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Compatible tight Riesz orders on groups of integer-valued functions

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A construction due to Reilly is extended to show that there is a correspondence between compatible tight Riesz orders on Z^X and non-principal filters on X. The maximal compatible tight Riesz orders are in one-to-one correspondence with non-principal ultrafilters, and are dual prime subsets of the positive set of Z^X. Conversely every dual prime algebraic Riesz order is maximal.

The lattice-ordered group Z^X of all functions defined on the set X and taking values in the totally-ordered group of rational integers Z admits no compatible tight Riesz order when X is finite. This can be seen by induction or, more conceptually the fact that Z^X then has no order-dense homomorphic image, and also from the fact that when X is finite all ultrafilters on X are principal. When X is infinite, however, there are compatible tight Riesz orders on Z^X : in the countably infinite case Reilly [4] has a construction that gives a compatible tight Riesz order for each non-principal ultrafilter on X.

For the definition of a compatible tight Riesz order on a lattice-ordered group see Wirth [5], where the following characterization occurs: a subset T of an abelian lattice-ordered group $(G, \preccurlyeq)$ is the strict positive set for a compatible tight Riesz order on G if and only if the following three conditions are satisfied:

(1) T is a proper dual ideal of $G^+ = \{g \in G : g \succcurlyeq 0\}$;

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(2) $T = T + T$;

(3) if $0 \preccurlyeq nx \preccurlyeq y$ for all $y \in T$, for all integers $n \geq 1$, then $x = 0$.

Throughout this paper a compatible tight Riesz order on Z^X will be identified with its strict positive set.

The positive set of Z^X (namely, the set of $f \in Z^X$ satisfying $f(x) \geq 0$ for all $x \in X$) is denoted by Z_+^X , and that of Z by Z_+ .

We define the mapping $\phi : Z^X \times Z \to Z^X$ by $\phi(f, m) = (|f|-m) \vee 0$, where $|f|$ is the function $x \to |f(x)|$ and m also denotes the constant function whose value at each point of X is $m \in Z$. For each $f \in Z^X$ the *zero set* of f is $Z(f) = \{x \in X : f(x) = 0\}$. The complement $X \backslash Z(f)$ of the zero set of f is the *support* of f , denoted by $\operatorname{supp}(f)$.

LEMMA 1. *If* $(f, m), (g, n) \in Z_+^X \times Z_+$ *then*

$$\phi(f, m) \wedge \phi(g, n) \geq \phi(f \wedge g, \max(m, n))$$

with equality if $m = n$.

Proof. For $(f, m), (g, n) \in Z_+^X \times Z_+$,

$$\begin{aligned}\phi(f, m) \wedge \phi(g, n) &= ((f-m)\vee 0) \wedge ((g-n)\vee 0) = ((f-m)\wedge(g-n)) \vee 0 \\ &\geq ((f-p)\wedge(g-p)) \vee 0 = ((f\wedge g)-p) \vee 0 = \phi(f\wedge g, p)\end{aligned}$$

where $p = \max(m, n)$. If $m = n$ then

$$\phi(f, n) \wedge \phi(g, n) = ((f-n)\wedge(g-n)) \vee 0 = ((f\wedge g)-n) \vee 0 = \phi(f\wedge g, n) .$$

In the following result the term "adjunction" is used in the sense of Mac Lane [3] (and in preference to the equivalent "dual Galois correspondence").

THEOREM 2. *There is an adjunction* $\alpha \vdash \beta$ *from the set of compatible tight Riesz orders on* Z^X *(ordered by inclusion) to the set of non-principal filters on* X *(ordered by inclusion).*

Proof. If F is a non-principal filter on X then $\beta(F)$ is defined to be the set $\left\{f \in Z_+^X : \operatorname{supp}\phi(f, m) \in F \text{ for all } m \in Z_+\right\}$. Since

$\text{supp}(0, 0)$ is empty the set $\beta(F)$ is properly contained in Z_+^X. Since F is non-principal there is a sequence $S_{-1}, S_0, S_1, \ldots$ of subsets $S_i \in F$ with $S_{-1} = X$, S_{i+1} properly contained in S_i, and $\bigcap_{i=0}^{\infty} S_i = \square$. If $f : X \to Z$ is defined by $f(x) = n$ if $x \in S_{n-1} \backslash S_n$, $n \geq 0$, then f is everywhere defined and

$$\text{supp}\phi(f, m) = \{x \in X : f(x) > m\} = \bigcup_{i=m-1}^{\infty} S_i \in F$$

for all $m \in Z_+$, so that $\beta(F)$ is not empty.

