# Formative feedback and the mindful teaching of mathematics

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We examine the use of formative assessment as a tool to assist teachers of mathematics to become more mindful developers of curricula. We focus on instructional design that is based on careful examination of student answers to questions. Empirical studies have shown the effectiveness of formative assessment for students, and recent theoretical work indicates that the positive feedback aspect of formative assessment stimulates self-regulation and transformation, processes that are regarded as critical to developing intelligence. We apply these ideas of formative assessment to teachers of senior mathematics as they rethink instructional design.

## Formative assessment

When teachers understand what students know and how they think, and then use that knowledge to make more effective instructional decisions, significant increases in student learning occur. Black and Wiliam (1998) examined approximately 250 studies and found that gains in student learning resulted from a variety of methods all of which had a common feature: formative assessment. This is a form of assessment that uses the data acquired to adapt instruction to better meet student need. They concluded that (1) improving formative assessment resulted in noticeable increases in student learning; (2) there is room for improvement; and (3) there are ways to improve the effectiveness of formative assessment.

### Formative assessment and feedback

The distinction between formative and summative assessment seems to have been made explicit first by Scriven (1967). He favoured summative assessment, but was aware of the preference for formative assessment of his colleague Cronbach (1957) who saw it as "part of the process of curriculum development" (Scriven, 1967, p. 41). Roos and Hamilton (2004, 2005) emphasise the intellectual debt of formative assessment to cybernetic theory, via the critical notion of positive feedback. They also link formative assessment to a mindful approach to teaching, curriculum development, and instructional design. Black and Wiliam (2003, p. 631) also emphasise this aspect of formative assessment, when they write: "more will be gained from formative feedback where a test calls for the mindfulness that it helps to develop."

The critical intellectual basis of formative assessment, according to Roos and Hamilton, is that positive feedback provides a stimulus to the activities of self-regulation and transformation — essential elements, in Piagetian epistemology, for generating higher aspects of intelligence (Piaget, 1963).

Formative assessment is a way of assisting students to develop, and to help them become more mindful and aware, through the processes of self-regulation and transformation. This is the strength of formative assessment, as evidenced empirically by the seminal Black and Wiliam (1998) meta-analysis, and theoretically by being based on positive feedback to provide a stimulus to these critical thought processes for students (Roos & Hamilton, 2004, 2005). Teachers can ask questions of students that provide data for new and improved forms of instruction, which was the basis of Cronbach's interest in formative assessment in the 1950s. What teachers can learn from student answers to questions may impact directly on the cohort of students being taught at the time, and it may also have effects on the next cohort to be taught the same content. While the latter outcome is not a common understanding of formative assessment, we argue that it is entirely consistent with formative assessment seen, in Cronbach's terms, as part of the process of curriculum development (Cronbach, 1957). We have argued, too, that the essential features of formative assessment are not simply real-time adjustments to curriculum as it is being delivered but are more fundamentally focused on feedback to provide mindful revision of curriculum.

## Formative assessment and questioning

Black and Wiliam (1998) found that not all classroom discussion and questioning results in improved learning: discussions and questions in which a teacher is looking for a particular response actually inhibits future learning by shutting down thoughtful unanticipated, but productive, attempts by students to come up with their own answers. A common problem of the use of questioning as a means of checking on learning is that it is often unproductive. Teachers frequently answer their own questions in less than a few seconds without providing students with sufficient time to think about a response, and regularly call on the same few students to answer questions (Rowe, 1972, 1987; Tobin, 1987).

## Interpreting answers as data

Royall (1997) has written eloquently about the dilemmas facing practitioners in interpreting data. He formulates three questions that have wide applicability to the interpretation of all data. These questions are often confounded, and Royall shows clearly how separating them can lead to significant insights into the interpretation of data. As practitioners, mathematics teachers gain data from their own students — in the form of test answers, classroom answers, and classroom conversations. Royall's analysis suggests to us that it would be very useful for mathematics teachers to ask themselves the following three questions when trying to asses student responses to questions:

- What do I believe now that I have this answer?
- What do I do now that I have this answer?
- What is this answer evidence for?

