A tale of divergence: Spectral analysis of concept maps of two undergraduate mathematics students

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To appear in the Journal of Mathematical Behavior, 2019

ABSTRACT

The purpose of this study was to delineate differences in mathematical knowledge construction between students who made substantive gains and those who made only limited gains over the course of a semester. We found that students with high gain —initial-test to final test—construct, organize, and restructure knowledge in ways that are qualitatively different from the processes utilized by those with low gain. Concept maps and their corresponding schematic diagrams, combined with spectral analysis of those schematic diagrams, triangulated and interpreted with other data, revealed radically different processes of knowledge construction and organization between the students with high and low gain. Concept maps and other data of two students, S3 and S25, representative of the high and low gain cohorts of a class, document the divergent performance that occurred over a semester. The significant finding is that the student with low gain seems unable to productively integrate new knowledge into existing structures.

Keywords: divergence, concept maps, functions, algebra, spectral analysis

1. INTRODUCTION

It is not uncommon for instructors of beginning undergraduate mathematics to see two, or more, students who begin the semester as instrumental and procedural learners, yet over time one seems to be able to develop conceptually and in terms of relational thinking and understanding, while the other remains stuck in an instrumental procedural mode. We approach the question of what modes of thinking and mental organization might be behind this phenomenon of divergent performance through an examination of two students' concept maps and the spectral analysis of those maps, triangulated with data collected throughout the semester. A significant difference of this study from other concept map studies is the use of spectral graph clustering of schematic diagrams of concept maps to document dynamic changes in students' organization of their internal knowledge representations.

The question of whether divergence occurs was determined by first ranking a class of 26 students based on the *z*-score of fractional *gain.* An individual *gain* was calculated for each student (based on the results of a pre-test at the beginning of semester to a closely related "final" score at the end of semester, namely, the average of each student's post-test score, departmental final exam score and open-response course final exam score. We calculated the *gain* in test scores from the initial to the final test: *gain = (final-test% - initial-test%) / (100% - initial-test%)* and ranked students by the *z*-score of their gain in test scores. We deemed students with *gain z*-score > 1 to have "high gain" and students with *z*-score < –1 to have "low gain" (z-score = (individual gain – class mean gain) / standard deviation).

We document and analyze the work of two students, who at the beginning of semester appear to be very similar in ability and attitudes to learning mathematics, both identifying as procedural and instrumental learners, yet who take widely divergent paths as the semester progresses. One student retained a belief that mathematics is procedural and instrumental in nature and is best learned that way, while the other student demonstrated a willingness to expand their thinking and buy into the broader goals of the course.

This study addresses the question: Do concept maps and schematic diagrams of those maps provide visual evidence of changes in knowledge structures that occur and, when triangulated with multiple sources of data collected over the semester, delineate differences between a student who makes substantive gains and one who makes only limited gains?

2. LITERATURE REVIEW

2.1 Divergence of performance

Divergence of performance and fragmentation of strategies have been studied extensively in elementary and secondary mathematics classrooms. Krutetskii and other Russian researchers characterized the divergence of performance of students, selected by their teachers as "gifted", "capable", "average", and "incapable", as a consequence of the qualitatively different strategies and mathematical abilities they identified (Krutetskii, 1976; Dubrovina, 1992; Shapiro, 1992). Gray & Tall (1994) concluded that the divergence of strategies was a consequence of students' ability to think flexibly, and that interpretations of mathematical symbolism as process or procepts results in a *proceptual divide* (i.e., a divergence between procedure and procept) — a bifurcation of strategy between flexible thinking and procedural thinking which distinguishes more successful students from those less successful (Gray & Tall, 1994, pp. 116, 132). Our experience is that students who exhibit more flexible mathematical thinking do not simply develop alternate ways of thinking, but actively modify pre-existing ways of thinking. A student who has a rich connected network of concepts and solution strategies available in memory is more likely to be able to recall a potentially useful prior strategy in a novel problem setting.

2.2 Concept maps

A *concept map* is a diagram representing the knowledge structure of a subject discipline as a labelled graph in which concepts are represented as labelled nodes and connections among concepts indicated by undirected lines (which may or may not be labeled). Ausubel's theory of hierarchical memory structure and work on learning was the basis for the use of concept maps in science education (Ausubel, Novak & Hanesian, 1978). Novak is credited with the development of this tool in the early 1980's as a way of "determining how changes in conceptual understanding were occurring in students" (1990, p. 937). Novak & Gowin believed that concept maps should consist of an expanding hierarchy of concepts, organized under more general, inclusive concepts (1984, p. 97).

Concept maps as an instructional tool for assessment, and as a research tool, were initially used in quantitative research in science education (Blunt & Karpicke, 2014; Carnot, Feltovich, Hoffman, Feltovich & Novak, 2003; Jackson & Trochim, 2002; Novak, 1990, 2010; Novak & Gowin, 1994; Ruiz-Primo & Shavelson, 1996; Safayeni, Derbentseva & Cañas, 2005).

In mathematics education, Skemp (1987) used the term *concept map* in the sense of a concept-dependency network to refer to the process of schema construction in which order is important, with the necessary direction being from lower to higher order concepts. He considered concept maps as a particular kind of schema and envisioned them as advanced organizers and as a means of analysis when planning a teaching sequence or for diagnosis (1987, p. 122). Park & Travers (1996) used them as a comparative research tool, contrasting the maps of students ("novices") with those of an "expert," a use of concept maps commonly described in the research literature of the sciences. They were used as a research tool to investigate community college students' mathematical knowledge within a narrow range of competence (Laturno, 1994) and as an instructional tool to document changes in the nature of middle school teachers' thinking about their assessment practices as a result of using decision-making case studies (Wilcox and Lanier, 2000). Concept maps have also been used as a means of documenting a change in knowledge as a result of instruction (Williams,1998; Hough, O'Rode, Terman & Weissglass, 2007). Wheeldon & Faubert

(2009) explored the use of qualitative research as part of a multistage data collection. Studies by McGowen (1998) and McGowen & Tall (1999) used concept maps created over time to document qualitative differences between students of different levels of achievement.

In the process of creating a concept map, a student is engaged in a metacognitive activity that shapes and modifies their understanding of what they know when the map is constructed. In this study, three concept maps on the topic of *Function,* together with schematic diagrams of those maps, were visible *representations* of the student's internal cognitive structure at a given moment in time. Concept maps created over time document the processes by which that structure changes as students organize and integrate new concepts and procedures into their existing conceptual frameworks.

3. METHODS

3.1 Participants

The subjects of the broader study were 26 undergraduates (78%) of the thirty-three initially enrolled at a large suburban community college and completed an Intermediate Algebra course. All students satisfied the prerequisite for this course (a grade of C or better in Introductory Algebra or entry based on a placement test score). Twenty-four of the 26 students were enrolled full-time (12 or more semester hours). Students were twenty years of age or younger, either recent high school graduates, or had attended the community college the previous year. Six students were taking the course for the second time. One student was taking the course for the third time. Based on class rank following course completion, two of the eight students in the extremes of the class are profiled.

3.2 Procedure and data collection

The need to collect data at different stages in the study was addressed by collecting data from all students throughout the semester since identifying which students would still be in class at the end of the 16-week semester was not possible. A pre-course self-evaluation survey used to establish prior variables of students' beliefs about their mathematical abilities confirmed the findings of an earlier field study of 237 students which found that students lacked self-confidence, had a negative attitude towards mathematics at the beginning of the semester, and feelings of high math and test anxiety. Data collected included: pre- and post-course test results and self-evaluation attitude surveys, journal problems, unit exams, a course open-response final exam (week 15), a departmental multiple-choice final exam, (week 16), written mid-term and end of semester self-evaluations and interviews.

3.3 Fractional gain and student ranking

Students were ranked after the semester ended by the *z*-score of fractional *gain* that a student obtained from initial test to "final" test (the average of the student's post-test score, departmental final exam score and open-response final exam score). This is an approximate numerical indicator of mathematical growth over time: *gain* = (final-test%–pre-test %) / (100%–pre-test %). The question of whether divergence occurred was determined by this ranking, which identified those with high gain*, i.e., gain* *z*-score > 1 (n = 3) and those with *low gain*, i.e., gain *z*-score < –1 (n = 5). Class rankings for all 26 students based on *Gain* are included in Appendix A: Table 1.

A statistical advantage of focusing on individual gains rather than on mean class gains (as in Hake, 1998) is that we can obtain a *distribution* of gains, of which the mean is just one summary. Further, knowing the distribution allows us to estimate confidence intervals for the mean of the individual gains, and to calculate gain *z*-scores. We interpret individual *gain* as a numerical indicator of the "size" of change from one test to a succeeding test. In the application of the individual *gain* statistic, we are interested in student attitudes and dispositions to learning mathematics, rather than comparing mean test scores before and after an instructional treatment. McGowen & Davis (2005) documented that the *gain* correlates only poorly with initial test scores ($r^2 \approx 0.2$). The *gain* statistic is one of a family of relative change functions (Tornqvist, Vartia & Vartia, 1985; Bonate, 2000) and is characterized by its preservation of the binary operation $x*y = x+y - xy$. In our context, the *gain* function provides *significant* extra statistical information beyond initial-test scores.

Finally, high initial test scores do not, in general, equate to low gains. In a study by McGowen & Davis (2005), approximately 18% of the cohort had below average initial-test scores and below average gains, and approximately 19% had above average initial-test scores and above average gains. While it is more likely is that a student with a below (*resp*. above) average initial-test score will have an above (*resp*. below) average gain, it is by no means a foregone conclusion.

3.4 Concept maps

Three concept maps, submitted at weeks 4, 9, and 15, provided documentation of how high and low gain students' knowledge structure changed over time. Any or all of the following factors can affect the visual representation of an individual student's developing cognitive structure and knowledge construction processes: (a) the amount of time a student is able to spend on constructing a map; (b) the amount of information and the number of connections between and among elements a student is able to record on a two-dimensional, standard-size sheet of paper; and, importantly, (c) the student's ability to categorize, organize, and reflect upon their perceptions, actions, and schemas employed.

Concept maps on the concept of *Function* recorded changes in each student's knowledge structure over the sixteen weeks of the course. Each concept map was collected a week after its assignment. The maps were not returned to the student but retained by the researcher as part of the data collection. The nature and extent of changes in knowledge construction, organization, and stability over the semester were analysed by their corresponding schematic diagrams.

3.5 Schematic diagrams

A simple pictorial technique (schematic diagram) was developed to expose the underlying structure and the changes in the maps that occurred over time of eight students (3 high gain, 5 low gain). A *schematic diagram* (an outline diagram) was created for each of the concept maps created by the high and low gain students, stripping the concept maps of semantic details to compare different students' processes of knowledge construction and how the organization of that knowledge changed over time. Schematic diagrams for the second and third maps for each student distinguished:

- elements from the previous concept map remaining in the same position.
- elements moved somewhere else, or recalled from an earlier map.
- new elements.