If $f \in \beta(F)$ and $g \geq f$ then for each $m \in Z_+$, $\text{supp}\phi(g, m) \supseteq \text{supp}\phi(f, m) \in F$ so $\text{supp}\phi(g, m) \in F$ and $g \in \beta(F)$. If $f, g \in \beta(F)$ and $m \in Z_+$ then

$$\text{supp}\phi(f \wedge g, m) = \text{supp}\phi(f, m) \wedge \phi(g, m) = \text{supp}\phi(f, m) \cap \text{supp}\phi(g, m) \in F$$

so $f \wedge g \in \beta(F)$. That is, $\beta(F)$ is a proper dual ideal of Z_+^X.

The criterion (3) for $\beta(F)$ to be a compatible tight Riesz order is satisfied since Z^X is archimedean. It remains to see that $\beta(F) = \beta(F) + \beta(F)$. One inclusion is immediate since

$$\text{supp}\phi(f+g, m) \geq \text{supp}\phi(f, m) \cap \text{supp}\phi(g, m)$$

for all $f, g \in Z_+^X$ and all $m \in Z_+$. If, on the other hand, $f \in \beta(F)$ then we define $g \in Z^X$ by $g(x) = [f(x)/2] + 1$, where, for a rational number ξ, $[\xi]$ is the integral part of ξ. Suppose that $m \in Z_+$ and $m \geq 2$. If $x \in \text{supp}\phi(f, 2m-2)$ then $f(x) > 2m - 2$ so that $f(x)/2 > m - 1$ and $[f(x)/2] \geq m - 1$. In this case $g(x) \geq m > m - 1$ sc that $\text{supp}\phi(f, 2m-2) \subseteq \text{supp}\phi(g, m-1)$ and, since F is a filter, $\text{supp}\phi(g, m-1) \in F$ for all $m \geq 2$. Then $\text{supp}(g, 0) \supseteq \text{supp}(g, 1) \in F$ so that $g \in \beta(F)$. Now we have to see that $h = f - g \in \beta(F)$. It follows, as above, that $\text{supp}\phi(f, 2m+2) \subseteq \text{supp}\phi(h, m-1)$ for $m \geq 1$ so that $h \in \beta(F)$. This establishes $\beta(F)$ as a compatible tight Riesz order on

Z^X .

Suppose that T is a compatible tight Riesz order on Z^X and $\alpha(T) = \{S \subseteq X : S \supseteq \text{supp}\phi(f, m) \text{ for some } f \in T, m \in Z_+\}$. Then $\alpha(T)$ is a filter on X since

$$\text{supp}\phi(f, m) \cap \text{supp}\phi(g, n) = \text{supp}\phi(f, m) \wedge \phi(g, n) \supseteq \text{supp}\phi(f \wedge g, \max(m, n))$$

for $f, g \in Z_+^X$ and $m, n \in Z_+$. If $\alpha(T)$ is a principal filter then there is an $x \in X$ such that $f(x) > m$ for all $m \in Z_+$, which is absurd.

