

ORTHOGONALITY RELATIONS ON ABELIAN GROUPS

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Abstract

An orthogonality relation is an abstract relation on a group having properties similar to the relation on a l -group given by $x \perp y$ if $|x| \wedge |y| = 0$. A group G with an orthogonality relation \perp is isomorphically represented as a subgroup of the group Γ of continuous global sections of a sheaf of groups. If the stalks of the sheaf are torsion-free and G has an element 1 satisfying $1^+ = (0)$ then Γ can be ordered so that it is an l -group and $x \perp y$ if and only if $|x| \wedge |y| = 0$ in Γ . An l -group G is *complemented* if for all $x, y \in G$ there is an $a \in x^+ \oplus x^{++}$ with $y \in a^{++}$: equivalent conditions are given for G to be complemented.

Introduction

A. I. Veksler (1967) introduced the concept of disjointness relation on a linear space, as an abstraction of the relation \perp on vector lattices defined by $x \perp y$ if $|x| \wedge |y| = 0$, and showed that for certain linear spaces E with a disjointness relation \perp it is possible to define a lattice-order on E such that $x \perp y$ if and if and only if $|x| \wedge |y| = 0$. In this paper the concept of disjointness, now called orthogonality, is extended to abelian groups. It is seen that every group G with an orthogonality relation is (isomorphic with) a subgroup of a group Γ of continuous global sections of a sheaf of groups with a Boolean base space. When the group G has an element 1 satisfying $1^+ = (0)$ and the groups comprising the stalks in the sheaf representation are torsion-free then these stalks can be totally-ordered so that Γ is a lattice-ordered group and $x \perp y$ if and only if $|x| \wedge |y| = 0$.

Finally, various characterizations are given of a class of l -groups that were called “weakly projectable” by Spirason and Strzelecki (to appear).

1. Orthogonality relations

Throughout, let G be an abelian group. A relation \perp on G is a *pre-orthogonality* relation if

- (1) $x \perp y$ implies $y \perp x$
- (2) $0 \perp x$ for all $x \in G$, where 0 is the identity of G
- (3) $x \perp x$ implies $x = 0$
- (4) $x \perp a, y \perp a$ implies $x + y \perp a$.

For each non-empty subset A of G define $A^\perp = \{x \in G: x \perp a\}$ for all $a \in A$, $A^{\perp\perp} = (A^\perp)^\perp$. If $A = \{x\}$ is a singleton set then denote $A^\perp, A^{\perp\perp}$ by $x^\perp, x^{\perp\perp}$. An *orthogonality relation* on G is a pre-orthogonality relation \perp for which $x \perp y$ is equivalent to $x^{\perp\perp} \cap y^{\perp\perp} = (0)$. Using Frink's axioms (1941) for a Boolean algebra, it is readily seen that when \perp is an orthogonality relation the class $B(G)$ of subsets of G of the form A^\perp , ordered by inclusion, is a complete Boolean algebra with $\bigcap_\alpha A_\alpha^\perp = (\cup A_\alpha)^\perp$ and $A^{\perp\perp}$ as the complement of A^\perp .

The Stone space of $B(G)$ (i.e., the set of prime ideals of $B(G)$ equipped with the hull-kernel topology) will be denoted by Q . For each $t \in Q$ a subset G_t of G is defined by $G_t = \{x \in G: x^{\perp\perp} \in t\}$. It is readily seen that each G_t is a subgroup of G and $\bigcap_{t \in Q} G_t = (0)$. Using the method of Dauns and Hoffman (1966) it can be seen that there is a sheaf of groups with base space Q and stalks $G/G_t, t \in Q$, and an isomorphism \wedge given by $\hat{x}(t) = x + G_t$ mapping G into the group Γ of continuous global sections of this sheaf, such that $x \perp y$ if and only if for each $t \in Q$ either $\hat{x}(t) = 0$ or $\hat{y}(t) = 0$. This sheaf representation is more special than the obvious representation of G as a sub-direct product of the groups G/G_t only in that Q has a topology and the disjoint union of the groups G/G_t is given the finest topology for which each \hat{x} is continuous. Γ then consists of all choice functions $Q \rightarrow \cup G/G_t$ that agree locally with some \hat{x} . Since Q is compact Hausdorff and has a base for the open sets consisting of the closed open sets $Q_{A_t^\perp} = \{t \in Q: A^\perp \notin t\}, A \subseteq G$, then for any $\sigma \in \Gamma$ there is a finite closed-open partition $Q_{A_1^\perp}, \dots, Q_{A_n^\perp}$ of Q , and $x_1, \dots, x_n \in G$ such that $\sigma = \sum I(Q_{A_i^\perp}; x_i)$ where

$$I(Q_{A_i^\perp}; x_i)(t) = \begin{cases} \hat{x}_i(t) & \text{if } t \in Q_{A_i^\perp} \\ 0 & \text{if } t \in Q_{A_i^\perp}^c \end{cases}$$