A comparison of a student's three concept maps and corresponding schematic diagrams, triangulated with other qualitative and quantitative data, provided evidence of the nature of students' knowledge construction and organization processes, and of the divergence that occurred. The mathematical quality of each schematic diagram was analyzed through spectral graph clustering, (discussed below) and confirmed the more subjective analyses of the concept maps and schematic diagrams.

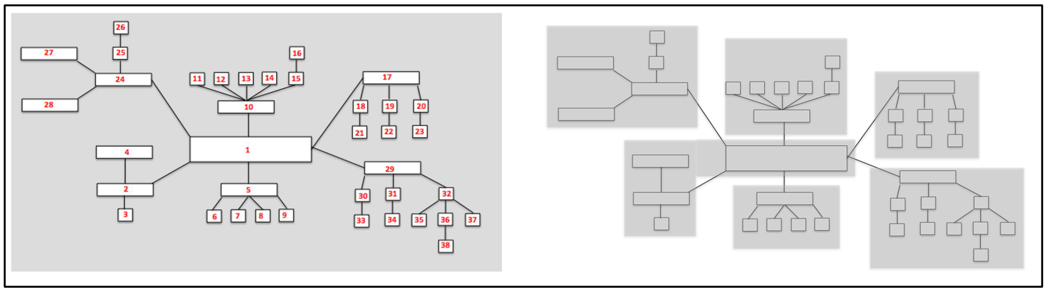
3.6 Spectral graph clustering

Schematic diagrams still retain some semantic information in terms of the shape and labelling of the vertices. By stripping away all semantic labeling and just keeping the vertices and connections between them, we obtain simple connected graphs. We analyze the structure of these graphs through spectral graph clustering (Von Luxburg, 2007). This is a graph-theoretic technique for decomposing a graph into chunks, or clusters, that are in some sense more similar within themselves than they are to other chunks. It is a technique that relies only on the abstract structure of a graph—the connections between vertices—and does not rely on any semantic labeling or interpretation of the graph vertices or edges.

Our application of spectral clustering to the schematic diagrams of student-generated concept maps provided us with a more principled and less subjective way to ascertain whether the graph-theoretic structure of these diagrams provides evidence of the stability of identifiable clusters in the student concept maps. In other words, spectral clustering is used in place of an instructor's or researcher's semantic interpretation of the student labelling of the vertices of the concept maps. It should be noted that one student's third concept map contained ambiguous connections between a few nodes, which meant that not all elements of semantic interpretation were eliminated when obtaining the simple connected graphs of this student.

We *arbitrarily* number the vertices of the graph arising from a student's concept map as 1, …, n. Then the (un-normalized) *Laplacian matrix* L of the graph is constructed as D-A, where D is a diagonal matrix with diagonal entry $d_{i,j}$ = degree of vertex i, and A is the incidence matrix of the graph: the i, j entry of A is 1 if vertices i, j are connected by an edge, and 0 otherwise (Brouwer & Haemers, 2011). The Laplacian matrix L is positive semi-definite so its eigenvalues are real and non-negative. The second smallest (that is, first non-zero) eigenvalue is referred to as the *Fiedler eigenvalue*, and also as the *algebraic connectivity* of the graph. An eigenvector for this eigenvalue is referred to as a *Fiedler eigenvector* (Fiedler, 1973, 1989). The eigenvectors of L can be viewed as real-valued functions on the vertices of the graph, and the graph is connected exactly when the Fiedler eigenvector does not obtain the value 0. Consequently, we can then decompose the vertices of the graph into those vertices where the Fiedler eigenvector tales a positive value, and those where it is negative, yielding a decomposition of the original graph into two sub-graphs. Provided these sub-graphs are connected, this decomposition process can be iterated on the sub-graphs. The iteration process constitutes a spectral clustering of the original graph. Figure 1 shows a numbered schematic diagram and the spectral clusters resulting from the decomposition process.

Figure 1: A Numbered Schematic Diagram & Its Decomposed Cluster Graph



4. RESULTS

Pre-and Post-Course Data

Results of the pre-course self-evaluation survey revealed that high gain students initially under-estimated their mathematical abilities and held more negative beliefs compared with low gain students, who tended to rate their abilities and confidence in their responses higher than their performance on the pre-test warranted, consistent with the Kruger-Dunning effect (Kruger & Dunning, 1999). Over the semester, high gain students experienced greater improvement.

High and low gain students' responses on the post-test, compared with their pre-test responses document the divergence of performance that occurred. The number of correct post-test responses compared with those on the pre-test of high-gain students revealed a significant difference ($\chi^2 = 86.176$), with two degrees of freedom ($p < 0.0001$). The difference in the number of correct post-test responses compared with the number of correct pre-test responses for the low-gain group, was not significant. Table 1 includes mean responses of the class, high and low gain cohorts.

Table 1: Mean Pre- & Post-test response of the class, high and low gain students, S3, S25

	PRE-TEST	POST-TEST	DIFFERENCE
Class	0.18	0.47	+ 0.29
High Gain	0.17	0.86	+ 0.69

S3	0.14	0.79	+ 0.65
Low Gain	0.16	0.19	+ 0.03
S25	0.14	0.21	+ 0.07

4.1. Interpreting the Minus Symbol

McGowen (2017) and McGowen and Tall (2010, 2013) identified multiple meanings of the minus symbol in new contexts and the need to interpret symbolism flexibly as major sources of difficulties for undergraduates. The pre-test revealed that interpreting function notation and recognizing the role of context when interpreting the minus symbol were sources of difficulty for all students in the class—and continued to be sources of difficulty throughout the semester for many of them. We consider two examples of divergent performance data of high and low gain students: (a) their ability to interpret the minus symbol and (b) use of function notation in different contexts and the stability of re-constructed schemas.

On the pre-test, all students were initially unsuccessful when asked to interpret the minus symbol in different contexts, i.e., evaluate $(–3)^2$ and $–3^2$; meaning of the notation $f(–x)$ and $–f(x)$; $(x – c)$; evaluate (a) $f(–3) = –x^2$ and (b) $f(h–1) = –x^2$. On the pre-test, 23 of the 26 students correctly evaluated $(–3)^2$. Only six students correctly evaluated $–3^2$ and $(–3)^2$. On the post-test, 24 students evaluated $(–3)^2$ correctly, only 16 correctly evaluated both $–3^2$ and $(–3)^2$.

4.2. Changing an inappropriate schema: Interpreting function notation

Students were asked to evaluate a quadratic function, $f(x) = –x^2 – 2x + 7$ =? given a negative numerical value input, f (–3), and a binomial input, (h–1), on various assessments during the semester. By week 3, high gain students were able to correctly and consistently evaluate a quadratic function for similar inputs, having constructed a more appropriate schema which remained stable throughout the semester. Students within 1 standard deviation of the *gain* mean, with additional experience, reflection, and practice, generally developed consistent performance by the time of the Unit I exam week 6. Low gain students were unable to reconstruct a more appropriate schema, much less stability, by week 16. Appendix B contains supplemental information expanding on these examples.

5. A TALE OF TWO STUDENTS' DIVERGENT PATHS

5.1 S3 and S25: Representatives of the high and low gain cohorts:

We profile two students of the extremes of a class in order to clearly delineate qualitative differences that occurred over the semester. S3 and S25 shared many attitudes and prior mathematical experiences. Both had 3 years of high school mathematics, Algebra I & II, and Geometry. S3 took Algebra II in the junior year (with no mathematics in the senior year), S25 in the senior year. Both began their college career enrolled in an individual, self-paced Beginning Algebra course. Both described their overall goal of learning was to connect new knowledge to prior knowledge and remained unchanged throughout the semester for S3 and S25. Both students attended class regularly and worked hard to keep up with assignments. Despite these initially similar experiences and attitudes to learning mathematics, as the semester progressed S3 and S25 exhibited increasing divergence in their mathematical problem solving strategies and skills and in their views about what it means to learn mathematics and use technology.

5.1.1 Views on what it means to learn mathematics. S3 and S25 expressed different attitudes to learning mathematics. S3 recognized a need to take responsibility for learning with understanding. Disappointed with results of an individual exam compared with a group exam done shortly before, S3 wrote in his mid-term self-evaluation:

> "I've never been very good at taking math exams, I find that a lot of what I know slips away when it's time to show and prove. To be honest, I wasn't as prepared as I could have been. After receiving a score of nearly proficient on the test I took it upon myself to go back to Section 1.4 in the book and review everything from that point on in order to fully understand the material. I did that because I realize that math is a very progressive subject, and if I were to continue forward with minimal understanding of

> the previous sections, I would surely have minimal understanding of the rest of the sections."

Though S25 describes similar prior experiences, the responsibility for learning and understanding mathematics was viewed differently, as described in the mid-term self-evaluation:

> "I thought math was about doing a lot of the same problems in order to have an understanding of what you were learning. I have always thought that doing mathematics meant doing operations, with formulas. I believe learning math with concrete functions, definitions, and examples is the best way for me to learn math."

5.2 Procedural Flexibility

Procedural flexibility is the ability to modify an existing way of thinking to conform to changing conditions and includes ability to switch from one interpretation of symbolic notation to another, depending upon the context. A comparison of the strategies employed by S3 and S25 when interpreting function notation is an example of the divergence in strategies and knowledge construction that occurred over the semester shown by their responses to the pre- and post-test question: *In the expression (x – c), is the value of c positive, negative, or unknown?* Initially, both S3 and S25 used two different schemas simultaneously on their responses—they used the minus symbol twice—first to indicate that *c* is negative, then as the subtraction operator. S3 perceived (*x*–*c*) as "the value for *c* is negative because of the – sign in front of *c*, then wrote, "*c* will subtract from any number that comes before the – symbol. S25 changed the subtraction sign to addition, placed a minus sign to the upper left of *c*, and wrote: "when you see a minus symbol in front of a letter, change signs and add." Both responses suggest the retrieval of two default schemas: (a) interpreting the minus symbol twice—once as subtraction and (b) to indicate a negative value, *c*.

S3's post-test response is consistent with a more flexible interpretation of the minus symbol by the end of the semester: "the value of *c* is neither because it may be positive or negative. If *c* were positive, it would become negative and if it were negative, it would become positive"—evidence of growth of procedural ability to interpret ambiguous notation. S25 continued to use the minus symbol twice; once to subtract and as the sign of *c*, indicating a negative-valued number and to repeat the rule: subtract, change signs and add.

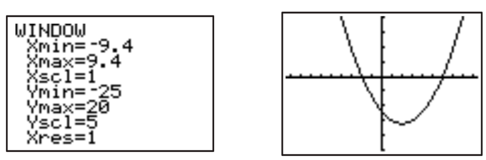
5.3. Responses to high level cognitive tasks

High level cognitive tasks (Stein, et. al., 1996, p. 467) require the integration and application of knowledge: selecting an appropriate alternative strategy, connecting symbolic and non-symbolic procedures to underlying concepts and meaning, translating between various representations, making and describing connections which required knowledge of several mathematical concepts (e.g. *x*-intercepts, zeros of the function, and factors). and justifying the responses.

