GENERATIVE CONCEPT IMAGES

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We report on part of a study of university students' developing understanding of sensitive dependence on initial conditions, one of the essential ingredients of chaos. We discuss how the teaching team's concept images of sensitive dependence on initial conditions changed as a result of their interactions with students. These interactions lead to a substantially simpler, more geometric, way of viewing sensitive dependence on initial conditions. We discuss this simpler way of interpreting sensitive dependence in terms of generative concept images, in which prototypical objects are analysed deeply and then modified by higher level operations.

INTRODUCTION

In dealing with mathematical definitions, students have to bring to mind what the definition is about for them. The process of using a definition in practice is heavily dependent on memory for various parts of mathematics. These memories can take the form of images, feelings, syntactic expressions, and procedures, among other things. Associated with a definition a student will have a host of memories to retrieve. Indeed, for a given definition, Tall and Vinner (1981; see also Vinner, 1983, 1991) associate with each student a concept image that consists of:

> "… the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes. It is built up over the years through experiences of all kinds, changing as the individual meets new stimuli and matures."

In this paper we consider the idea of concept image in the context of a somewhat difficult definition – that of sensitive dependence on initial conditions, a basic ingredient of chaos. We argue that in order for students to focus productively on the idea of sensitive dependence it is useful to structure their concept images by assisting them to analyse examples deeply, and by providing them with higher-order operations that allow them to generate further examples. This process works, we argue, because it allows recollections of highly concrete imagery and relieves the strain on working memory.

SENSITIVE DEPENDENCE ON INITIAL CONDITIONS

Over the past 25 - 30 years the idea of sensitive dependence on initial conditions for a continuous function has become a central notion of chaos (Devaney, 1989; see also Holmgren, 1996; Elaydi, 1999). Conceptions of sensitive dependence appear

2006. In Novotná, J., Moraová, H., Krátká, M. & Stehlíková, N. (Eds.). *Proceedings 30th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 393-400. Prague: PME.

explicitly in Guckenheimer (1979), Ruelle (1979) and Kaplan and Yorke (1979). Prior to these authors May (1976, p. 466) wrote:

> "… it may be observed that in the chaotic regime arbitrarily close initial conditions can lead to trajectories which, after a sufficiently long time, diverge widely."

which he related to Lorenz's (1963) "butterfly effect". Li and Yorke (1975), who first used "chaos" in a technical mathematical sense, did not include sensitive dependence on initial conditions as a criterion for chaos. So explicit mention of sensitive dependence on initial conditions as an ingredient of chaos appeared somewhere between 1975 and 1979. Many popular and technical articles (Gleick, 1987, p. 8; Stewart, 1989, p. 113; Ruelle, 1991, p. 40; Feigenbaum, 1992, p. 6; Peitgen *et a*l, 1992, p. 48 and p. 511; Froyland, 1992) contain concept images, in the sense of Tall and Vinner (1981), but not concept definitions. Devaney (1989) places sensitive dependence on initial conditions as a major ingredient of chaos. Three points make it likely that this definition and associated conceptions will be difficult for students to appropriate:

1. Historically the definition did not come easily to workers in the field.

2. Popular and technical accounts have sometimes simplistic, confusing, or contradictory concept images.

3. The definition, as phrased in Devaney (1989), contains a mixture of five existential and universal quantifiers: a function $f: X \to X$, where X is a metric space with metric *d*, has sensitive dependence on initial conditions if $\exists \delta > 0$ such that $\forall x \in X, \forall \varepsilon > 0, \exists y \in X, \exists$ positive integer n with $d(x, y) < \varepsilon$ and $d(f^n(x), f^n(y)) > \delta$ (here, f^n denotes the nth iterate of *f*, namely, $f^1 = f$ and $f^{n+1} = f \circ f^n$ for $n \geq 1$).

Definitions at this level of mathematics are critically important because they are organizing principles for a variety of phenomena, and because they distinguish subtly different examples. Defined notions constitute the "advanced" in advanced mathematical thinking. However working directly from a definition can be difficult: for example, it is tedious and messy to show directly from the definition that the quadratic function $Q:[0,1] \to [0,1]$ defined by $Q(x) = 4x(1-x)$ has sensitive dependence on initial conditions. Generally, mathematicians work with "higher-level" operations (see Thurston, 1997, p. 118) when trying to settle issues such as this. For example, it is much easier, as we will see below, to establish, that the tent function $T:[0,1] \to [0,1]$ defined by $T(x) = 1 - |1 - 2x|$ has sensitive dependence, via a precise graphical analysis, and then to relate the behaviour of T to that of Q by a change of coordinates.