A necessary and sufficient condition for $\hat{G} = \{\hat{x}: x \in G\}$ and Γ to be isomorphic is $G = A^\perp \oplus A^{\perp\perp}$ for all $A^\perp \in B(G)$.

2. Lattice-ordering Γ

It will be assumed in this section that \perp is an orthogonality relation on the abelian group G .

LEMMA 2.1. *The quotient groups G/G_t are all torsion-free if and only if $x^\perp = (mx)^\perp$ for all x in G and all integers $m > 1$.*

PROOF. Assume $x^\perp = (mx)^\perp$ for all x in G and all $m \geq 1$. If $mx \in G_t$ then $x^{\perp\perp} = (mx)^{\perp\perp} \in t$ so $x \in G_t$ and G/G_t is torsion free. If each G/G_t is torsion-free and $y \perp mx$ then $y \notin G_t$ implies $mx \in G_t$ so $x \in G_t$ and thus $y \perp x$.

PROPOSITION 2.2. *Let G have an orthogonality relation \perp satisfying $x^\perp = (mx)^\perp$ for all $x \in G$, $m \geq 1$. For each $t \in Q$ let P_t be a positive set defining a total order on G/G_t compatible with the group structure. Define a partial order on Γ by $\sigma \geq 0$ if $\sigma(t) \in Q$. Then Γ is lattice-ordered if for each $x \in G$ the set $\text{Pos}(x) = \{t \in Q: \hat{x}(t) > 0\}$ is open in Q .*

PROOF. Each G/G_t is abelian and torsion-free by lemma 2.1 and therefore admits a total order P_t . Suppose that $\text{Pos}(x)$ is open for each $x \in G$. Then the set $\{t \in Q: \hat{x}(t) \geq 0\} = Q \setminus \text{Pos}(-x)$ is closed and contains $\text{Pos}(x)$ yet if $\hat{x}(t_0) = 0$ then there is a basic closed-open neighbourhood $Q_{A^+} = \{t \in Q: A^\perp \notin t\}$ of t_0 such that $\hat{x}(t) = 0$ for all $t \in Q_{A^+}$ so that $Q_{A^+} \cap \text{Pos}(x)$ is void. Hence $\text{Pos}(x)$ is closed and therefore closed-open. The function $\hat{x} \vee 0$ mapping Q into $\bigcup_{t \in Q} G/G_t$ according to $t \mapsto \max\{\hat{x}(t), 0\}$ is the supremum of \hat{x} and 0 in the ordered group of all (not necessarily continuous) global sections. However, $\hat{x} \vee 0 = I(\text{Pos}(x); x)$ where

$$I(\text{Pos}(x); x)(t) = \begin{cases} \hat{x}(t) & \text{if } t \in \text{Pos}(x) \\ 0 & \text{if } t \notin \text{Pos}(x), \end{cases}$$

so $\hat{x} \vee 0 \in \Gamma$. For $\sigma = \sum_i I(Q_{A_i^+}; x_i) \in \Gamma$, where $\{Q_{A_i^+}\}$ is a finite closed-open partition of Q ,

$$\begin{aligned} \sigma \vee 0(t) &= \max\{\sigma(t), 0\} \\ &= \sum_i I(Q_{A_i^+} \cap \text{Pos}(x_i); x_i) \end{aligned}$$

is in Γ , so that Γ is lattice-ordered.