The extent to which S3 and S25 engaged with high level cognitive tasks is shown by their responses to two questions on the course open-ended final exam which required knowledge of several mathematical concepts (e.g. *x*-intercepts, zeros of the function, and factors). Responses of S3 and S25 to post-test Question 14 are given in Table 2.

15. The graph of a quadratic function appears below.

Figure 3: Post-test question 14



a. What are the zeros of this function?

b. What are the factors of this function?

c. Write the algebraic representation of this function.

Table 2: Responses to Post-test Question 14

Question	S3 Responses	S25 Responses
a. What are the zeros?	*x*-intercepts –2 and 6	two real solutions; (6,0) (2,0)

b. What are the factors?	use opposite of x-intercepts, $(x - 0)$; 2 and −6, $(x + 2)(x - 6)$	What are the values /numbers
c. Algebraic representation?	multiply factors; $f(x) = x^2 - 4x - 12$	Made no attempt to answer

5.4. Different strategies based on different initial perceptions

Another question on the open-response final exam involved solving a 3 × 3 linear system in contextual form:

> *A toy rocket is projected into the air at an angle. After 6 seconds, the rocket is 87 feet high. After 10 seconds, the rocket is 123 feet high. After one-half minute, the rocket is 63 feet high.*
>
> *a. The model for the rocket's motion is $h=at^2 + bt + c$ where h is the height in feet of the rocket after t seconds. Using the given information, find the values for a, b, and c so the function models the situation. Briefly explain what you did.*
>
> *b. Approximate how long it will take for the rocket to hit the ground. Explain how you arrived at your answer.*
>
> *c. What representation did you choose to investigate this problem? Why?*
>
> *d. Describe the process you used to find the answers to part (a) and to part (b).*

Students had the option of (a) setting up a system of three linear equations in three unknowns and solving the system algebraically or (b) solving the system using the list, matrix and regression features on the graphing calculator to calculate parameter values for an appropriate algebraic model of the problem. Once the parameters and algebraic representation of the problem were determined, students also had options for determining when the projectile would hit the ground. They could (a) enter and graph the algebraic representation and use the ROOT or ZERO option to find the solution; (b) display table values of the function and locate the input value when the output value was zero; (c) use the TRACE command on the graph and approximate the answer; or (d) solve the equation algebraically, using the quadratic form with determined parameter values.

S3 initially focused on and circled the equation, writing: "quadratic regression," then entered the ordered pairs (time, height) in two lists on the graphing calculator, and selected the quadratic regression option to determine parameter values, *a*, *b* and *c*—an example of filtering out "the superficial to concentrate on the more abstract qualities…able to relate to a hierarchy of ideas" (Gray, Pitta & Tall, 1997, p. 124). S3 graphed the discrete points, entered the algebraic model based on the regression values, and tested it graphically against a plot of the discrete points. S3 wrote: "The line falls directly on the ordered pair," recording parameter values of A = –.5, B =17, and C = 3 and wrote an equation using the displayed calculator values.

S3 recalled and used prior strategies in novel problem settings, using both symbolic and non-symbolic procedures effectively to translate between various representations. S3 chose to find parameter values using the graphing calculator features of a list and quadratic regression rather than set up and solve the system algebraically, explaining: "I used the table values to find the output at zero and the graphical [representation]. I felt more comfortable with it. I didn't feel confident trying to investigate algebraically." Despite a lack of confidence in algebraic skills to solve the linear system algebraically, S3 demonstrated high-level cognitive thinking.

S25 initially focused on and circled the time values stated in the problem, writing: "I choose these numbers because *h* equals the height in feet at *t* seconds. The numbers in front of *t* equal the seconds." The process used to find parameter values is described as: "quadratic formula because I had to find *a, b*, *c* and used the quad. equation." For choice of representation, S25 wrote: "Evaluate because I need to find the output." The initial choice and use of only the time values resulted in S25's retrieval of a prior, instrumentally-learned schema of memorized procedures associated with quadratic equations (find the discriminant; use the discriminant value in the quadratic formula to "solve" the equation)—ignoring problem data which did not fit this retrieved schema—an example of prior knowledge interfering with assimilating new knowledge into an existing schema. Describing the work, S25 wrote:

"In part a, I choose the amount of seconds it would take for the rocket to reach its height and in part b, I plugged those #'s into the quadratic formula. [It will take] almost a minute long: 123 – 63 = 60 – 63; I subtracted largest height and 63 feet–for ½ minute and then another 63 feet for rest of time."

Based on this work, one might assume that S25 does not know how to solve linear systems using the matrix and/or regression features of the graphing calculator. In fact, of the five linear system problems on the final departmental multiple choice exam, S25 correctly answered four of them, able to solve a 2 × 2 inconsistent linear system and a 3 × 3 system, using the matrix features of the graphing calculator. It should be noted that none of the departmental final exam linear system problems required more than procedural knowledge. None were contextual problems and all but the 2 × 2 inconsistent linear system were multiple choice format questions, written in traditional symbolic form of textbook exercises.

S25 noting that the model was quadratic and recognizing the need to have consistent units, (changing one-half minute into 30 seconds), demonstrated proficiency in attention to some details. Confused about which variable was to be found, S25 solved the quadratic for y instead of for x. Once a default quadratic equation schema was retrieved and actions based on that schema initiated, S25 focuses on "getting the answer." Davis (1984, p. 65) describes the situation in which a student fails to match initial perceptions that cues retrieval of an appropriate schema:

"If no appropriate input can be obtained from the present 'primitive' input source, a frame (schema) will typically make a default evaluation…. Once an instantiated frame (schema) is judged acceptable, nearly all subsequent information processes used this instantiated frame as a data base. The original primitive data is thereafter ignored."

Noticeably lacking is any strategy of checking one's work, interpreting results against the constraints of the problem, or testing answers to determine if they make any sense. Gray, Pitta & Tall (1997) contend that: "different perceptions…, whether mental or physical, are at the heart of different cognitive styles that lead to success and failure" (p. 117). S25's work on the final open response rocket problem is indicative of the compartmentalization of knowledge in which procedures are linked to a particular concept—in this instance the use of matrices to solve linear systems only when S25's perception cues recall of the linear systems schema. Additional data on S3's and S25's engagement with high level cognitive tasks are in Appendix C.

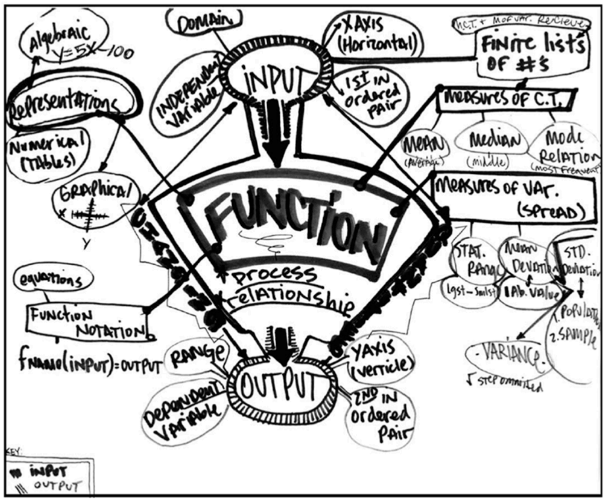
6. CONCEPT MAP EVIDENCE OF DIVERGENCE

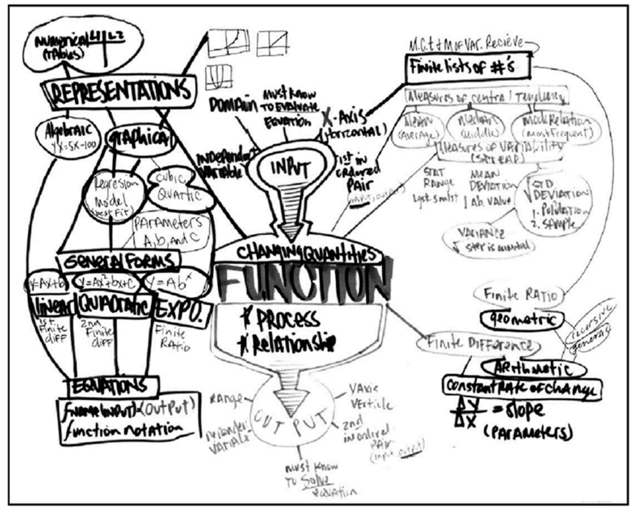
A comparison of their three concept maps revealed qualitative differences and the divergence that occurred between S3 and S25. S3's initial week 4 core organizational structure was retained and visible in the 2nd and 3rd concept maps. Each subsequent map of S25 had a different structure, retaining few, if any concepts from the week 4 or week 9 map in week 15. What elements were retained were relocated in a different arrangement. Spectral graph analysis of S25's corresponding schematic diagrams mathematically confirmed the differences and divergence in the underlying organizational structures of S3's and S25's concept maps. Triangulated with other data, the detailed analyses comparing the concept maps, corresponding schematic diagrams, and spectral graphs of S3 and S25 revealed radically different processes of knowledge construction over time.

6.1. Concept maps of S3. S3 visualized an initial anchoring concept of *Function* that reflected a unique way of thinking and organizing knowledge which was retained in week 9 and week 15 maps. Each map increased in complexity and interiority over the semester—visual evidence of the development of an organized, stable cognitive structure. Topics that were not a subject of study after week 4 (*measures of central tendenc*y and *measures of variability*), remain virtually unchanged on S3's week 9 map (Figure 4).

Figure 4. Week 4 and week 9 Concept Maps, S3

Week 4 **Week 9**





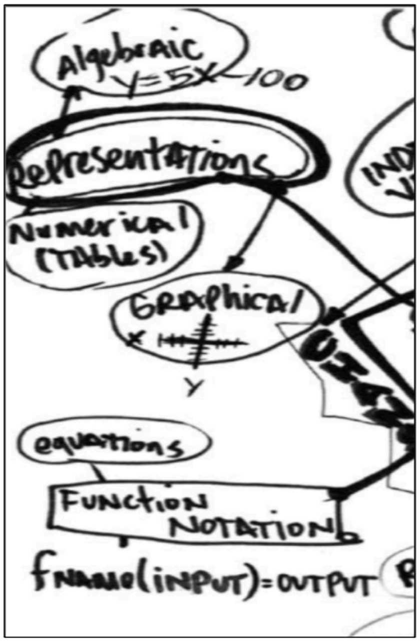
The growing complexity of mathematical knowledge and new connections is described in S3's written mid-term self-evaluation:

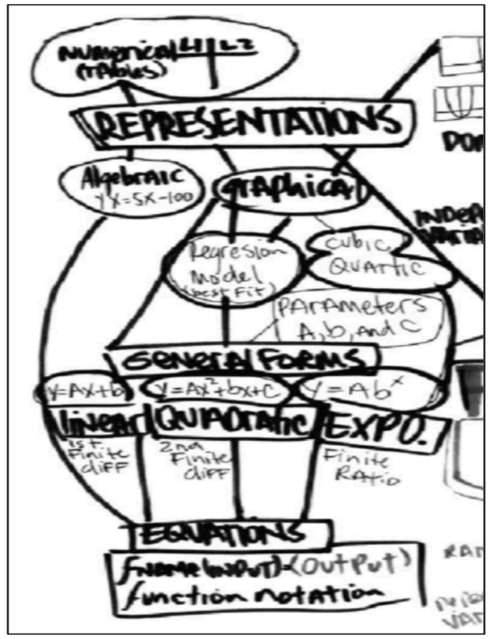
> "I think it's interesting how we learned to find finite differences and finite ratios early on and then expanded on that knowledge to understand how to find appropriate algebraic models. As we move on in class I've been noticing more and more the connection between what I have on my paper and what I have on my view window."