We argue that a powerful way of connecting formative assessment, with its demonstrable benefits in terms of student learning, with teacher questioning is to link the two via the approach a teacher takes to student answers. Utilising Royall's data analysis questions in the context of formative assessment leads us to formulate to the following general feedback model:

1. A teacher asks questions of students, in class or in tests.
2. Students answer questions, verbally or in writing.
3. The teacher analyses student answers using one or more techniques of data analysis.
4. The teacher reflects on changing beliefs, need for action, or evidence for or against an existing assumption.
5. The teacher modifies and redesigns curriculum as a response to the analysis of student answers.

## Fragmentation of student knowledge

### Preliminary observations

A common observation in our own teaching of mathematics is that student knowledge is commonly fragmented and not connected, and that this generally indicates a superficial degree of concept learning (McGowen & Davis, 2005). An analysis of final examination data of 148 students enrolled in an Introductory Algebra class in 2003 at two-year college (College A) in the midwest of the United States documented noticeable differences in the percentage of correct students' responses to pairs of related questions dealing with slopes of linear equations and linear inequalities. The question:

> Determine the slope/intercept form of the equation $2x + 3y = 6$

was answered correctly by 87% of students, yet only 49% correctly identified the graph of a linear equation with undefined slope. Only 41% of the 148 students correctly answered both questions. On two related questions dealing with linear inequalities, 89% of the students correctly graphed the inequality: $-3 \leq x < 5$ but only 48% could correctly locate the point $(x, y)$, if $x > 0$ and $y < 0$. Only 41% correctly answered both questions.

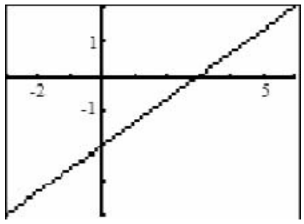
These observations are consistent with fragmentation of student knowledge and lack of flexibility in mathematical thinking that both authors have seen in similar contexts (Davis & McGowen, 2002; McGowen & Davis, 2002; McGowen & Davis, 2005). Flexibility of mathematical thinking also seems to be an issue that is paid little attention by a majority of mathematics teachers (Warner, Davis & Coppolo, 2002; Warner, Davis, Alcock & Coppolo, 2002).

### Further evidence of fragmentation of student knowledge

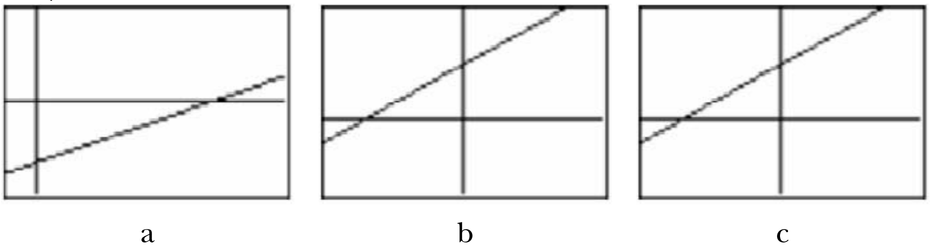
This phenomenon was further investigated at a large mid-western university in the United States (University B) in 2004, where 140 students enrolled in an introductory algebra course were asked the following three related questions on linear equations and slope on a survey at the beginning and at the end of the semester. Student responses to the three questions were examined singly and in combination.

1. Find the equation of the line with slope –7 and a vertical intercept at 3.
2. Which of the following equations has the given graph? Circle and justify your choice.

   a. $3x + 2y = 6$ b. $2x + 3y = 6$ c. $2x - 3y = 6$ d. $3x - 2y = 6$ e. none




3. Which of these graphs could be the graph of $y = 40 - 0.7x$? Circle and justify your choice.




Student responses, summarised in Table 1, indicate that the students entering the developmental algebra course had or recalled little prior knowledge about linear equations and slope.

By the end of the semester, though there had been some improvement in the number of students who correctly answered each of the questions, the overall results, summarised in Table 2, are disheartening.

Less than 20% of the 140 students were able to write a correct linear equation given the slope and the *y*-intercept by the end of the semester. Further examination of the data tells us that of the 140 students, 46 made no attempt to answer Question 1. When students' responses to the first two questions are examined jointly we find that only 4 students (3%) got both questions 1 and 2 correct. Only one of the 140 students answered all three questions correctly.