THEOREM 2.3. *If G has an orthogonality relation \perp satisfying $x^\perp = (mx)^\perp$ for all x in G , $m \geq 1$ and an element 1 satisfying $1^\perp = (0)$ then the group Γ can be lattice-ordered so that $\sigma \perp \tau$ if and only if $|\sigma| \wedge |\tau| = 0$ for σ, τ in Γ .*

PROOF. Since $1^\perp = (0)$ then $1 \notin G_t$ for all $t \in Q$. For each $t \in Q$ there is a positive set in G/G_t such that for each $x \in G$ the set $\text{Pos}(x) = \{t \in Q: \hat{x}(t) > 0 \text{ in } G/G_t\}$ is open. Consider the positive sets $\{m1 + G_t: m = 0, 1, 2, \dots\}$: for $x \in G$, $\text{Pos}(x) = \bigcup_{m=0}^\infty \{t \in Q: \overline{x - m1}(t) = 0\}$ which, as a union of open sets, is open. The class K of strings (P_t) such that each P_t is a positive set in G/G_t containing $1 + G_t$ and such that $\text{Pos}(x) = \{t \in Q: 0 \neq x + G_t \in P_t\}$ is open for each $x \in G$ is inductive when ordered by $(P_t)_{t \in Q} \geq (P'_t)_{t \in Q}$ if $P_t \supseteq P'_t$ for all $t \in Q$, so that maximal elements of K follow from the axiom of choice. Let $(P_t)_{t \in Q}$ be maximal in K . For $x \in G$ the closure $\overline{\text{Pos}(x)}$ of $\text{Pos}(x)$ is closed-open in Q . If $t \in Q$ define $P'_t = \{x + G_t: t \in \overline{\text{Pos}(x)}\}$. If $x + G_t \in P'_t \cap (-P'_t)$ then $t \in \overline{\text{Pos}(x)} \cap \overline{\text{Pos}(-x)}$ which, however, is void since $\overline{\text{Pos}(x)}$, $\overline{\text{Pos}(-x)}$ are disjoint open sets with open closures. For each $t \in Q$ the set $P'_t \cup \{0\}$ is therefore a positive set in G/G_t containing P_t . Furthermore, for $x \in G$,

$$\begin{aligned} \text{Pos}'(x) &= \{t \in Q: x + G_t \in P'_t\} \\ &= \{t \in Q: t \in \overline{\text{Pos}(x)}\} = \overline{\text{Pos}(x)} \text{ is closed-open.} \end{aligned}$$

The maximality of $(P_t)_{t \in Q}$ in K therefore gives that $\text{Pos}(x)$ is closed-open for each $x \in G$. Now suppose that for some $t_0 \in Q$, P_{t_0} does not define an isolated order on G/G_{t_0} , so there is an $a + G_{t_0} \in P_{t_0}$ and an integer $m_0 \geq 2$ such that $m_0 a + G_{t_0} \in P_{t_0}$.

Then, $t_0 \in \text{Pos}(m_0 a)$ so there is an open neighbourhood Q_{A^+} of t_0 contained in $\text{Pos}(m_0 a)$. Also, $t_0 \notin \text{Pos}(a)$ so there is an open neighbourhood Q_{B^+} of t_0 contained in $Q \setminus \text{Pos}(a)$. Then for $t \in Q_{A^+} \cap Q_{B^+}$, P_t does not define an isolated order on G/G_t . Consider the string $(P'_t)_{t \in Q}$ with

$$P'_t = \begin{cases} P_t & \text{if } t \in Q_{A^+} \cap Q_{B^+}, \\ \{ma + y + G_t: m = 0, 1, 2, \dots, y + G_t \in P_t\} & \\ \text{if } t \in Q_{A^+} \cap Q_{B^+} \end{cases}$$