An examination of an enlarged portion of the left-side of weeks 4 and 9 maps documents this growing complexity. Week 9 map shows concepts *Representation (upper left)* and *Equations (lower left*) acquired greater complexity and new connections: *Equations* to (a) 1st *finite difference*, 2*nd finite difference*, and *finite ratio,* and (b) to *General Forms*: *liner*, *quadrati*c, and *exponential* (Figure 5).

Figure 5: Comparison of the left portion of week 4 and week 9 Concept Maps, S3.

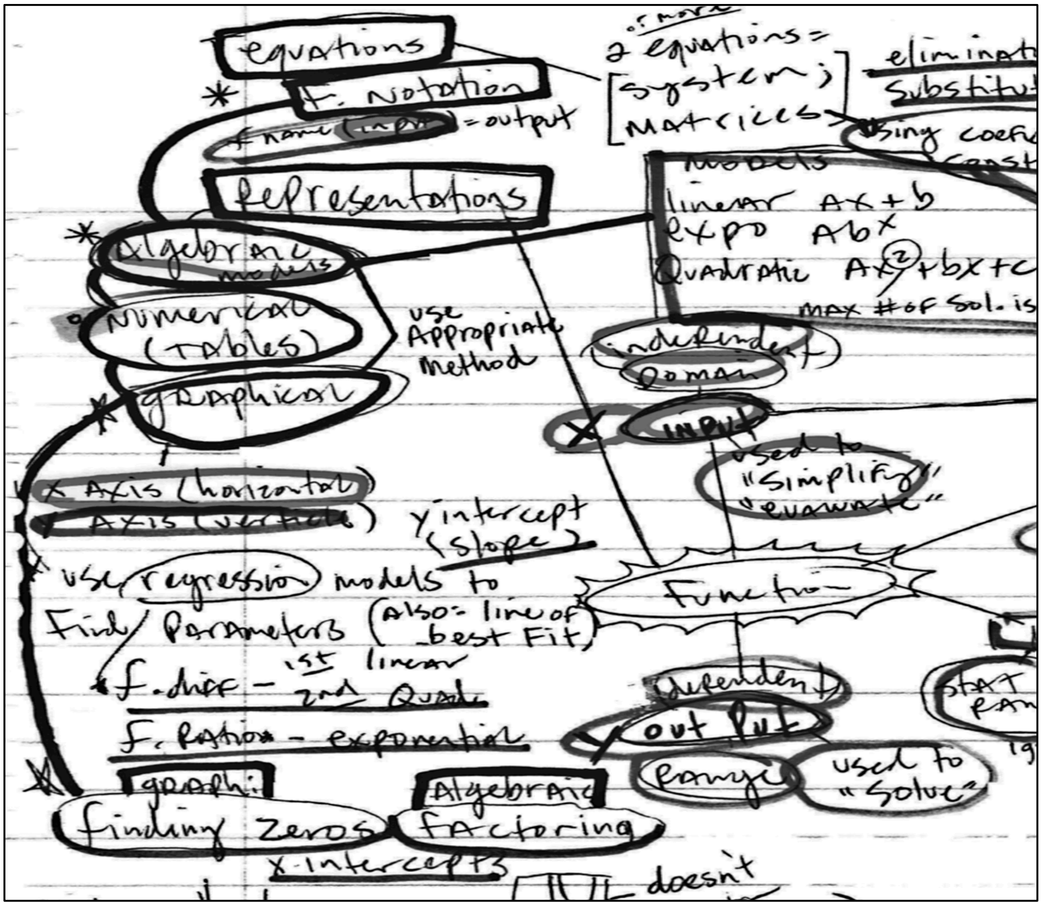
Week 4 **Week 9**





The anchoring concept of *Function*, with *Input* (above) and *Output* (below) remains the core structure of S3's week 15 map. New mental connections had been formed in the week 15 map, including concepts and procedures associated with *graphica*l*: x-axis (horizontal); y-axis (vertical); use regression models to: find parameters (finite differences-*1^{st}*-linear;* 2^{nd}*-quadratic, finite ratios (exponential); finding zeros (x-intercepts, factoring (algebraic); y-intercept; slope, also line of best fit).* Some restructuring has occurred. *Equations,* though still connected to the concepts *algebraic, numerical and graphical,* has been shifted from the lower left of week 4 and week 9 maps to the upper left above *Representation*s. New concepts: *systems* and *matrices* have been linked to *Equations.* The phrase, *use appropriate method* has been added and is lined to all three representations: *algebraic*, *numerical (table*s*)*; and *graphica*l. An enlargement of the left-side portion of S3's week 15 map shows these new connections and complexity more clearly.

Figure 6. Week 15 Concept Map (partial), S3.



On the semester final self-evaluation, S3 wrote:

> "While creating my (final) concept map on function, I was making strong connections in the area of representations. Specifically, between algebraic models and the graphs they produce. I noticed how both can be used to determine the parameters, such as slope and the *y*-intercept. At the beginning of the semester, I had a grasp of basic algebra—ex: finding slope, combining like terms, solving, evaluating, substituting and factoring…I feel that I now have a better understanding of these processes. Looking at lists of data and using finite differences to find slope $\Delta y / \Delta x$ has clarified my understanding of slope (rate of change). I was impressed by the link between finding the zeros (*x*-intercepts) on a graph and/or finding them algebraically by factoring (upper left-side). It's very efficient to graph an equation to find whether or not it has a real solution(s). I think the most memorable information from this class would be the use and understanding of function

notation. A lot of emphasis was put on input and output which really helped me comprehend some algebraic processes such as solving for *x*."

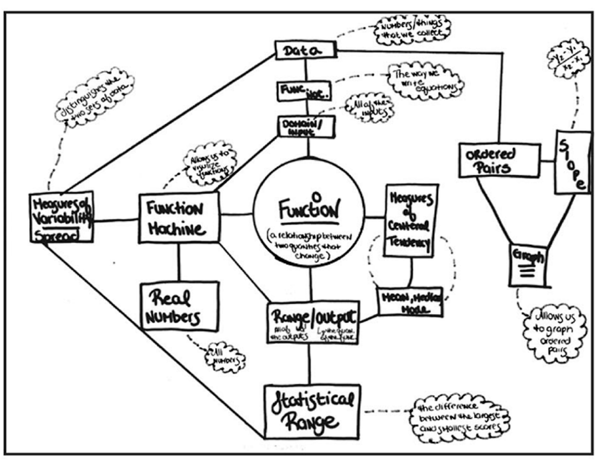
This is an articulate description of meaningful learning as characterized by Clarke, D., Helme, S. and Kessel, C. (1996), i.e., by evidence that indicates the student is not simply accumulating isolated facts and procedures but: (a) claims to have learned something new; (b) can articulate what it is they think they have learned with some degree of clarity and accuracy; (c) can demonstrate formation of links with an existing framework that the student already possesses and (d) has experienced a change from insecurity to confidence.

6.2. Concept Maps of S25

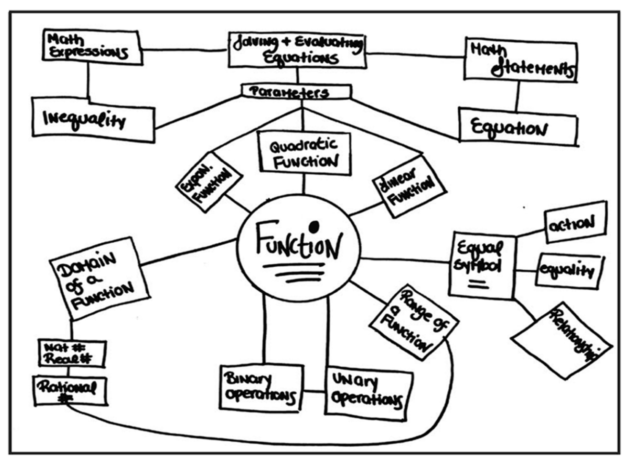
The concept maps of S25, typical of the maps of the low gain students, provide a sharp contrast to those of S3 (Figure 7).

Figure 7. Week 4 and Week 9 Concept Maps, S25.

Week 4



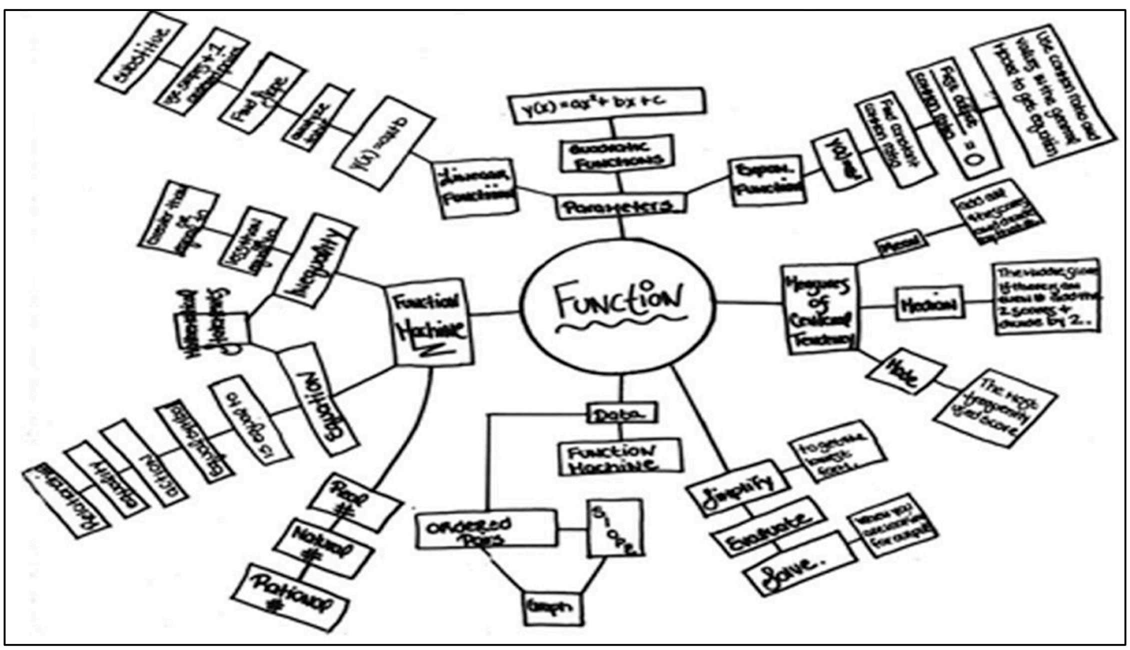
Week 9



The week 4 map includes definitions (clouds) of a few concepts (*range*, *output*, *domain*, *input*). S25's week 9 map shows a new, differently organized structure consisting only of the names of concepts (*domain, equations, parameters, inequality, linear*, *exponential*, and *quadratic function*) and processes (*solving, evaluating equations.* The element, *parameter*s, is linearly linked to each of the following: *inequality*, *equation*, *solving and evaluating equations.* No definitions are included. Except for *Function*, *domain* and *range*, no elements from week 4 appear on the week 9 map. The sparseness of the week 9 concept map suggests that S25 is not building on prior knowledge, given the absence of many of the Week 4 map elements. S25's final concept map reveals no links of procedures to concepts such as: *factors*, *zeros of the function*, or *roots of the equation* or connections to alternative representations such as graphical or numerical tables—all of which were the subject of investigation throughout the semester.

