

PERMUTATION REPRESENTATION OF GROUPS WITH BOOLEAN ORTHOGONALITIES

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(Received 20 June 1977; revised 1 October 1979)

Communicated by E. Strzelecki

Abstract

Reduced rings and lattice-ordered groups are examples of groups with Boolean orthogonalities. In this note we show that any group with a Boolean orthogonality satisfying a finiteness condition introduced by Stewart is isomorphic with a group of homeomorphisms of a topological space, in which two homeomorphisms are orthogonal if and only if they have disjoint supports.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*): primary 54 H 10; secondary 06 F 15, 54 H 15, 57 S 99.

Introduction

Groups with Boolean orthogonalities were introduced by Davis (1971a) principally to obtain representation and extension theorems for lattice-ordered groups and reduced rings in a uniform way. Since then Boolean orthogonalities on groups and rings have been studied by Davis (1971b), (1975) and Stewart (1975). A universal algebraic study of algebras with Boolean orthogonalities has been undertaken by Cornish (1975) and Cornish and Stewart (1977).

In this note we show that every group G with a Boolean orthogonality \perp satisfying a certain finiteness condition can be faithfully represented as a group of permutations $\hat{G} = \{\hat{g}: g \in G\}$ of a set X in such a way that $g \perp h$ holds if and only if $\{x \in X: x\hat{g} = x\} \cup \{x \in X: x\hat{h} = x\} = X$, for all $g, h \in G$.

Boolean orthogonalities on permutation groups

Let G denote a group and \perp a binary relation on G . For a non-empty subset S of G we define $S^\perp = \{g \in G: g \perp h \text{ for all } h \in S\}$ and $S^{\perp\perp} = (S^\perp)^\perp$. If $S = \{h\}$ is a singleton set then we denote S^\perp and $S^{\perp\perp}$ by h^\perp and $h^{\perp\perp}$ respectively.

The relation \perp is a *Boolean orthogonality* on G if for each non-empty subset S of G the set S^\perp is a subgroup of G and, in addition,

- (1) \perp is a symmetric relation,
- (2) if $g \perp g$ then $g = 1$,
- (3) if $g \perp h$ then $k^{-1}gk \perp k^{-1}hk$ for all $k \in G$,
- (4) if $g^{\perp\perp} \cap h^{\perp\perp} = 1$ then $g \perp h$.

THEOREM 1 (Davis (1971b)). *Let G be a group with a Boolean orthogonality \perp . Then $B(G) = \{S^\perp: S \text{ is a non-empty subset of } G\}$, ordered by inclusion, is a complete Boolean algebra with $A \wedge B = A \cap B$ and $A \vee B = (A^\perp \cap B^\perp)^\perp$ for all $A, B \in B(G)$.*

Let G be a subgroup of the symmetric group on a set X . The support of $g \in G$ is the set $\text{supp}(g) = \{x \in X: gh \neq x\}$. We shall say that G is a *d-group* (for “directed group”) if whenever $g_1, g_2 \in G$ and $\emptyset \neq \text{supp}(g_1) \cap \text{supp}(g_2)$ there is an $h \in G$ with $\emptyset \neq \text{supp}(h) \subseteq \text{supp}(g_1) \cap \text{supp}(g_2)$.

PROPOSITION 2. *Let G be a subgroup of the symmetric group on X . Define the binary relation \perp on G by $g \perp h$ if $\text{supp}(g) \cap \text{supp}(h) \neq \emptyset$. If G is a d-group then \perp is a Boolean orthogonality on G . Conversely if \perp is a Boolean orthogonality and for each $(x, g) \in X \times G$ with $xg = x$ there is an $h \in G$ with $x \in \text{supp}(h)$ and $\text{supp}(g) \cap \text{supp}(h) = \emptyset$, then G is a d-group.*

PROOF. Suppose G is a d-group. The relation \perp is clearly symmetric and satisfies (2) of the definition of a Boolean orthogonality. Furthermore each S^\perp is clearly a subgroup of G and condition (3) follows from the fact that $\text{supp}(k^{-1}gk) = \text{supp}(g)k$ for all $g, k \in G$. Suppose that $g, h \in G$ and $g \perp h$ does not hold. Then $\text{supp}(g) \cap \text{supp}(h) \neq \emptyset$ so there is a $k \in G$ with $\emptyset \neq \text{supp}(k) \subseteq \text{supp}(g) \cap \text{supp}(h)$. If $g' \in g^\perp$ then $\text{supp}(g') \cap \text{supp}(k) \subseteq \text{supp}(g') \cap \text{supp}(g) = \emptyset$ so $g' \in k^\perp$. That is $g^\perp \subseteq k^\perp$. Similarly $h^\perp \subseteq k^\perp$ so $1 \neq k \in g^{\perp\perp} \cap h^{\perp\perp}$, and therefore \perp is a Boolean orthogonality.