Each P'_t is a semigroup properly containing P_t . If P'_t is not a positive set for some $t \in Q_{A^+} \cap Q_{B^+}$ then $n_1 a + y_1 + G_t = -n_2 a - y_2 + G_t$ for some integers $n_1, n_2 \geq 1$, and $y_1 + G_t, y_2 + G_t \in P_t$. Then $(n_1 + n_2)a + G_t = -(y_1 + y_2) + G_t \in -P_t$ and $m_0(n_1 + n_2)a + G_t \in P_t$ (since $m_0 a(t) > 0$ for $t \in Q_{A^+} \cap Q_{B^+}$) so that $m_0(n_1 + n_2)a + G_t = 0$ which means $a \in G_t$ since G/G_t is torsion-free. This contradicts the choice of $a \in G$ so that each P'_t is a positive set in G/G_t . For $x \in G$ put $\text{Pos}'(x) = \{t \in Q: 0 \neq x + G_t \in P'_t\}$. Suppose $t_1 \in \text{Pos}'(x)$. If $t_1 \in Q_{A^+} \cap Q_{B^+}$ then $t_1 \in \text{Pos}(x)$ so there is an open neighbourhood of t_1 contained in $\text{Pos}(x)$ and hence in $\text{Pos}'(x)$. If $t_1 \in Q_{A^+} \cap Q_{B^+}$ then $\hat{x}(t_1) = \overline{ma + y}(t_1) \neq 0$, so there is an open neighbourhood Q_{C^+} of t_1 such that $\hat{x}(t) = \overline{ma + y}(t) \neq 0$ for $t \in Q_{C^+}$. Also, there is an open neighbourhood Q_{D^+} of t_1 such that $\hat{y}(t) > 0$ for $t \in Q_{D^+}$. Thus $Q_{C^+} \cap Q_{D^+} \supseteq \text{Pos}'(x)$. The set $\text{Pos}'(x)$ is therefore open for each $x \in G$. This contradicts the maximality of $(P_t)_{t \in Q}$ in K so that each P_t defines an isolated order on G/G_t . Now suppose that for some $t \in Q$, P_t does not define a total order on G/G_t , so that for some $a \in G$, $a + G_t \in P_t \cup (-P_t)$. That is $t \in \text{Pos}(x) \cap \text{Pos}(-x)$ which is an open set, so for t_1 in some open neighbourhood Q_{A^+} of t , P_t does not define a total order on G/G_{t_1} . As before, consider the string $(P'_t)_{t \in Q}$ with

$$P'_t = \begin{cases} P_t & \text{if } t \in Q_{A^+}, \\ \{ma + y + G_t: m = 0, 1, 2, \dots, y + G_t \in P_t\} & \\ \text{if } t \in Q_{A^+} \end{cases}$$

Each P'_t is a semigroup containing P_t in G/G_t . If P'_t is not a positive set for some $t \in Q_{A^+}$ then, as before, $ma + G_t \in -P_t$ for some $m \geq 1$. Since P_t defines an

isolated order on G/G_t , then $a + G_t \in -P_t$, contrary to assumption. The maximality of $(P_t)_{t \in Q}$ therefore gives that each P_t defines a total order on G/G_t . The group Γ is therefore lattice-ordered by $\sigma \geq 0$ if $\sigma(t) \in P_t$ for all $t \in Q$. Since $|\sigma|(t) = \max\{\sigma(t), -\sigma(t)\}$ in G/G_t and $\sigma \wedge \tau(t) = \min\{\sigma(t), \tau(t)\}$ in G/G_t , then $|\sigma| \wedge |\tau| = 0$ if and only if for each $t \in Q$ either $\sigma(t) = 0$ or $\tau(t) = 0$.

3. Lattice-groups

In this section G will denote an abelian lattice-group. The relation defined on G by $x \perp y$ if $|x| \wedge |y| = 0$ is an orthogonality relation with

$$x^{\perp\perp} \cap y^{\perp\perp} = (|x| \wedge |y|)^{\perp\perp}$$

$$x^{\perp\perp} \vee y^{\perp\perp} = (|x| \vee |y|)^{\perp\perp},$$

and the subgroups G_t are prime lattice-ideals. The results 3.1, 3.3 are due to Spirason and Strzelecki (to appear).

PROPOSITION 3.1. *A subgroup $I \subseteq G$ is of the form G_t for some $t \in Q$ if and only if*

- (1) $x \perp y$ implies $x \in I$ or $y \in I$
- (2) $x \in I$ implies $x^{\perp\perp} \subseteq I$
- (3) $x \in I$ implies $x^\perp \neq (0)$

An immediate corollary to this is that every minimal prime lattice-ideal of G is of the form G_t , for some $t \in Q$. Recall that a prime lattice-ideal M is *minimal prime* if M is minimal in the class of prime lattice-ideals of G : a necessary and sufficient condition for a prime lattice-ideal M to be minimal prime is that for each $x \in M$ there is a $y \notin M$ with $x \perp y$.

DEFINITION 3.2. The lattice-group G is said to be *complemented* if for all $x, y \in G$ there is an $a \in x^{\perp\perp} \oplus x^\perp$ such that $y \in a^{\perp\perp}$.