Week 15 map contains fewer elements (5) branching from *Function* than the Week 9 map (8) and is even more linearly organized. *Linear* and *exponential functions* on this map contain only computational procedural steps used to determine parameters of those functions (Figure 8).

Figure 8. Week 15 Concept Map, S25.



Basic functions studied in the course: *linear*, *quadratic*, and *exponential*, each linked directly to *Function* on the week 9 map, are now incorporated under the element *Parameter.* Notably missing are the concepts of *quadratic formula* and *discriminant*—frequently recalled prior knowledge of terms and procedures used by S25 throughout the semester. Also missing are the steps of a computational procedure for finding the parameters of a quadratic function. S25 recalled prior knowledge of computational procedures associated with solving quadratic equations (quadratic formula, finite ratios, discriminant) but usually used these procedures incorrectly and did not associate them with quadratic functions on either map of week 9 and 15.

S25's concept maps and written comments provide little evidence of meaningful learning. Limited growth and a lack of understanding of linear, quadratic, and exponential functions re confirmed in S25's final self-evaluation comments:

> "My greatest weakness on my concept map is understanding linear, quadratic, and exponential functions. After using my notes and the book I was able to put together some of the pieces of my confusion. I can get the formulas but when I have a specific problem I don't know which formula to choose to complete the problem."

S25 attributes the lack of success to having no prior knowledge of linear, quadratic, and exponential functions to build on. Despite claims of having no prior knowledge to build on, S25's demonstrated problematic prior knowledge of quadratic procedures offers a more likely explanation:

> "… when looking at a graph I can't tell if it is linear, quadratic, or exponential. This was the first time I have worked with these functions and I think that may have been part of the problem because I had no prior knowledge to build on. There are other areas that I feel that I have a strong understanding about as well. I choose *mathematical statements* because I was able to use my past knowledge and the new knowledge I have obtained this semester to have a better understanding."

The claim of better understanding of mathematical statements is also contradicted by S25's week 15 concept map and final portfolio evaluation. Without realizing it, at different times under different circumstances, S25 retrieved and used one of two different interpretations of *solve*. On week 15 concept map, *solve* links to: *when you are looking for output.* A week later, in the final portfolio self-evaluation, S25 wrote: *solve: know output, find input*. Unaware of using two different meanings for *solve*, S25 expressed surprise when the inconsistency was pointed out in week 16 interview.

S25's attitude became increasingly negative and confidence in the correctness of answers decreased over the semester.

> "I know, throughout the semester, I have learned many new math strategies, but making sense of them, I'm not so sure about. Perhaps I can make sense out of some like solving an equation but my confidence in what I can and cannot do is very low."

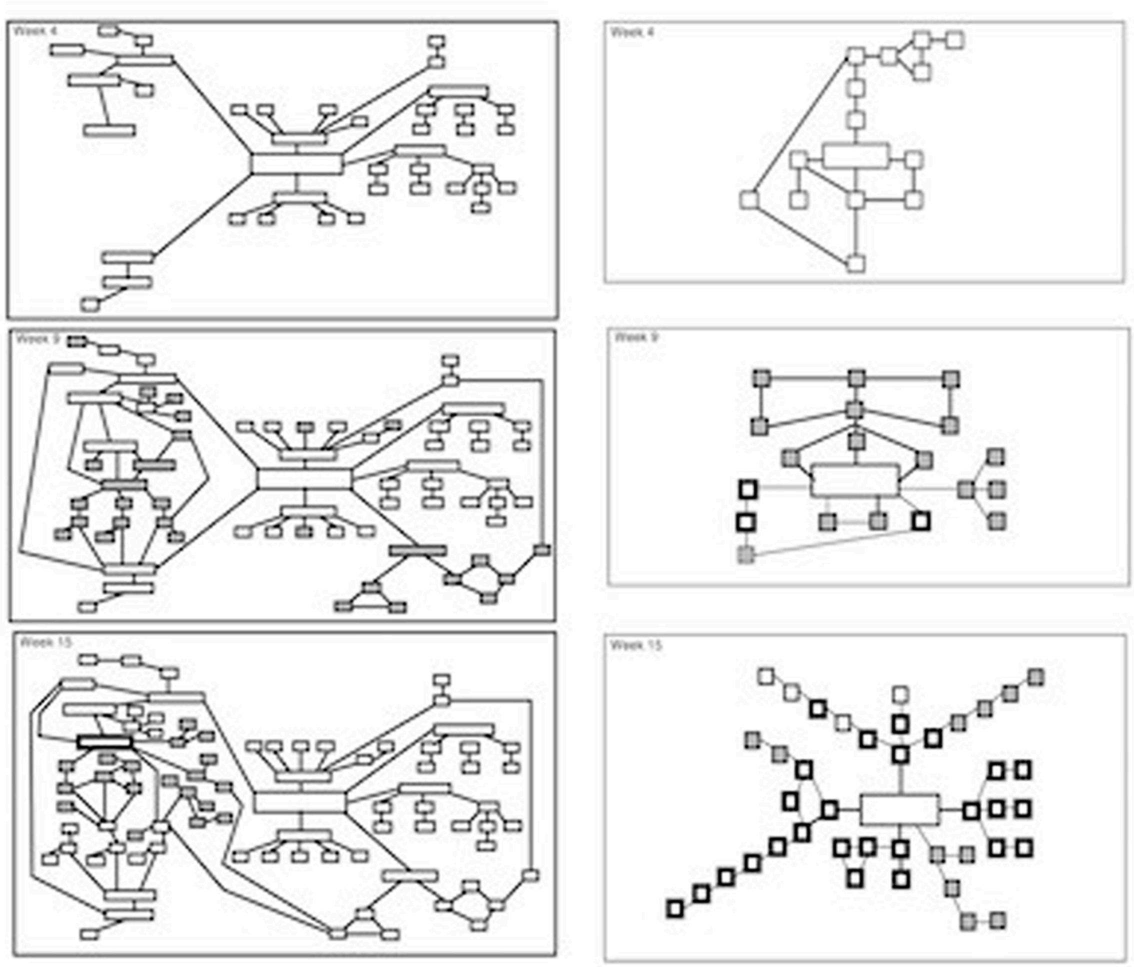
7. SCHEMATIC DIAGRAMS: UNDERLYING CONCEPT MAP STRUCTURES

The idiosyncratic nature of individual students' concept maps makes it challenging to compare one student's maps created over time or to compare the maps of one student with those of another student. Schematic diagrams provided a visual means of comparing different students' processes of knowledge construction and structural organizational changes over time.

Schematic diagrams of the three concept maps of each student are shown ordered top to bottom (the week 4, week 9, week 15), indicating underlying organizational structure and how it changed over time (Figure 9). Main categories (larger rectangles) and associated sub-categories (small rectangles and squares) in the week 4 schematic diagrams are un-shaded. New concepts, processes, and representations not included on an earlier map are indicated by gray-colored rectangles in Week 9 and Week 15. A boldly-outlined rectangle represents an element that was present on an earlier map and is now in a different category or relative position. The schematic diagrams of S3's three concept maps clearly show the underlying core structure of the week 4 map, retained and enriched in week 9, and grown even more in complexity and connections by week 15. The schematic diagrams of the three concept maps of S25 show a different organizational structure in each map. No core structure from week 4 is retained in either week 9 or week 15. Appendix D has the schematic diagrams of the three maps of two additional students, S1 (high gain) and S26 (low gain).

Figure 9. Schematic Diagrams of Concept Maps of S3 and S25.

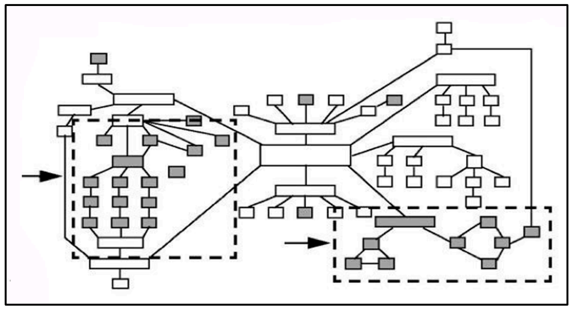
S3 **S25**



7.1. S3'S Schematic Diagrams

As one's eye moves over the schematic diagrams of S3 one is struck with a sense of familiarity. An enlarged image of S3's week 9 schematic diagram clearly shows the retained week 4 anchoring structure, with new bits and pieces of knowledge integrated into and added onto the initial framework. In the week 9 structure, the elements, *representations* and *equations* (left side) have become more complex and grown in interiority, with new, shaded elements and connections (Figure 10).

Figure 10. S3's Schematic Diagram of Week 9 Concept Map



The schematic diagrams of S25's concept maps reveal three distinct structures. No core structure is retained from week 4 to week 9 to week 15—an indication that a new cognitive framework was constructed by week 9 and yet another, distinct framework by week 15.

