Department of Mathematical Sciences
University of Delaware

M210 DISCRETE MATHEMATICS
Graph Theory Group Work

NAME:	Solutions

Instructions

1. You are to work in groups of two or three. You will turn in one worksheet for the group.
2. This will count for **two** quiz grades.
3. Any proofs from class can be assumed. Remember, error on the side of caution. If you are unsure if you can assume something - put an appendix of the proof if there is time.
4. You have a stack of blank paper. Label the pages with the question you are answering. Give each question its own page for the sake of ordering. Make sure your final answers are legible and you have shown your work. (Give me some reasoning for your answers so even if you are wrong I can give you partial credit.

Problem 1

Show that every simple graph has two vertices of the same degree.

We will use the pigeonhole principle. Assume that a graph has n vertices. Each of those vertices can have degree $0, 1, \ldots, n-1$. If any vertex has degree $n-1$ then it is adjacent to every other vertex. Hence, no other vertex can have degree 0. So, there are only $n-1$ possibilities for the degrees and n vertices. Hence, by the pigeonhole principle at least two have to have the same degree.

Problem 2

Show that if n people attend a party and shake hands with others (but not with themselves), then at the end, there are at least two people who shook hands with the same number of people.

This problem is equivalent to problem one. We can draw a graph were the vertices represent the people and the edge represent the handshakes. We know from problem one that two vertices have the same degree which means that two people shook hands with the same number of people.

Problem 3

Prove that a complete graph with n vertices contains $\dfrac{n(n-1)}{2}$ edges.
By definition of a complete graph we know that the degree of every vertex in a complete graph is $n-1$. From the handshake lemma we know that $2|E| = \sum_{v \in V} \deg(v) = n(n-1)$ because there are n vertices each with degree $n-1$. So $|E| = \dfrac{n(n-1)}{2}$.

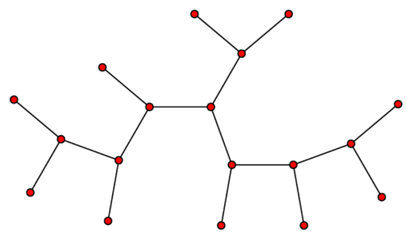
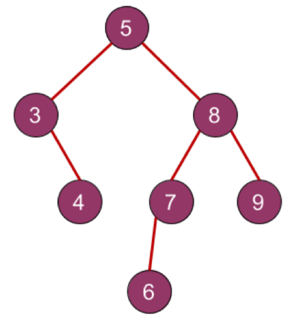
Problem 4

Prove that if u is a vertex of odd degree in a connected graph then there exists a path from u to another vertex v of the graph that also has odd degree.

From class we know that there are an even number of vertices of odd degrees which means if we have a vertex of odd degree then we have at least two. By definition of connected there is a path between any two vertices of the graph. This means that there is a path between the two vertices of odd degree.

Problem 5

A tree is a graph that starts a one vertex (the root) and all the vertices adjacent to the root are not adjacent to each other. You can continue this process to make as many branches as possible. Another definition is a graph that has no cycles. Here are some examples:

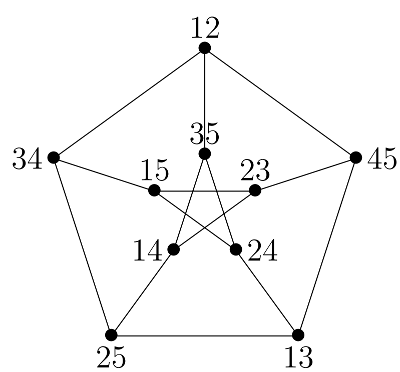



Show that any tree with at least two vertices is bipartite.

Show that a graph is bipartite if and only if it has no odd cycles. (This was a practice problem I assigned). Since a tree by definition has no cycles then there are no odd cycles then by the previous statement the tree is a bipartite graph.

Problem 6

The Petersen graph is famous in graph theory. Below is a picture of it. The vertex set is formed by all subsets of 2 elements (or 2-subsets) from the set $\{1, 2, 3, 4, 5\}$ and two 2-subsets are adjacent if and only if they are disjoint. To simplify things, I will write 12 to represent $\{1, 2\}$ so 12 is adjacent to 34 since $\{1, 2\} \cap \{3, 4\} = \emptyset$, but 12 is not adjacent to 14 since $\{1, 2\} \cap \{1, 4\} \neq \emptyset$. How many vertices and edges does the Petersen graph have ? **You may not just count them. You must reason it out by how they are defined.** Does it have any cycles of length 5 ? Does it have any cycles of length 6 ? What is the smallest number of colors that you can assign to its vertices such that any two adjacent vertices get different colors ?



