $h \circ g$ (including their domains and co-domains).

3. a) [3 points] Find all integers x such that $0 \leq x \leq 100$ and $12x \equiv 4 \pmod{22}$.

b) [2 points] Find all integers $-50 \leq x \leq 50$ such that

$$x \equiv 2 \pmod{3}$$
$$x \equiv 3 \pmod{4}.$$

4. a) [3 points] Is there a graph with degree sequence $5, 5, 5, 3, 2, 2, 1$? Is there a graph with degree sequence $6, 5, 4, 3, 2, 1, 1$? If yes, draw the graph. If no, explain why. Find the tree whose Prüfer code is $(4, 4, 3, 3, 2, 7)$.

b) [2 points] Find the last two digits of 7^{2006}. Find the last two digits of 4^{2006}.

Name:

Final Exam, Math 210 - Spring 2019

Duration: 2 hours
Please turn off your cellphones and laptops, close your books and notebooks.
To get full credit you should explain your answers for questions 2-8.
Each question is worth 5 points so 40 points=100%.

1. [5 points] (TRUE/FALSE) For each of the following questions, determine whether it is true or false. You do not need to justify your answers. Each question is worth 1 point.

a) A function $f : A \to B$ is surjective if for any $x \in A$, there is $y \in B$ such that $y = f(x)$.

b) A function $f : A \to B$ is injective if for any $a, b \in A$, $f(a) \neq f(b)$ implies that $a \neq b$.

c) If A, B and C are sets such that $A \setminus B = A \setminus C$, then $B = C$.

d) If n is an integer, then n^2 cannot equal $2 \pmod 4$.

e) $\binom{6}{0}2^6 - \binom{6}{1}2^5 + \binom{6}{2}2^4 - \binom{6}{3}2^3 + \binom{6}{4}2^2 - \binom{6}{5}2^1 + \binom{6}{6}2^0 = -1$.

#	Score
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Total	

2. a) [3 points] Prove by mathematical induction that

$$1^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

for any natural number n.

b) [2 points] A store sells stamps in packages of 5 and packages of 8 and you want to buy n stamps. Prove that for every natural number $n \geq 28$, this store can sell you exactly n stamps.

3. a) [3 points] How many non-empty subsets does the set $\{1, 2, 3, 4, 5, 6, 7\}$ have ? How many subsets with 2 elements does the set $\{1, 2, 3, 4, 5, 6, 7\}$ have ? What is the sum of the elements of all the subsets of 2 elements of $\{1, 2, 3, 4, 5, 6, 7\}$?

b) [2 points] Use induction on n to prove that

$$(1 + x)^n \geq 1 + nx$$

for any $x \geq -1$ and any natural number n.

4. a) [3 points] Let $f : \mathbb{N} \to \mathbb{Z}$, $f(n) = n \cdot (-1)^n$ and $g : \mathbb{Z} \to \mathbb{N} = |n| + 1$. Prove f is injective, but not surjective. Prove that g is surjective, but not injective. Determine $(f \circ g)(-4)$ and $(g \circ f)(3)$.

b) [2 points] Let $f : A \to B$ and $g : B \to C$ be two functions.
If $g \circ f$ is injective and f is surjective, then prove that g is injective.
If f is injective and g is injective, then prove that $g \circ f : A \to C$ is injective.

5. a) [3 points] Let $f : \mathbb{R} \to \mathbb{R}$ defined as

$$f(x) = \begin{cases} x + 3 & \text{if } x \leq 2 \\ 5 + \sqrt{x-2} & \text{if } x > 2. \end{cases}$$

Prove that f is injective and surjective. Determine the inverse f^{-1} of f (including its domain and co-domain).

b) [2 points] How many integers between 1 and 300 are divisible by 3, but not divisible by 2 nor by 5 ?

6. a) [3 points] There are 5 blue fish and 3 red fish in a bowl. A sample of 5 fish is selected. How many samples are possible ? How many such samples of 5 fish contain exactly 2 red fish ? How many samples of 5 fish contain at least 3 blue fish ?

b) [2 points] Prove that if $d|ab$ and $\gcd(d, b) = 1$, then prove that $d|a$. Show that if p is a prime and $p|ab$, then $p|a$ or $p|b$.

7. a) [3 points] For $n \geq 0$, the $(n+1)$-th row of the Pascal triangle consists of the $n+1$ integers

$$\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n-1}, \binom{n}{n}.$$

Prove that each row of the Pascal triangle is a palindrome, namely that $\binom{n}{k} = \binom{n}{n-k}$ for every integers $n \geq k \geq 0$. Show that $\binom{n}{k} = \binom{n}{k-1} + \binom{n-1}{k-1}$. What is the sum of the entries in the $(n+1)$-th row of the Pascal triangle ? Explain your answers.

b) [2 points] Find $\gcd(153, 42)$ and determine one pair of integers (m, n) such that

$$\gcd(153, 42) = 153m + 42n.$$

8. a) [3 points] What is the maximum number of edges in a graph (with no loops nor multiple edges) on 7 vertices ? For $k \in \{6, 10, 20\}$, draw a connected graph with 7 vertices and exactly k edges. You need to draw 3 graphs.

b) [2 points] Find the Prüfer code of the tree with vertex set $\{1, 2, 3, 4, 5, 6\}$ whose edges are $12, 23, 24, 35, 56$. Find the tree whose Prüfer code is $(2, 1, 2, 3, 5)$.