This simple analysis raises the obvious question of why so few students could answer both of these related questions. It is evidence that students do not see questions 1 and 2 as being intimately connected. Depending on an instructors' thinking, it might also lead one to believe that students are unable to apply what they do know when confronted with a different context — an indication that what little they have learned and remembered is fragmented and unconnected and leads one to believe that their knowledge of slope and linear equations is quite fragile. The data indicates that an instruc-

*Table 1. Pre-test correct responses to questions on linear equations (n=140)*

| QUESTION | # CORRECT | % CORRECT |
|---|---|---|
| 1. Given slope and *y*-intercept, write equation of the line. | 0 | 0% |
| 2. Given graph, select the matching linear equation. | 3 | 2% |
| 3. Given the linear equation, select the matching graph. | 12 | 9% |

*Table 2. Post-test correct responses to questions on linear equations (n=140)*

| QUESTION | # CORRECT | % CORRECT |
|---|---|---|
| 1. Given slope and *y*-intercept, write equation of the line. | 24 | 17% |
| 2. Given graph, select the matching linear equation. | 11 | 8% |
| 3. Given the linear equation, select the matching graph. | 54 | 39% |

tor teaching a course in which related questions are pertinent, needs to make a concerted effort to draw out the explicit connections between the questions.

It might be argued that a possible explanation for the very small number of correct responses to Question 2 is that the question is misleading and that a different choice of graph and equation would have yielded better results. Attributing student difficulties to misleading choices in Question 2 fails to consider responses to related questions that, when examined together reveal other possible underlying sources of error. On Question 1, which asked students to write the equation of a line given the slope and *y*-intercept, 34 students drew a graph and plotted the slope as the *x*-intercept or, in a few instances, as the *y*-intercept. Many of these same students labelled the *x*- and *y*-coefficients as intercepts on the graph of Question 2. Only 41 students attempted to write an equation and 17 of them were not successful.

Students' written comments further support the conclusion that many of them interpreted the slope as an ordered pair and plotted it as an intercept or equated the *x*- and *y*-coefficients in the equation with the intercepts on the graph. The following were the most common responses:

"Positive 3 at *x*-axis (independent); negative 2 at *y*-axis (dependent)."

"Positive $3x$ is visible and $-2y$."

"The *x*-intercept is $3x$; the *y*-intercept is $-2y$. "

## Student tests questions and formative assessment

The fact that differences in correct answers between deliberately included related questions were found on the survey administered at University B, and similar differences were found on items typically found on course exams which could also be grouped at College A, suggests that this type of analysis is both a formative assessment tool when used to modify current instruction and help raise students' awareness of their own areas of strengths and weaknesses, and as a basis for informed curricular decisions and revision. Correctly answering a majority or all of a grouping of related questions on a given topic is an indicator of the ease or difficulty a student has in adapting to changing problem conditions and in reorganising one's work. A possible step to improving student success could be to incorporate formative assessment practices that provide teachers with data on what students know and how they understand what they are learning. Teachers could then make more informed, principled decisions about courses of action that result in improved student learning and higher rates of success.

## A pilot formative assessment project

These experiences led to the design of a pilot formative assessment project, September–December 2006, in which three teachers at College A agreed to investigate the extent to which introductory algebra students demonstrate understanding of, and ability to apply, both concepts and skills in different contexts in the topic areas of: (a) linear equations and slope; (b) linear systems, and (c) linear inequalities. Each assessment consisted of related multiple-choice questions administered prior to introduction of the topic during the Fall 2006 semester. After completion of each formative assessment and analysis of the data, the researcher and the three instructors were to meet to (a) discuss the results, (b) develop possible follow-up instructional tasks based on information provided by the assessments, and (c) review the next proposed formative assessment prior to its administration. Related questions similar to the formative assessment items were to be included on the departmental introductory algebra final exam.

### Sample results and analysis from the pilot formative assessment project

***(A) Student answers***

Analysis of related questions on the topics of linear equations/slope, linear systems, and linear inequalities provided evidence of students' fragmented learning and failure to utilise knowledge and skills learned in one context in a different situation. On the first pre-test, students did not distinguish intercepts from $x$- and $y$-coefficients. Several students insisted that an intercept is a number value, not a point/ordered pair. Lack of a principled understanding of the concept of slope was also documented. Some students wrote the slope as an ordered pair and did not view slope as a ratio. Others used the value of the slope as the $x$-intercept value.

Student responses to five related questions dealing with linear equations, intercepts and slope were examined singly and in combination. On the pre-test, 22% of 92 students correctly answered 3 of the 5 questions. Only one student correctly answered all five questions.