A major difficulty in working from definitions is that a definition has to relate to examples. However, if a student does not have a useful concept image what, for them, are the examples actually examples of? Both Skemp (1982) and Steffe (1990) cite examples as a necessary, but not sufficient, condition for students to operate successfully with a definition:

> "Concepts of a higher order than those which a person already has cannot be communicated … by a definition, but only by arranging to encounter a suitable collection of examples." (Skemp, 1982, p. 32)

> "Providing a definition can be orienting but using it can be very problematic especially if it has not been the result of experiential abstraction." (Steffe, 1990, p. 100)

Already in 1908 Poincaré observed:

> "What is a good definition? For the philosopher or the scientist, it is a definition which applies to all the objects to be defined, and applies only to them; it is that which satisfies the rules of logic. But in education it is not that; it is one that can be understood by the pupils" (Poincaré, 1908)

We argue that Poincaré, Skemp and Steffe's observations can be considerably strengthened by the process of formation of generative concept images, in which a simple, but typical object is analysed in depth, and appropriate higher level operations are used to generate further examples.

STUDENT DIFFICULTIES WITH SENSITIVE DEPENDENCE

We consider several university students' difficulties with the definition and concept of sensitive dependence on initial conditions. The students were enrolled in a course on one-dimensional dynamical systems at LaTrobe University, Melbourne, Australia. The majority of students were in the final year of their undergraduate degree. The class also included graduate students, and a 1st year student. The class was divided in two, with each group attending a one-hour weekly class for 26 weeks. The course team consisted of two faculty members, two graduate students, and a teaching assistant. All sessions were video-taped for later analysis and to assist the teaching team in real-time development of the course. Students were expected and encouraged to make presentations of their attempts at problem solutions to the rest of the chaos class. They were encouraged to write on an overhead projector and to talk aloud to their solution or attempted solution. Videotapes of class sessions were analysed each week for student difficulties, and exercises and teaching materials were adjusted accordingly.

Student engagement with sensitive dependence on initial conditions began with a lecture on a *Mathematica* experiment. Among the students present, Alice was particularly well qualified in mathematics. She had studied theoretical computer science at doctoral level at Moscow State University, and was, at the time of the chaos course, enrolled in a PhD in mathematics. Nonetheless, Alice experienced considerable difficulty in making sense of the written definition of sensitive dependence on initial conditions. The point that concerned her most about the definition of sensitive dependence was "$\exists$positive integer n". In the following excerpts we see the confusion of Alice, and her classmates Theo and Cherie, in trying to come to terms with the formal definition of sensitive dependence.

Alice: I just can't understand when you said "there exists an integer n" … why can't we pick up some very good integer n, for example 0, is it perfectly O.K.? For each x and each y you can pick up n_0 and for each δ.

Greg (teacher): But δ is independent of x and y.

Alice: I don't understand Veronica's (a teacher) explanation. Why we pick up two zero points and say this is not expansive - zero points of n iteration? Why we didn't stop at n -1 iteration? ….. What does it mean "there exists an integer n"?

….

Theo: δ … it can be greater than δ when n-1, but when get to n can be less than, no greater than ... sorry.

Jack (teacher): But it must be true for all x and y - so it must also be true for x and y close together.

Alice: So this n - fixed for all x and y, or for every pair of x and y?

Jack (teacher): Each pair of distinct x and y.

Alice: Now for every pair of x and y we choose n?

Jack (teacher): No!

Alice: It's fixed for all pairs?

Greg (teacher): The δ is fixed for all n.

Alice: I asked about n, I didn't ask about δ.

Cherie: You can pick any x and y and you've got a δ fixed, and you can iterate however as many times as necessary and there will exist a particular n - you don't have to specify for every pair - such that this is true.

Alice: I don't understand Veronica's (a teacher) explanation.

Veronica (teacher): If you fix the δ, after the iteration, with every pair of x, y in the [0,1] … must be greater than δ.

Greg (teacher): Ask yourself this question: in the tent map could δ be a half?

Alice: No questions!

Students were asked to prepare a statement of their understanding of sensitive dependence for the following class. They took turns to read out their explanations to the rest of the class. Two typical examples are:

Donald: What I wrote sounds a bit bizarre. I wrote that a function has sensitive dependence at a point if, if there is another point near that point so that the functions of the two points are a certain distance apart … after some number of iterations. Like, for example, … there is sensitive dependence at that point of you can find a point close to it so that after some number of iterations the functions of the two points will be some distance apart. Does that make sense?