Finally we see that the mappings α, β , which are clearly order-preserving, provide us with an adjunction. Suppose that $\alpha(T) \subseteq F$, where T is a compatible tight Riesz order and F is a non-principal filter on X . If $f \in T$ then $\text{supp}\phi(f, m) \in \alpha(T)$ for all $m \in Z_+$ by definition of $\alpha(T)$, so $\text{supp}\phi(f, m) \in F$ for all $m \in Z_+$. That is, $f \in \beta(F)$. On the other hand, suppose that $T \subseteq \beta(F)$. If $S \in \alpha(T)$ then $S \supseteq \text{supp}\phi(f, m)$ for some $f \in T$, $m \in Z_+$. Since $f \in \beta(F)$ we have $\text{supp}\phi(f, n) \in F$ for all $n \in Z_+$. In particular, $S \in F$ so we have $\alpha(T) \subseteq F$ if and only if $T \subseteq \beta(F)$.

We shall assume that the adjunction $\alpha \vdash \beta$ between the set of compatible tight Riesz orders on Z^X and the set of non-principal filters on X is the one described in Theorem 2.

DEFINITION 3. A compatible tight Riesz order T on Z^X is *prime* if for all $f, g \in Z_+^X$, $f \vee g \in T$ implies $f \in T$ or $g \in T$. Further we say that T is *algebraic* if $T = \beta\alpha(T)$ (of course, $T \subseteq \beta\alpha(T)$ in any case).

THEOREM 4. *There is a one-to-one correspondence between non-principal ultrafilters on* X *and maximal compatible tight Riesz orders on* Z^X . *In particular, every maximal compatible tight Riesz order on* Z^X *is of the form* $Z_+^X \backslash P_U$ *where* P_U *is a non-minimal prime subgroup of* Z^X *defined in terms of the non-principal ultrafilter* U , *so every maximal compatible tight Riesz order is algebraic and prime. Conversely every* *prime algebraic compatible tight Riesz order is maximal.*

Proof. The one-to-one correspondence between ultrafilters on X and maximal compatible tight Riesz orders follows immediately from the existence of the adjunction $\alpha \vdash \beta$ so does the fact that maximal compatible tight Riesz orders are algebraic. If $\mathcal{U}$ is a non-principal ultrafilter on X then $P = \left\{f \in \mathbb{Z}^X : Z\phi(f, m) \in \mathcal{U} \text{ for some } m \in \mathbb{Z}_+\right\}$ is a proper convex sublattice subgroup of $\mathbb{Z}^X$ which is prime but not minimal prime since $P_{\mathcal{U}}$ properly contains the prime subgroup $P_0 = \{f \in \mathbb{Z}^X : Z(f) \in \mathcal{U}\}$. Then

$$\beta(\mathcal{U}) = \left\{f \in \mathbb{Z}_+^X : \operatorname{supp}\phi(f, m) \in \mathcal{U} \text{ for all } m \in \mathbb{Z}_+\right\} = \mathbb{Z}_+^X \backslash P_{\mathcal{U}}$$

and $\beta(\mathcal{U})$ is prime since $P_{\mathcal{U}}$ is a join sublattice of $\mathbb{Z}^X$.

Suppose conversely that T is a prime algebraic compatible tight Riesz order on $\mathbb{Z}^X$. We shall see that $\alpha(T)$ is a prime, and therefore maximal, filter on X . It is sufficient to show that if $A \cup B \in \alpha(T)$, where $A, B \subseteq X$ and $A \cap B = \square$, then $A \in \alpha(T)$ or $B \in \alpha(T)$. Suppose $A \cup B \supseteq \operatorname{supp}\phi(f, m)$ for some $f \in T$, $m \in \mathbb{Z}_+$, but $A, B \notin \alpha(T)$. Then the sets $A' = \{x \in A : f(x) > m\}$, $B' = \{x \in B : f(x) > m\}$ are non-empty. We define $g, h : X \to \mathbb{Z}$ as follows:

$$g(x) = \begin{cases} 0 & \text{if } x \in A' \\ f(x) & \text{if } x \notin A' \end{cases}, \quad h(x) = \begin{cases} 0 & \text{if } x \in B' \\ f(x) & \text{if } x \notin B' \end{cases}.$$

Then

$$\operatorname{supp}\phi(g, m) = \{x \in X : g(x) > m\} = B' \subseteq B$$

and

$$\operatorname{supp}\phi(h, m) = \{x \in X : h(x) > m\} = A' \subseteq A .$$

Further, $g \vee h = f \in T$ so either $g \in T$ or $h \in T$ $\left(\text{since } g, h \in \mathbb{Z}_+^X\right)$. Thus $B \in \alpha(T)$ or $A \in \alpha(T)$. Since $\alpha(T)$ is a prime filter on X it is an ultrafilter, and $T = \beta\alpha(T)$ is a maximal compatible tight Riesz order.