The Petersen graph has $\binom{5}{2} = 10$ vertices. Each vertex has 3 neighbors because there is $\binom{3}{2}$ ways to choose a 2-subset that is disjoint from the current one. Therefore the number of edges is $\dfrac{10 * 3}{2} = 15$. The Petersen graph contains cycles of length 5 (like 12,45,23,15,34, 12) and cycles of length 6 (like 12,45,23,14,25,34,12).

Since the Petersen graph contains a 5 cycle we need at least 3 colors to color the vertices. Now color 12, 13, 14, 15 with color blue, 23,24,25 with color red, and 34,35,45 with color gold and we have a coloring of three colors. (There are more ways to color the graph.)

Problem 7

For each of the following degree sequences, prove that a simple graph with 7 vertices exists or does not exist by either drawing such a graph or showing mathematically that such a graph is impossible.

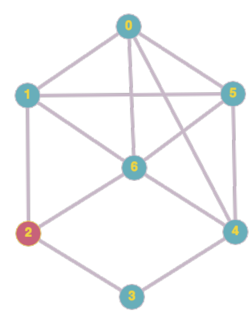
1. 1,1,2,3,4,5,6.
2. 3,3,3,3,3,3,6.
3. 2,3,4,4,4,4,5.

We are told the graphs must be simple graphs, so there are no loops or repeated edges.

(a) The maximum degree of any vertex is 6 (connected to every other vertex). Suppose there was a graph with this degree sequence. Then the degree 6 vertex is connected to every other vertex – in particular the 2 pendant vertices. If we removed this maximum degree vertex and all edges associated with it we would get a degree sequence $0, 0, 1, 2, 3, 4$. We can ignore the two isolated vertices and just think about the sequence $1, 2, 3, 4$. Since the maximum degree of any vertex in a simple graph containing 4 vertices is 3, this degree sequence is impossible. It follows that $0, 0, 1, 2, 3, 4$ is impossible, and so in turn is $1, 1, 2, 3, 4, 5, 6$. Hence no simple graph with degree sequence $1, 1, 2, 3, 4, 5, 6$ exists.

(b) This is the degree sequence of the graph W_6.

(c) Here is one. There are more.



Problem 8

A simple graph is called r-*regular* if every vertex has degree r. Let G be a r-regular graph on n vertices with e edges. Prove one of n or r must divide e.

Using the Handshaking theorem, we have

$$2e = \sum_{v \in V} deg(v) = nr.$$

It follows that one of n or r must be even. If n is even, then we can write $n = 2k$ for some integer k and then $e = kr$, so that $r|e$. If r is even, then the same argument just used proves $n|e$. In either case, one of n or r divides e, as required.

Problem 9

Determine exactly when a complete bipartite graph $K_{m,n}$ has an Euler path or Euler circuit.

To have an Euler circuit all the vertices must have even degree. This can only happen when both m and n are even. So $K_{m,n}$ has an Euler circuit if and only if m and n are even.

These are all trivially also Euler paths. It remains to see if there are any complete bipartite graphs admitting Euler paths and no Euler circuits. By Euler's result we require exactly two odd degree vertices. Suppose only one of m or n is odd (without loss of generality we can say its n). Since each vertex in one set is connected to every vertex in the other, the m vertices in one set all have degree n, while the n vertices in the other set all have degree m. So we must have $m = 2$ and n is odd (or the reverse), otherwise we would have m ore than two vertices of odd degree. Now suppose both m and n are odd, then all $m + n$ vertices in the graph have odd degree. For there to be an Euler path, exactly 2 vertices have odd degree, so $m + n = 2$. Hence $m = n = 1$ and it is clear $K_{1,1}$ has an Euler path.

Problem 10

Consider a graph G and let R be a relation on the vertices of G, V, defined by uRv if $u = v$ or there is a path from u to v. Show that R is an equivalence relation on V.

Reflexive: By the definition of the relation $u = v$. So it is reflexive.
Symmetric: If uRv then there is a path from u to v since take the path backwards and you have a path from v to u. So, vRu.
Transitive: Suppose uRv and vRw. Then there is a path from u to v and from v to w. So, connect the two paths and you get a path from u to w. So, uRw.

Hence, this is an equivalence relation.