7. SPECTRAL GRAPH ANALYSIS

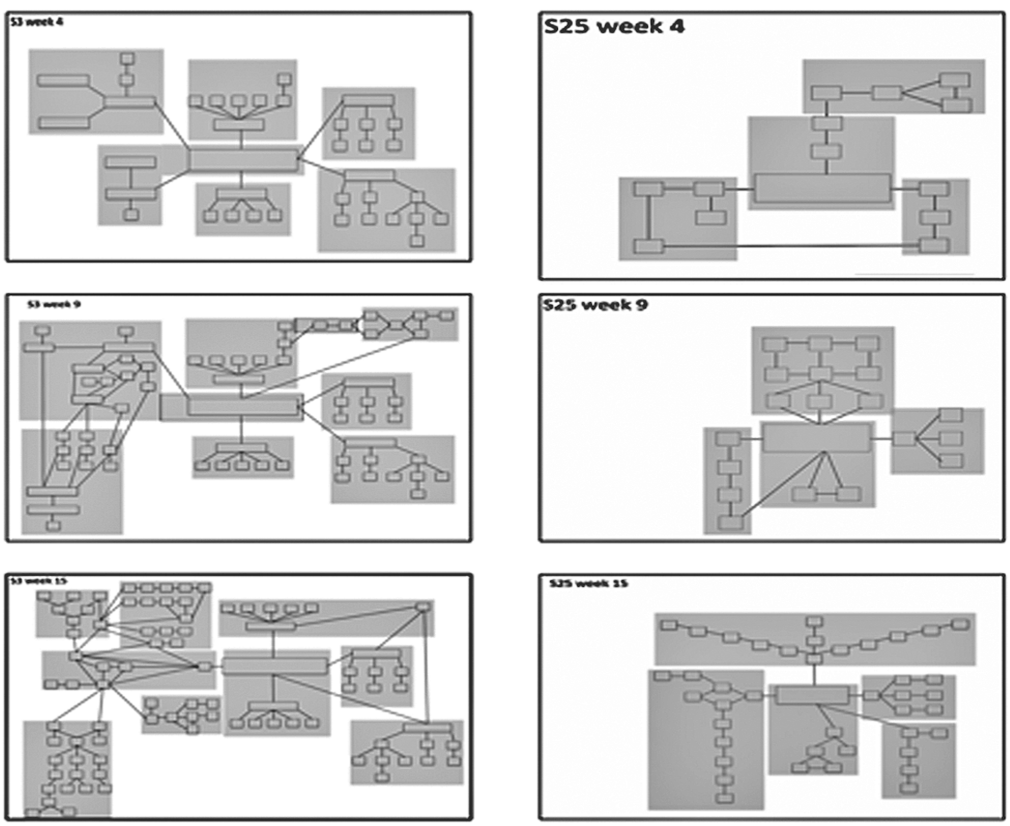
The growing complexity and stable structure of the spectral graph clusters of S3's schematic diagrams compared with the sparseness and lack of a stable organizational structure in the spectral graph clusters of S25's schematic diagrams confirm the findings based on analysis of S3's and S25's concept maps and schematic diagrams, triangulated with other data. When examined together with the original concept maps, the simple, graph clusters, absent any semantic labeling, are seen to closely resemble the schematic diagrams obtained from the labeled original concept maps contain vertices that are meaningfully connected. (Figure 11).

Spectral clustering of S3's week 4 schematic diagram reveals six *connected vertices* (i.e., all vertices within a cluster are connected). Week 9 diagram retains the basic Week 4 overall structure is retained, and contains seven clusters. The upper and lower left clusters of Week 9 build on the Week 4 organizational structure, having grown in complexity. Right center and the two lower right clusters remain unchanged. Week 15 includes nine clusters, including restructured left side clusters and new vertices. Some vertices shifted from one cluster to another.

The abstract graph of S25's Week 4 schematic diagram contains four connected clusters; Week 9, 4 clusters, and Week 15, 5 clusters. The Week 9 spectral graph has a new organizational structure. The center and upper right clusters of Week 4 are replaced with two center clusters of new vertices. Week 15 has yet another new organizational structure. Some vertices from weeks 4 and 9 are now included. The center right cluster structure contains completely different vertices from week 4. Basically, the spectral cluster analysis indicates that S25 does not retain the structure of identifiable clusters from one concept map to the next and has not developed an organized, coherent cognitive structure that builds on prior knowledge. Spectral clusters of the words, listed according to the corresponding numbered schematic diagram used to create the spectral clusters of S3's and S25's Week 4 concept map, are given in Table E1 of Appendix E.

Figure 11. Spectral Clustering of Schematic Diagrams

S3 **S25**



8. CONCLUSIONS & DISCUSSION

Two students identified in terms of achievement and ranked by the z-score of their fractional *gain* in test scores are profiled—one deemed to have “high gain” (*Gain* = 0.8006) and z-score = 1.5471, the other with “low gain” (*Gain* = 0.3067) and z-score –1.2897. Detailed analyses of three concept maps and corresponding schematic diagrams of S3 and S25, triangulated with quantitative and qualitative data delineates differences in their attitudes, beliefs, processes of knowledge construction, prior knowledge and strategies— documenting the divergence that occurred between them over a semester.

8.1 Findings of qualitatively different processes of knowledge construction

S3 was able to modify an existing way of thinking, demonstrated procedural flexibility of mathematical thinking, completed high level cognitive tasks, recalled and used prior strategies in novel problem settings, using both symbolic and non-symbolic procedures to translate between various representations. S3 demonstrated a consistency of performance on tasks involving linear and quadratic functions presented in different contexts and formats (open response, multiple choice), able to make and describe connections which required knowledge of several mathematical concepts (e.g. x-intercepts, zeros of the function, and factors), and justified responses.

S25 remained bound by a prior arithmetic interpretation of the minus symbol—unable to modify an existing inappropriate schema when interpreting the minus symbol in different contexts. Usually only one representational form was selected and used by S25 to investigate and solve a problem—invariably the more familiar symbolic procedure. S25’s weakest, inflexible performances included interpreting the minus symbol in various contexts and function notation, interpreting graphs. and tasks involving algebraic representations of quadratic functions.

S25 focused on only bits and pieces of a problem (the rocket problem, for example). When faced with tasks involving the graphing calculator, responded: "I can't deal with all of this, I'll only deal with this piece"—suggesting that this student might be classified at the uni-structural or, possibly, the (unconnected) multi-structural level, according to the SOLO taxonomy of Biggs and Collis (1982, p. 25), who proposed a hierarchy of "pre-structural, uni-structural, multi-structural, relational, extended abstract" levels to evaluate the quality of learning in many subject areas.

S25's failure to flexibly interpret and use ambiguous symbolism, resulted in "the uses of procedures that need to be remembered as separate devices in their own context" (Gray & Tall, 1994, pp. 20-21). Often, both previously and newly learned procedures were recalled imperfectly, resulting in computational errors. S25 demonstrated a lack of consistency of performance across different formats or on procedural activities in which a function rule was not stated, such as evaluating compositions of functions.

S25 demonstrated little growth and what growth occurred was unstable. Most surprising was the finding, documented in the concept maps, spectral graph analyses of the corresponding schematic diagrams, work, written comments and interviews, that S25 did not build successfully on prior knowledge. Robert Davis once described students like S25: "These students formed cognitive collages in which their limited understanding of mathematics terms; the fabric of words and expressions by which we communicate, shape, and modify our understandings to construct new collages, is too delicate, like lace or open cutwork, too fragile to hold the various bits and pieces together." (Davis, 1984; p. 154).

8.2. Concept maps documentation of divergent knowledge construction

Analyses of the concept maps and corresponding schematic diagrams, triangulated with written comments of S3 and S25, document the divergence that occurred over the semester. Spectral graph analysis of the schematic diagrams of their concept maps—a technique independent of semantic labeling and interpretation—mathematically confirmed the core structure and growing complexity of S3's knowledge structures and the absence thereof of those of S25.

S3's concept maps show that, initially, new knowledge was organized into a core cognitive structure that remained relatively stable over a semester. The set of schematic diagrams of S3's concept maps revealed a common characteristic: the week 9 and week 15 maps retained the basic structure of the week 4 map—a strong indication of retention in memory of a basic cognitive structure, in that students had no access to physical records of any of their previously constructed concept maps. once submitted.

The cognitive structure of S25's week 4 concept map was subsequently replaced by a new, differently organized, structure at week9 and at week 15. Concept map lacked essential linkages to related concepts and procedures. S25's schematic diagrams also exhibited a common characteristic: a new structure replaced the previous structure in each subsequent map, with few, if any elements of the previous map retained in the new structure. The few elements that were retained were generally reassembled into new local hierarchies or networks of concept images and schemas. Most concepts that appeared on an earlier map were not included in the subsequent map. Those prior concepts that were retained appear in different locations, associated with different elements on a later map. Spectral graph analysis of the schematic diagrams provides confirmation of the differences and divergence documented in the concept maps of S3 and S25 by ascertaining the stability of identifiable clusters on the students' concept maps.

S3 started as a self-described instrumental learner yet learned meaningfully through building rich connections between various mathematical topics. S25 began as a very instrumental and procedural learner and stayed that way. S3 and S25 were, mathematically speaking, marching to the beat of different drums. Both were determined to succeed in the course, so what prompted S3 to focus on learning more coherently and flexibly while S25, apparently, did not?

S3's attitudes and beliefs underwent a change during the semester. Those of S25 were impacted to a far lesser extent. S3 focused on understanding *why,* S25 focused on understanding *how to*—clear-cut examples of the difference between relational understanding and instrumental learning described by Skemp (1987, pp. 166–172). S3 was willing and able to change beliefs; S25 held fast to earlier beliefs—despite growing frustration and lack of progress. Early and later responses characterize this student and reflect the value attached to repetition and procedural rules which shaped the construction and organization of S25's concept images and schemas. Offered the same opportunities as S3, S25 stayed on the path of divergence towards the proceptual divide.

The divergence in knowledge construction and organization is far greater than the divergence measured by an ability to get a correct answer, or the grades of students at the end of term. What is remarkable and concerning for a teacher assessing students at the start of a course, is that the students who enter a class with very similar mathematical backgrounds, attitudes, and apparent abilities have such widely divergent learning trajectories over the course of a semester. The possibility that students like S3, construct, organize, and restructure their conceptual structures in ways that are qualitatively different from students such as S25, has profound implications for those of us who are attempting to deal with the practical problems of attending to the cognitive, social and human needs of our students.

8.3. Questions Unanswered and Possible Avenues for Future Research

Findings of the qualitative differences in S3's and S25's processes of knowledge construction, documented in their concept maps, raise questions for which we currently have few to no answers. It is not enough to just document the existence of the phenomenon: possible causes of it need to be examined. Could there be something else going on that drove this divergence? Could clues from neuroscience help us better understand the divergent learning trajectories of students like S3 and S25? Recent work in applications of neuroscience to mathematics education (De Smedt et al., 2010; De Smedt & Verschaffel, 2010) suggest that, at the very least, we should be asking the question of what neuroscience, especially the neuroscience of memory, might bring to a deeper understanding of the dynamics that drive such wide divergences in learning mathematics. It is now widely understood that the hippocampus is implicated in the formation, but not the storage or retrieval, of relational memories (Konkel & Cohen, 2009). Might it be that, given an appropriate context, one student was more capable than the other in the formation, and subsequent retrieval and expression, of relational memories? These are speculative questions, the answers to which may yet be far off.