THEOREM 3.3. *G is complemented if and only if every subgroup $G_t \neq G$ is a minimal prime lattice-ideal.*

The class of minimal prime lattice ideals of G will be denoted by $M(G)$. For $x \in G$ the class $M(x) \subseteq M(G)$ is defined by $M(x) = \{M \in M(G) : x \notin M\}$. The set $\mu_G = \{M(x) : x \in G\}$ forms a closed-open base for the open sets for a Hausdorff topology on $M(G)$. In fact, for $x, y \in G$,

$$M(x) \cap M(y) = M(|x| \wedge |y|)$$

and

$$M(x) \cup M(y) = M(|x| \vee |y|)$$

The class of subgroups $G_t \neq G$ is denoted by $V(G)$. For $x \in G$, the class $V(x)$ is defined by $V(x) = \{G_t \neq G: x \notin G_t\}$. For $x, y \in G$,

$$V(x) \cap V(y) = V(|x| \wedge |y|)$$

and

$$V(x) \cup V(y) = V(|x| \vee |y|).$$

The set $v_G = \{V(x): x \in G\}$ then forms a compact-open base for the open sets for a topology on $V(G)$, which is compact if and only if G has an element 1 satisfying $1^\perp = (0)$. Further it is readily seen that for $x, y \in G$, $V(x) = V(y)$ if and only if $x^\perp = y^\perp$.

THEOREM 3.4. *The following are equivalent:*

- (1) G is complemented
- (2) $V(G) = M(G)$
- (3) v_G is relatively complemented
- (4) $V(G)$ is a Hausdorff space
- (5) Each $V(x) \in v_G$ is closed in $V(G)$.

PROOF. If every $G_t \neq G$ is minimal prime and $G_{t_1} \neq G_{t_2} \notin V(G)$ then there is an $x \in G_{t_1}$ such that $x \notin G_{t_2}$, and a $y \in G_{t_1}$ such that $x \perp y$. Then $G_{t_1} \in V(y)$, $G_{t_2} \in V(x)$ and $V(y) \cap V(x)$ is void. Conversely, if $V(G) \neq M(G)$ then there is a $G_t \neq G$ that is not minimal prime. Then G_t contains a minimal prime G_{t_1} , and G_t, G_{t_1} cannot be Hausdorff-separated. Thus (2) is equivalent to (4). Suppose that G is complemented. Take $V(x) \in v_G$ and $V(y) \subseteq V(x)$ so that $y^{\perp\perp} \subseteq x^{\perp\perp}$. Then there exist $a \in y^\perp, b \in y^{\perp\perp}$ such that $x^{\perp\perp} \subseteq (a + b)^{\perp\perp} = a^{\perp\perp} \vee b^{\perp\perp}$. Then

$$\begin{aligned} (|x| \wedge |a|)^{\perp\perp} &\subseteq x^{\perp\perp} \text{ and } (|x| \wedge |a|)^{\perp\perp} \vee y^{\perp\perp} = (x^{\perp\perp} \wedge a^{\perp\perp}) \vee y^{\perp\perp} \\ &= (x^{\perp\perp} \vee y^{\perp\perp}) \wedge (a^{\perp\perp} \vee y^{\perp\perp}) = x^{\perp\perp}. \end{aligned}$$

Also, $|x| \wedge |a| \perp y$ so that $(|x| \wedge |a|) \cap V(y)$ is void, and $V(|x| \wedge |a|) \cup V(y) = V(x)$. That is v_G is relatively complemented. On the other hand if v_G is relatively complemented and $x, y \in G$ then $V(x) \subseteq V(|x| \wedge |y|)$ so there is an $x' \in G$ such that $x' \perp x$ and $(x')^{\perp\perp} \vee x^{\perp\perp} = x^{\perp\perp} \wedge y^{\perp\perp}$. Then $y \in (x')^{\perp\perp} \vee x^{\perp\perp} = (x' + x)^{\perp\perp}$ so that G is complemented. That is (1) is equivalent to (3).

Suppose that each $V(x)$ in v_G is closed and $G_t \neq G$. If $x \in G_t$ then $t \in V(x)$ so there is a basic open set $V(y)$ with $t \in V(y) \subseteq V(G) \setminus V(x)$. That is $y \perp x$ and $y \notin G_t$, so that G_t is minimal prime. If each $G_t \neq G$ is minimal prime then $V(G) = M(G)$ so that each $V(x) = M(x)$ is closed.

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