COROLLARY 5. *If* X *has cardinality* $k \geq \aleph_0$ *then there are* 2^{2^k} *maximal compatible tight Riesz orders on* Z^X .

Proof. There are 2^{2^k} non-principal ultrafilters on Z^X (Bell and Slomson [1], p. 108).

COROLLARY 6. *If* T *is an algebraic tight Riesz order on* Z^X *then* $T = Z_+^X \setminus \cup \left\{P_{U_i} : i \in I\right\}$ *for some class* $\{U_i : i \in I\}$ *of non-principal ultrafilters on* X .

For a given ultrafilter U on X the totally-ordered group Z^X/P_0 has the same first-order properties as Z , since this group is an ultrapower of Z . If U is a non-principal ultrafilter the totally-ordered group Z^X/P_U must be dense however, since $Z_+^X \setminus P_U$ is a compatible tight Riesz order.

THEOREM 7. *Let* G *be an abelian lattice-ordered group and* I *a convex sublattice subgroup of* G *. Then* $T = G^+ \setminus I$ *satisfies* $T = T + T$ *if and only if* G/I *is a dense lattice-ordered group.*

Proof. Suppose that G/I is dense. If $t \in T$ and $t + I > 0$ in G/I then there is an $x \in G$ satisfying $t + I > x + I > 0$. With $y = (x \vee 0) \wedge t$ we have $t > y > 0$ so $y \in G^+ \setminus I$, $t-y \in G^+ \setminus I$ and $t = y + (t-y) \in T + T$. The reverse inclusion is immediate. On the other hand suppose that $T = T + T$ and $x + I > 0$ in G/I . Then $x \in I$ and $x + y > 0$ for some $y \in I$ so $x + y = t_1 + t_2$ where $t_1, t_2 \in T$. Then $0 < t_1 + I < x + I$ so G/I is dense.

The problem of characterizing the groups Z^X/P_U , for U a non-principal ultrafilter on X does not seem to be easy. Without something like Gödel's axiom of constructability even the cardinality of Z^X/P_U is unclear. As a group we can write Z^X/P_U as $\left(Z^X/P_0\right)/ P_U/P_0$ and then use known results on the cardinality of ultrapowers in an attempt to determine the cardinality of Z^X/P_U .

When X is countable (non-finite) the cardinality of the ultrapower Z^X/P_0 is $2^{\aleph_0}$, for each ultrafilter U on X (Bell and Slomson [1], p.129). As Reilly [4] has remarked the prime subgroup P_U covers P_0 in Z^X so P_U/P_0 is isomorphic with a subgroup of the real numbers.

The totally-ordered groups Z^X/P_U admit more interpolation than that implied by density. A result of Gillman and Jerison [2] (Lemma 13.7), is valid with the real numbers R replaced by Z and then says that if A, B are countable subsets of Z/P_U with $A < B$ then $A \leq g \leq B$ for some $g \in Z^X/P$. We summarize these properties of Z^X/P_U in the following statement.

PROPOSITION 8. *If* X *is a countable set and* U *is a non-principal ultrafilter on* X *then the group* $G_U = Z^X/P_U$ *has the following properties:*

(1) $\aleph_0 \leq |G_U| \leq 2^{\aleph_0}$;

(2) G_U *is dense;*

(3) *if* $A, B \subseteq G$, $|A \cup B| \leq \aleph_0$ *and* $A < B$ *then* $A \leq g \leq B$ *for some* $g \in G_U$;

(4) G_U *is a quotient of an ultrapower of* Z *by a real group.*

References

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