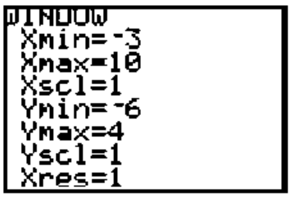
7. The equation of the line with slope –3 and $y$-intercept 5 is:
   a. $y = -5x + 3$  b. $y = 5x - 3$  c. $y = -3x + 5$  d. $y = 3x - 5$

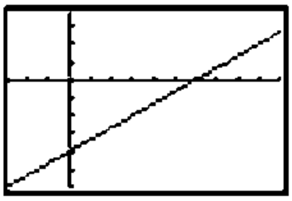
4. The $x$-intercept of the equation $2x - 7y = 12$ is:
   a. (0, –12/7)  b. (–12/7, 0)  c. (0, 6)  d. (6, 0)

5. Given the equation $y = mx + b$, the vertical-intercept is represented by:
   a. $y$  b. $m$  c. $x$  d. $b$

3. Given the equation $Ax + By = C$, the slope of the equation is:
   a. $-A/C$  b. $-A/B$  c. $C/A$  d. $C/B$

9. Given the view window and graph shown below, the equation of the line is:



   a. $6x + 4y = 12$  b. $2x - 3y = 12$  c. $6x - 4y = 12$  d. $3x + 2y = 12$

*Table 3. Related pretest responses to questions on linear equations, intercepts and slope.*

| QUESTION | n = 92 | % CORRECT |
|---|---|---|
| 7. Given slope –3 and $y$-intercept 5, select linear equation. | 65 | 71% |
| 4. Determine the $x$-intercept of the equation $2x - 7y = 12$? | 38 | 41% |
| 5. What is the vertical intercept of $y = mx + b$? | 37 | 40% |
| 3. What is the slope of $Ax + By = C$? | 24 | 26% |
| 9. Given the view window and graph, what is the equation? | 13 | 14% |
| | | |
| 3 of 5 questions answered correctly | 20 | 22% |
| 4 of 5 questions answered correctly | 6 | 7% |
| All five questions answered correctly | 1 | 1% |

Asked to identify the relation between slopes of parallel lines, 72% of 71 students who took the second pre-test on linear systems gave a correct response. Only 41% were able to correctly identify the correct relationship between slopes of perpendicular lines and only 22 students (31%) answered both questions correctly.

Sixty-eight percent of the 71 students correctly solved the system:

$$3x - y = -2$$
$$3x - y = 6$$

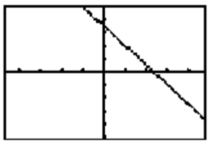
but only 35% of the students were able to determine the point of intersection of:

$$y = 2 - x$$
$$3x + 2y = 2$$

Given the graph of two intersecting linear equations, 8 students (11%) correctly estimated the point of intersection.

Three related questions addressed generalisations about linear inequalities.

6. If $y = ax + b$, what can be said about the signs of $a$ and of $b$ in the graph below?



a. $a > 0$ and $b > 0$ b. $a < 0$ and $b > 0$ c. $a > 0$ and $b < 0$ d. $a < 0$ and $b < 0$

5. If $a$ and $b$ are any real numbers such that $0 < ab < b$, what must be true of $ab$?
a. $0 < ab < b$ b. $b < ab < 0$ c. $0 < b < ab$ d. $ab < a < 0$

10. What are the values of a and b, where $a|b - 2| < 0$ ?
a. $a < 0$ and $b$ is any number greater than 2.
b. $a < 0$ and $b$ is any number greater than or equal to 2.
c. $a < 0$ and $b$ is any number except 2.
d. $a < 0$ and $b$ is any number less than 2.

*Table 4. Responses to related pretest questions: Testing generalisations about inequalities.*

| QUESTION | n = 73 | % CORRECT |
|---|---|---|
| 6. Given $0 < a < 1 < b$, what must be true of $ab$? | 47 | 64% |
| 5. What are the signs of $a$ and $b$, given graph of $y = ax + b$? | 27 | 37% |
| 10. What are the values of $a$ and $b$ whenever $a\|b - 2\| < 0$? | 13 | 18% |
| | | |
| 2 of 3 questions correct | 17 | 23% |
| All three questions correct | 5 | 7% |