Suppose conversely that \perp is a Boolean orthogonality and for all $(x, g) \in X \times G$ with $xg = x$ there is a $g' \in G$ with $x \in \text{supp}(g')$ and $\text{supp}(g) \cap \text{supp}(g') = \emptyset$. Assume that $g, h \in G$ and $\text{supp}(g) \cap \text{supp}(h) \neq \emptyset$. Since \perp is a Boolean orthogonality we can find $1 \neq k \in g^{\perp\perp} \cap h^{\perp\perp}$.

Suppose $x \notin \text{supp}(g)$. Then there is a $g' \in G$ with $x \in \text{supp}(g')$ and $\text{supp}(g) \cap \text{supp}(g') = \emptyset$, so $\text{supp}(g') \cap \text{supp}(k) = \emptyset$ and therefore $x \notin \text{supp}(k)$. Similarly $x \notin \text{supp}(h)$ implies $x \notin \text{supp}(k)$ so $\emptyset \neq \text{supp}(k) \subseteq \text{supp}(g) \cap \text{supp}(h)$. That is, G is a d -group.

Let G be a group of permutations of X . If G is a d -group then we shall equip G with the Boolean orthogonality \perp defined by $g \perp h$ if and only if $\text{supp}(g) \cap \text{supp}(h) = \emptyset$. We let Σ_G denote the topology on X having the sets $\text{supp}(g)$, $g \in G$, as sub-basic open sets. The condition that G be a d -group is almost the condition that the sets $\text{supp}(g)$, $g \in G$, are *basic* open sets for the Σ_G -topology. We give a name to this latter phenomenon: G is a d^* -group if the sets $\text{supp}(g)$, $g \in G$, form a base for the Σ_G -topology on X . Thus, G is a d^* -group if and only if

- (1) for all $g, h \in G$, if $x \in \text{supp}(g) \cap \text{supp}(h)$ then there is a $k \in G$ with $x \in \text{supp}(k) \subseteq \text{supp}(g) \cap \text{supp}(h)$,
- (2) no point of x is fixed by all $g \in G$.

LEMMA 3. Let G be a d^* -group on the set X . For all $g \in G$, $g^{\perp\perp} = \{k \in G: \text{supp}(k) \subseteq \text{supp}(g)\}$, where \overline{S} denotes the closure of $S \subseteq X$ for the Σ_G -topology.

PROOF. Take $g \in G$ and suppose $\text{supp}(k) \subseteq \overline{\text{supp}(g)}$. If $g \perp h$ then $\text{supp}(g) \cap \text{supp}(h) = \emptyset$ so $\text{supp}(h) \cap \text{supp}(g) = \emptyset$ and therefore $\text{supp}(h) \cap \text{supp}(k) = \emptyset$. That is, $k \in g^{\perp\perp}$.

Suppose on the other hand that $k \in g^{\perp\perp}$. If $\overline{\text{supp}(g)} = X$ there is nothing to show. Suppose otherwise that O is a non-empty open set contained in $X \setminus \text{supp}(g)$. Then $O = \bigcup \{\text{supp}(h): \text{supp}(h) \cap \text{supp}(g) = \emptyset\}$ so $\text{supp}(k) \cap O = \bigcup \{\text{supp}(k) \cap \text{supp}(h): \text{supp}(h) \cap \text{supp}(g) = \emptyset\} = \emptyset$. That is, $\text{supp}(k) \subseteq \text{supp}(g)$.

COROLLARY 4. Let G be a d^* -group on the set X . Then the following are equivalent, for all $g, h \in G$:

- (1) $g \perp h$,
- (2) $\overline{\text{supp}(g)} \cap \overline{\text{supp}(h)} = \emptyset$,
- (3) $\overline{\text{supp}(g)} \cap \overline{\text{supp}(h)}$ has empty interior for the Σ_G -topology.

We note that every group is a d -group in its right regular representation. The Boolean orthogonality defined in proposition is, however, trivial in this case in the sense that $g \perp h$ holds if and only if $g = 1$ or $h = 1$.

We note also that not every permutation group is a d -group. For instance if G acts sharply doubly transitively on X then G is not a d -group. Indeed, since no permutation can have a one-element support the symmetric group on three or more letters is not a d -group.

Permutation representation

Throughout this section G will denote a group with a Boolean orthogonality \perp .

A subgroup H of G is a π -subgroup if $h_1^{\perp\perp} \vee \dots \vee h_n^{\perp\perp} \subseteq H$ for each finite subset $\{h_1, \dots, h_n\}$ of H .

The π -subgroups of G ordered by inclusion clearly form a complete lattice.