Michael: A mapping has sensitive dependence at a point if for some number of iterations, if after some, … if for some iterations the function at some nearby point is greater than, than some fixed distance away.

We see, in these two statements, clear examples of what Tall and Vinner (1981) refer to as the students' concept image of sensitive dependence.

WIGGLY ITERATES: REFINING THE CONCEPT IMAGE

As we mentioned in the introduction, checking directly from the definition that a given function has sensitive dependence can be tedious, messy, and difficult. This is not generally how mathematicians work:

"As with most raw definitions, direct use … is rare." (Thurston, 1997, p. 118)

To provide the students in the course with a technique for establishing sensitive dependence the teaching team began with an analysis of the tent function $T:[0,1]\to[0,1]$ defined by $T(x)=1-|1-2x|$. The iterates T^n of T are particularly easy to understand geometrically.

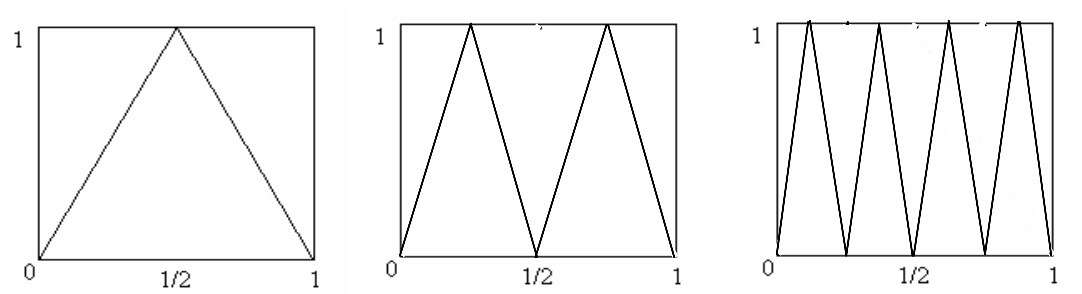

Figure 1: Graphs of the first, second, and third iterates of $T(x)=1-|1-2x|$ on [0, 1]

The n^{th} iterate of T consists of n squashed up copies of T – squashed in the base, but not in height – and the distance between successive zeros of T^n is $1/2^{n-1}$. The slope of the individual linear segments of T^n has absolute value 2^n. These facts are proved easily by induction on n. From this it is clear why T has sensitive dependence on initial conditions: we can choose $\delta=\frac{1}{2}$, for example (any positive number less than 1 will do), and then for any point $x\in[0,1]$ and $\varepsilon>0$, we simply take a $y\neq x$ within ε of x that is not a zero of some iterate of T, and iterate enough times so that the steepness of T^n ensures that $T^n(x)$ and $T^n(y)$ are at least $\delta=\frac{1}{2}$ apart.

The teaching team for the chaos course planned to write a single chapter on sensitive dependence on initial conditions. Chapters were planned to be around 10 pages in length, with no more than one graphic image per page. Because of student difficulties with the idea of sensitive dependence, the proposed chapter was expanded to three chapters. The initial definition, which seemed more or less clear to the course instructors, was found to be difficult for the students to comprehend. Consequently, the teaching team undertook a substantial analysis of the difficulties, and attempted to view them from the students' perspectives. This involved detailed analysis of student responses in class, their attempts at problem solving, and an effort to involve them in the ways in which professional mathematicians attempt to come to grips with the meaning of sensitive dependence.

GENERATIVE CONCEPT IMAGES

A common model for relating concept definitions and concept images involves:

(a) exemplifying the definition with examples (models of the definition), or

(b) organizing different examples by a definition, in the process abstracting or extracting appropriate relevant features,

or both. However, there is no guarantee that a student will:

(a) isolate the relevant parts of the definition as features of any given example, since they can easily focus on what, for the teacher, are irrelevant aspects of a given example, or

(b) extract the appropriate features that the examples are deemed to have in common.

The difficulty for a student in exemplification is to know how to interpret the definition in a particular example. The difficulty in organization is to know why just certain features of the presented examples are picked out as relevant and made into a definition. Tall (1986) refers to the process of extraction from numerous examples as the formation of a generic concept image:

> "… a generic concept being defined as one abstracted as being common to a whole class of previous experiences."