### *(B) Teacher data*

By December 2006 none of the three teachers in the project had incorporated curricula changes in their classes as a result of analysing student responses to the grouped questions. One of the teachers, Lisa, wrote a note to the other two teachers in which she implicitly addresses some of the issues raised by Royall, and indicates why addressing those issues can be a problem in the course of a semester. Lisa gave her students a quiz on the relationship between the coefficients of $x$ and $y$ in an equation for a straight line, and the $x$- and $y$-intercepts, following class discussion and assignment of an intervention task. The excerpts below come from an email Lisa sent to the two other teachers in the study, Diane and Nancy, following that quiz:

> I'd like to share some results that I think are pretty exciting. As you know, I spent part of my two-hour class last Wednesday trying to address some of our mutual concerns about students mistaking the coefficients of $x$ and $y$ in a general form equation for the $x$- and $y$-intercepts. I'm attaching the handout, although I did provide grids — love those graph patches!
>
> I began with a discussion about the concept of intercepts and how to use the "cover-up" method to find them and use them for graphing the lines. We did the first one together, them they did the rest of the graphs themselves. Then we talked about #5, and concluded that the coefficients were, of course, not the intercepts. Last, we "brainstormed" lists for the second page questions. I was surprised how many things they wanted to add to each list, even though some were duplicates (worded differently). I felt good about the lesson, but I decided to check their understanding with a brief quiz on Friday.
>
> The quiz was about systems as well as the concerns above. I was extremely pleased by the results. … [All but one] of them got the questions asking for the intercepts and slope of a line whose equation was given in general form. Finally! I hope it lasts through to the final (and beyond)!

We infer from Lisa's final statement that the quiz results led her to believe that most of the students who took the test now understood the relationship between horizontal and vertical intercepts and slope, and the coefficients in an equation of a straight line. What also seems reasonable to infer from her comments is that she did not yet see this data as providing evidence that the students had a stable, durable understanding.

Lisa then addressed a difficulty in utilising classroom data to alter curriculum in real-time, namely that time constraints can inhibit or prevent a teacher from taking action to alter curriculum during semester:

"My final thought today is that I wish I had the time to analyse and address all of the needs of my students in all my classes."

Despite her feeling she did not have time to analyse the data in order to address student needs in real-time, she felt that the method of analysis was helping her with her instruction:

"I do feel that this project is helping me pinpoint where I should focus my instruction."

Lisa, like the other two teachers in this project, did not alter her curriculum during the semester as a result of student answers to questions, but she did become mindful of a need to change some aspects of her teaching. The student data she gathered led her to believe something about those students, and lead her to think about what she should do. The fact that anticipated curricula changes are postponed to the next semester does not negate the critical formative aspect of the assessment, namely that she had become more aware of student difficulties, what the data from students was telling her in terms of what she believed about their knowledge, and what she should do as a result.

## Conclusions

Related grouped questions, dealing with varied aspects of a particular mathematical concept, can provide classroom teachers with valuable data to help them understand the coherent strength or fragility of their students' knowledge of that concept. However, it is by no means automatic that a teacher will have time to analyse that classroom data in real-time in order to alter curriculum to better meet student needs. What may have to happen, due to time and other constraints, is that a teacher may have to postpone analysing such data until the end of a semester, and postpone curriculum change until a following semester. From the perspective we have described in this article this is still a powerful form of formative assessment because it promotes curriculum change based on mindful reflection on the feedback from carefully structured questions.

When a teacher has time to analyse student data they implicitly or explicitly address Royall's questions of what they believe now they have this data, what they should do now they have this data, and what this data is evidence for. We argue that by learning to make these distinctions explicit, a teacher can become more mindful and aware of what curriculum changes the data suggests, and so engage in productive curriculum change in accordance with Cronbach's understanding of the term "formative assessment".

The need to align our perspectives about mathematics and the learning of mathematics in order to more effectively communicate with our students is essential if we expect to reduce the growing number of students who are not successful. These students do not reason as we expect them to, nor do they retain knowledge and skills from their previous courses that we assume they can demonstrate. Research does not provide us with "a best method of instruction," but it has identified many of the transitional contexts in which students' prior knowledge is no longer adequate and provides us with questions that help us better understand how our students are thinking. Classroom data can be used to alter what is taught, to whom, and when. Data from class tests that examine student answers to questions in pairs, or in other joint combinations, as indicated above, can substantially assist a teacher to make informed decisions about instructional design. Carefully chosen questions, examined this way, can stimulate teachers to be more self-regulatory about instructional design, and help transform their thinking on instruction that is better aligned to student needs.

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