Another intriguing possibility comes from recent empirical and theoretical research in working memory (Vandierendonck, 2014; Zokaei, Ning, Manohar, Feredoes & Husain, 2014). Working memory (WM), (Baddeley, 2003), is a short-term memory buffer that enables the use of stored information in real-time settings. There are at least two forms of attention that are highly relevant to working memory. One is *executive attention* which is concerned with overall carrying out, including planning, of a real-time task. Another is *selective attention* which is invoked when attention during a task is oriented to specific aspects, objects or representations involved in the task to the exclusion of others. Vandierendonck (2014) argues that in many circumstances selective attention can produce an overload on working memory that inhibits executive attention. In other words, focus on a specific aspect of a task can interfere, in real time, with executive control of the execution of the task. To our minds this is an excellent description of what took place in S25's solution of the toy rocket problem. Zokaei, et. al. (2014) argue that: "The *context* in which people are required to hold more than one item in WM makes a difference to how flexibly they can switch attention to non-focused items and improve their precision of recall" (p. 10). Again, we see reflections of this finding in relation to S25's problem solving abilities: S25 does not seem to be able to readily switch attention to non-focused items in a task to recall them with any overall relevance or precision.

Another possibility, also related to student memory, is the issue of practicing retrieval. Karpicke (2012) and Karpicke & Grimaldi (2012) demonstrate empirically that practicing retrieval of material produces meaningful, long-term learning. This finding is of more immediate use to teachers who can, through the use of frequent and periodic quizzes, promote the active retrieval of previously encountered material. If, as Karpicke & Grimaldi say, this promotes meaningful, long-term learning, that should become evident in student concept maps over time, something that is eminently testable. Blunt & Karpicke (2014) have utilized student concept maps as the *medium* for practicing retrieval and find that: "… results show that concept mapping can indeed serve as an effective task when it is implemented as a retrieval-based learning activity. The key element for promoting meaningful learning was not the format of the activity; it was the requirement to engage in active retrieval practice during learning" (p. 857).

The significant finding of this study is that low gain student, S25, unlike high gain student S3, did not build on prior knowledge and integrate new knowledge into a core cognitive structure that remained relatively stable over time. Currently, the conundrum of students like S25, who claim to want to connect new mathematical knowledge to old, yet appear unable to integrate new knowledge into existing structures, except in a very limited way, is still an unsolved problem.

ACKNOWLEDGMENTS

We thank Dr. John Mason and Dr. Eddie Gray for their constructive suggestions on an earlier version of this paper.

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APPENDIX A: Student Rankings:

Table A1. Pre-to-Total Final Gain* & Z-Score

RANK GAIN*	RANK POST	PRE-TEST	POST TEST	COURSE FINAL	DEPT FINAL	FINAL	GAIN	Z-SCORE
1	**1**	0.1429	0.9286	0.95	0.9286	0.9357	**0.9250**	**2.2619**
2	**2**	0.2143	0.8571	0.89	0.8235	0.8569	**0.8179**	**1.6465**
3	**3**	**0.1429**	**0.7857**	**0.88**	**0.8214**	**0.8290**	**0.8006**	**1.5471**
4	13	0.0714	0.5714	0.91	0.6786	0.7200	0.6985	0.9607
5	4	0.2857	0.7857	0.72	0.7857	0.7638	0.6693	0.7934
6	5	0.1429	0.6429	0.72	0.7857	0.7162	0.6689	0.7909
7	10	0.0714	0.5714	0.78	0.7143	0.6886	0.6646	0.7663
8	6	0.2857	0.5714	0.87	0.7857	0.7424	0.6393	0.6211
9	7	0.1429	0.5714	0.75	0.7500	0.6905	0.6389	0.6185
10	16	0.0714	0.4286	0.75	0.6786	0.6190	0.5897	0.3363
11	9	0.2143	0.5000	0.85	0.6786	0.6762	0.5879	0.3256
12	15	0.1429	0.3571	0.85	0.5357	0.5810	0.5111	-0.1154
13	14	0.2857	0.5000	0.75	0.6786	0.6429	0.5000	-0.1792
14	12	0.3571	0.5000	0.85	0.6429	0.6643	0.4778	-0.3068
15	11	0.1429	0.3571	0.72	0.5714	0.5495	0.4744	-0.3260
16	19	0.0714	0.3571	0.65	0.5000	0.5024	0.4641	-0.3854
17	20	0.1429	0.3571	0.72	0.5357	0.5376	0.4606	-0.4058
18	21	0.0714	0.2857	0.72	0.4643	0.4900	0.4508	-0.4620
19	17	0.1429	0.3571	0.62	0.5357	0.5043	0.4217	-0.6291
20	8	0.4286	0.5714	0.72	0.7143	0.6686	0.4200	-0.6387
21	18	0.2143	0.3571	0.72	0.5357	0.5376	0.4115	-0.6874
23	**22**	0.2857	0.2143	0.75	0.6186	0.5276	**0.3387**	**-1.1059**
22	**23**	0.1429	0.2143	0.65	0.4286	0.4310	**0.3361**	**-1.1206**
24	**25**	0.0714	0.1429	0.6	0.3929	0.3786	**0.3308**	**-1.1512**
25	**24**	**0.1429**	**0.2143**	**0.61**	**0.3929**	**0.4057**	**0.3067**	**-1.2897**
26	**26**	0.1429	0.1429	0.6	0.2143	0.3190	**0.2056**	**-1.8704**
	MEAN	**0.1758**	**0.4670**	**0.7538**	**0.6228**	**0.6146**	**0.5312**	
	STDEV	**0.0953**	**0.2159**	**0.1010**	**0.1652**	**0.1516**	**0.1741**	

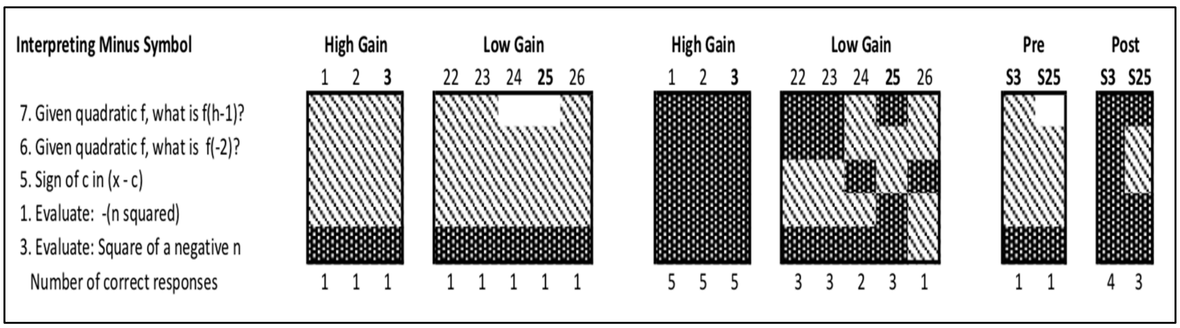
APPENDIX B: High and low gain students' strategies and divergence

A given ranked student's responses (column) indicates: correct response (black); incorrect response (stripe); no attempt (white) to question.

I. Pre- to post-test divergence in ability to interpret the minus sign in context.

High and low gain students' pre- and post-test responses interpreting the minus symbol in different contexts are shown in Figure B1.

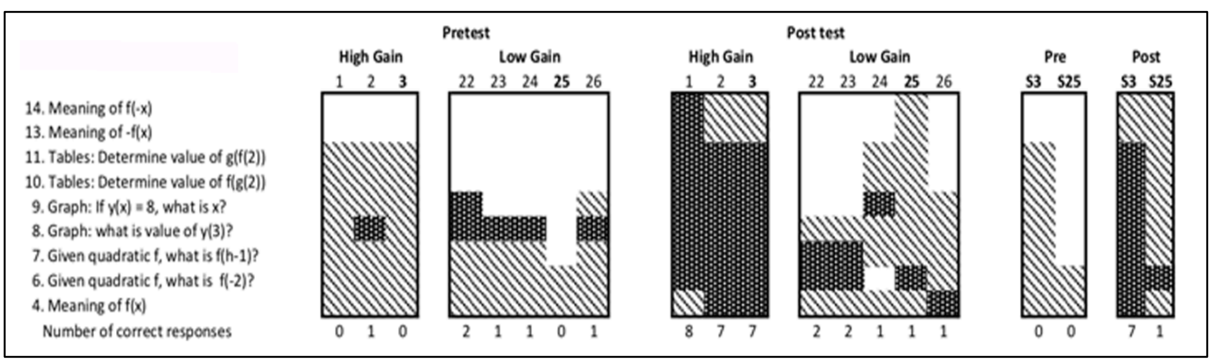
Figure B1. High & low gain responses interpreting the minus symbol.



II. Ability to interpret function notation in different contexts

Pre- and post-test responses of high and low gain students to questions interpreting function notation in different contexts are shown in Figure 6.

Figure B2. High & low gain pre- and post-test responses interpreting function notation.



III. Resistance to changing belief about a minus symbol in front of a variable.

Post-test responses of both high gain and low gain students are shown in Table B1 indicting how resistant to change is the belief that a minus sign in front of a variable means "negative."

Table B1: Post-test responses to questions requiring interpretation of the minus sign

QUESTION	STUDENT	POST-TEST RESPONSE
Q1: -5^2	S2	square –5; Answer: –25
Q1: -5^2	S22	take –5 X –5 to get answer –25
Q5:(x – c)	S4	The value of *c* is positive because *x* is in front of the –sign; if just –*c,* then *c* would be negative.
Q5:(x – c)	S25	Subtract, change to $x + {}^{-}c$; *c* is negative
Q13: $-f(x)$	S2	negative whole output
Q13: $-f(x)$	S3	$-f(x)$ makes the whole function negative
Q13: $-f(x)$	S25	*f* of the function is negative

IV. Stability of reconstructed schemas of high and low gain students

At various times during the semester, students were asked to evaluate a quadratic function, $f(x) = x^2 - 2x + 7$, given a negative numerical value input, $f(-3)$, and a binomial input, $(h-1)$. The stability of re-constructed schemas of the entire class member are shown in Figure B3. Those of high and low gain students are shown in Figure B4.