He then suggests that this process of extraction, or abstraction, rather than that of working with a "formal" definition, is more likely to be helpful for a student:

> "Mathematicians analyze concepts in a formal manner, producing a hierarchical development that may be inappropriate for the developing learner. Instead of clean, formal definitions, it may be better for the learner to meet moderately complicated situations which require the abstraction of essential points through handling appropriate examples and non-examples."

Thurston (1997) states however that mathematicians rarely work from raw definitions. From what then do they work? Our answer is that they work from a deep analysis of relatively simple, but prototypical, objects. This is not the same thing as exemplification, since it involves a deep analysis of why the object is an instance of the definition, and what general features of the object can be used as a concept image. Nor is it the same as extraction, because the features of the definition are built into a specific more easily analysed prototypical object, and into the higher order operations that transform that object. In the case of sensitive dependence on initial conditions the tent map is such an object. A deep, but technically straightforward, geometric analysis of its relevant features leads to a visual interpretation of sensitive dependence that carries over to other functions via higher level operations. The quadratic map $Q:[0,1]\to[0,1]$ defined by $Q(x)=4x(1-x)$, for example, can be seen to have sensitive dependence on initial conditions, because a change of coordinates, effected by the transformation $\alpha:[0,1]\to[0,1]$ defined by $\alpha(x)=\sin^2(\frac{\pi x}{2})$, transforms the tent map into the quadratic map – in that $\alpha\circ T=Q\circ\alpha$ – and a change of coordinates preserves sensitive dependence. The two higher level operations – change of coordinates, and preservation of sensitive dependence under change of coordinates – require some technical checking. However, when this relatively straightforward checking is carried out, once and for all, a student's thought processes can operate at a more economical level, and so take less space in working memory. A student then has the possibility of no longer being overwhelmed by a morass of technical details. What formerly seemed difficult now becomes transparent, because of a powerful and generative concept image that involves the action of higher level operations on deeply analysed prototypical objects.

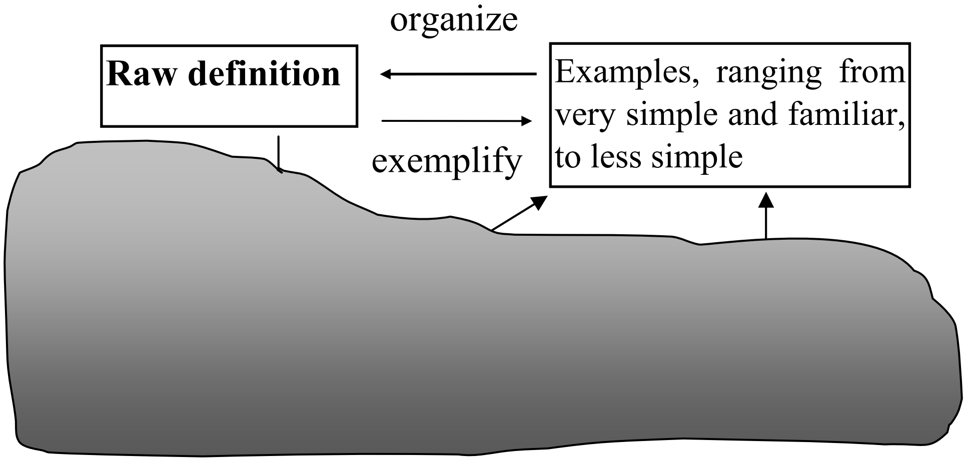

Figure 2. A model for the development of generative concept image and higher level operations: shaded areas contribute to formation of a generative

Tall and Vinner (1981) note that:

> "When a student is given a formal concept definition, the concept definition image that it forms in his (*sic*) cognitive structure may be very weak. … a weak understanding of the concept definition can make … formal proof … very hard ..."

Later, in referring to the connection between mental imagery and concept image they write:

> " [*the students*] are then in the situation where they may have a strong mental picture yet the concept definition image is weak. They understand the statements of theorems as being obvious, but cannot follow the proofs."

We propose that focusing on the relationship between concept definition and examples, through exemplification and organization generally produces a form of concept image that is weak, in the sense of Tall and Vinner, and not usually generative. This form of concept image generally only allows a student to mechanically check the conditions of a definition in particular examples, and possibly draw limited logical conclusions, via simple proofs, from the concept definition. This is in contrast to a form of concept image that we term generative, in which a student has the possibility of higher level operations on deeply analysed prototypical instances of the concept definition. A generative concept image allows a student the possibility of using their imagination in proposing likely general facts that flow from the concept definition. This is because their working memory is not overloaded with tedious checking of minutiae, so that through higher operations they can both generate, and focus on, deeper aspects of more complicated examples.

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