Figure B4. Stability of Class Responses to Questions involving minus symbol

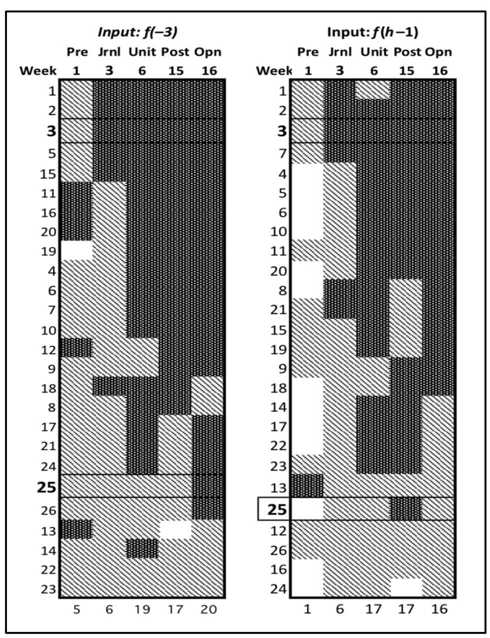
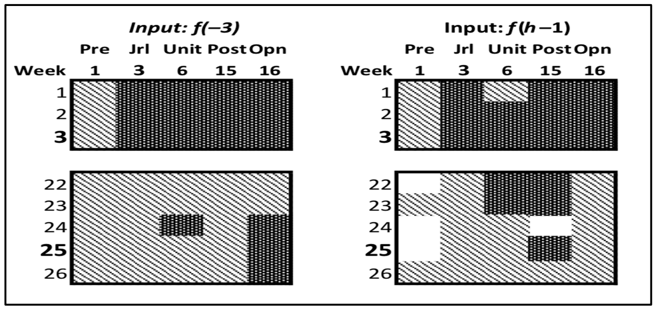


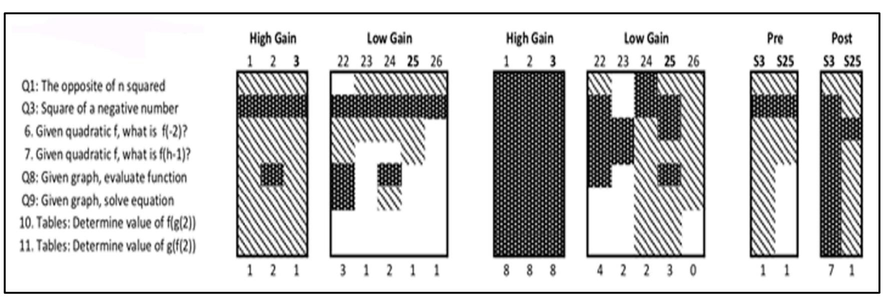
Figure B3. Stability of reconstructed schemas of high and low gain students



IV. Flexible thinking: Reversibility of Thought

Ability of students in each group of extremes to think flexibly and switch one's train of thought from a direct process to its reverse process was examined. Pre- and post-test questions related to reversing a process included (a) the direct arithmetic process involving the minus sign, together with its reverse process, (i.e., square a negative value and determine the additive inverse of a value squared (Q1 and Q3)); (b) a symbolic functional representation involving the minus symbol in squaring a negative-valued input and squaring an algebraic input (binomial) (Q6 and Q7); (c) evaluate a function and solve an equation given graphical representation (Q8 & Q9); and (d) determine the value of f(g(x)) and g(f(x)), given two tables of values. High and low gain responses are shown in Figure 7.

Figure B5. Reversibility of Thought



APPENDIX C. S3's and S25's engagement with high level cognitive tasks

S3's and S25's responses to a journal problem and to pre- and post-test questions provide yet other examples of the extent to which they engaged or did not engage in high level tasks. The Week 5 journal problem involved both symbolic and non-symbolic representations (algebraic, numerical tables) and required use procedures connected to concepts: input/output values with ordered pairs, finite differences with parameters, and justification of the specific algebraic model selected.

Week 3 Journal Problem: Given the table of data, determine whether this data is modeled by a linear, exponential or quadratic function. Why? Find the specific algebraic model for the data. Show all work.

Table C1: Journal 5

INPUT	OUTPUT
0	3
1	4
2	9
3	18
4	31

S25 wrote: "This data is modeled by a quadratic function because the common ratio beginning with the second output is constant." S25 work consisted of: (Quadratic formula); $y(x) = ax^2 + bx + c$." S25 circled a, b, and c. S25 was one of two low gain students who calculated second finite differences. No student in this cohort provided a specific equation or work to determine the parameters of the equation.

S3 included work, a specific algebraic equation for the given data, an explanation for determining the value of parameter, a, and justification for choice of algebraic model. After completing the table, S3 wrote: “Quadratic, because the second finite difference is constant.”

Table C2: S3 Journal 5

INPUT	OUTPUT	Δ	ΔΔ
0	3	1	4
1	4	5	4
2	9	9	4
3	18	13	
4	31		

$y(x) = ax^2 + bx + c$; $c = 3$, output when input =
$A = 2$, because ½ of 4 =2 (½ 2nd finite diff)
$y(x) = 2x^2 + bx + 3$; $4(1) = 2(1)^2 + bx + 3$
$4 = 2 + b + 3$; $4 = 5 + b$; $-1 = b$
$y(x) = 2x^2 - x + 3$

S3’s and S25’s responses to Q10 and Q11, asks for evaluation of the direct process $f(g(1))$ and its reverse process, $g(f(5))$, given tables of values for functions f and g.

Table C3: Composition of functions

Given the following tables for functions f and g:

x	$f(x)$		x	$g(x)$
1	3		–2	3
2	–1		–1	1
3	1		0	5
4	0		1	2
5	–2		2	4

10. What is the value of f (g(1))?

Pre-test: S3: I’m really not sure how these two tables relate to one another.

S25: No attempt at response.

Post-test: S3: $g(1) = 2$; $f(g(1))$ is equal to $f(2)$; $f(2) = -1$, so $f(g(10)) = -1$.

S25: I think of $g(1)$ meaning when x is at 1, what is g. $f(2)$.

11. What is the value of g(f (5))

Pre-test: S3: –1 comes to mind because its opposite of 2of the table of functions.

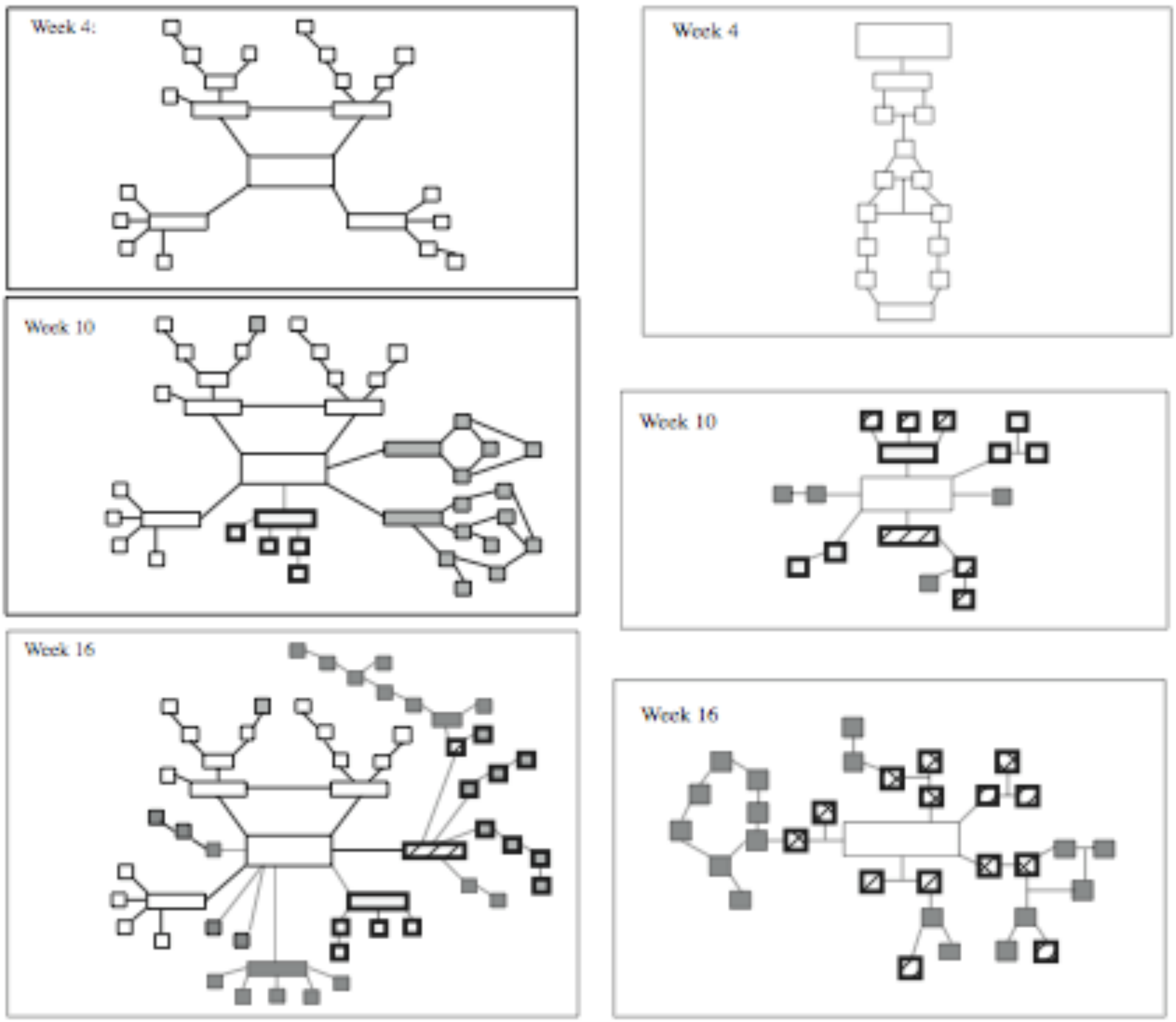
S25: No attempt at response.

Post-test: S3: $x = 3$. f(5) = –2; $g(-2) = 3$. Finding the output of $f(5)$ is –2. Then inputing –2 in function $g(x)$, and using table values to find the output.

S25: What is x when f is at 5. $f(-2)$.

APPENDIX D: SCHEMATIC DIAGRAMS CONCEPT MAPS OF S1 AND S26

FIGURE D1: Schematic Diagrams ofConceptMapsofS1 andS26.



Main categories (larger rectangles) and associated sub-categories (small rectangles and squares) in the week 4 schematic diagrams are un-shaded. New concepts, processes, and representations not included on an earlier map are indicated by gray-colored rectangles in Week 9 and Week 15. A boldly-outlined rectangle/square represents an element that was present on an earlier map and is now in a different category and/or relative position.

APPENDIX E: S3'S & S25'S SPECTRAL CLUSTERINGS & WEEK 4 MAP TERMS

Table E1. S3: Week 4 Concept map terms by spectral cluster

Cluster 1 (Upper left)	Cluster 2 (Lower left)	Cluster 3 (Upper center)	Cluster 4 (Lower center)	Cluster 5 (Upper right)	Cluster 6 (Lower right)
Representations	Function (process, relationship)	Function	Output	M.C.T.	Meas. of var. (spread)
Numerical (Tables)	Function notation	Input	Range	Mean	Stat Range
Algebraic,	Equations	Independent Variable	Dependent Variable	Median	Mean Deviation
y = 5x – 100	*F*name(Input) = output	Domain	2^{nd}, Ord. pair	Mode	Std. Deviation
Graphical		x-axis (horizontal)	y-axis (vert)	average	Largest-smallest
		1^{st}, Ord. pair		Middle	Abs value
		Finite lists		most frequent	Population
		M.C.T.			Sample
					Variance, sq. root

Table E2. S25: Week 4 Concept map terms by spectral cluster

Cluster 1 (Upper left)	Cluster 2 (Lower left)	Cluster 3 (Upper center)	Cluster 4 (Lower center)
Data	Function	Function Machine	Measures of Central Tendency
Ordered Pairs	Domain/Range	Real Numbers	Mean, Median and Mode
Graph	Function Notation	Measures of Variability	Range/Output
Slope		Statistical Range	