LEMMA 5. *Let H be a π -subgroup of G and g an element of G . Let $\langle H \cup \{g\} \rangle$ be the smallest π -subgroup of G containing $H \cup \{g\}$. Then $\langle H \cup \{g\} \rangle = \cup \{F^{\perp\perp} \vee g^{\perp\perp} : F \subseteq G, F \text{ finite}\}$.*

PROOF. If $F = \{h_1, \dots, h_n\}$ is a finite subset of H then $F^{\perp\perp} = h_1^{\perp\perp} \vee \dots \vee h_n^{\perp\perp}$ so $F^{\perp\perp} \vee g^{\perp\perp} \subseteq \langle H \cup \{g\} \rangle$. If $k_1, k_2 \in G$ then $k_1^{\perp} \cap k_2^{\perp} \subseteq (k_1 k_2^{-1})^{\perp}$, since sets of the form S^{\perp} are subgroups of G , so $k_1 k_2^{-1} \in k_1^{\perp\perp} \vee k_2^{\perp\perp}$. Consequently if $k_1 \in F_1^{\perp\perp} \vee g^{\perp\perp}$ and $k_2 \in F_2^{\perp\perp} \vee g^{\perp\perp}$, where F_1 and F_2 are finite subsets of H , then

$$k_1 k_2^{-1} \in k_1^{\perp\perp} \vee k_2^{\perp\perp} \subseteq F_1^{\perp\perp} \vee F_2^{\perp\perp} \vee g^{\perp\perp} = (F_1 \cup F_2)^{\perp\perp} \vee g^{\perp\perp}.$$

Hence $\cup \{F^{\perp\perp} \vee g^{\perp\perp} : F \subseteq G, F \text{ finite}\}$ is a subgroup of G , and therefore by its definition a π -subgroup, contained in $\langle H \cup \{g\} \rangle$, so equality holds.

We call the meet-irreducible elements of the lattice of π -subgroups of G the *prime π -subgroups of G* . Thus, a π -subgroup P of G is a prime π -subgroup if and only if for all π -subgroups H_1, H_2 of G , $H_1 \cap H_2 \subseteq P$ implies $H_1 \subseteq P$ or $H_2 \subseteq P$. Equivalently, a π -subgroup P is prime if and only if for all g, h in G , $g^{\perp\perp} \cap h^{\perp\perp} \subseteq P$ implies $g \in P$ or $h \in P$.

We shall now see that when G satisfies a certain finiteness condition for \perp , the prime π -subgroups of G intersect in 1. This fact allows us to represent such groups as d^* -groups of homeomorphisms.

DEFINITION. The Boolean orthogonality \perp on G is *finite* if for all $g, h \in G$ there is a finite subset F of G with $g^{\perp\perp} \cap h^{\perp\perp} = F^{\perp\perp}$.

In the context of Boolean orthogonalities on rings this finiteness condition was introduced by Stewart (1975).

PROPOSITION 6. *Let G be a group with a finite Boolean orthogonality \perp and suppose $g_0 \neq 1$ in G . Then there is a subgroup of G maximal with respect to being a π -subgroup of G not containing g_0 . Any such π -subgroup is prime.*

PROOF. The set of π -subgroups of G is plainly inductive so if $g_0 \neq 1$ then there is a subgroup H of G maximal with respect to being a π -subgroup not containing g_0 . Assume that H is not prime. Then there exist $g, k \notin H$ such that $g^{\perp\perp} \cap k^{\perp\perp} \subseteq H$. Since $g_0 \in \langle H \cup \{g\} \rangle \cap \langle H \cup \{k\} \rangle$ there are finite subsets F_1, F_2 of H such that

$$g_0 \in (F_1^{\perp\perp} \vee g^{\perp\perp}) \cap (F_1^{\perp\perp} \vee k^{\perp\perp}) \\ = (F_1^{\perp\perp} \cap F_2^{\perp\perp}) \vee (F_1^{\perp\perp} \cap k^{\perp\perp}) \vee (F_2^{\perp\perp} \cap g^{\perp\perp}) \vee (g^{\perp\perp} \cap k^{\perp\perp}).$$

Since \perp is finite, there is a finite subset F of G with $F^{\perp\perp} = g^{\perp\perp} \cap k^{\perp\perp}$. Since $g^{\perp\perp} \cap k^{\perp\perp} \subseteq H$ then $F \subseteq H$ and $g_0 \in H$ —a contradiction, so H is a prime π -subgroup.

COROLLARY 7. *If G is a group with a finite Boolean orthogonality π then $\cap \{P \subseteq G: P \text{ is a prime } \pi\text{-subgroup}\} = 1$.*

It is easily seen from condition (3) for a Boolean orthogonality that each conjugate of a (prime) π -subgroup of G is again a π -subgroup of G .

Let G and H be groups with Boolean orthogonalities, both denoted by \perp . A group homomorphism $f: G \rightarrow H$ is a \perp -homomorphism if $g \perp h$ in G implies $gf \perp hf$ in H . A \perp -isomorphism $f: G \rightarrow H$ is a bijection such that both f and f^{-1} are \perp -homomorphisms.

We now let $X = \cup \{G/P: P \text{ is a proper prime } \pi\text{-subgroup of } G\}$ be the disjoint union of left cosets of proper prime π -subgroups of G .

If H is a subgroup of the symmetric group on X and also a d -group we shall equip H with the Boolean orthogonality \perp defined by $g \perp h$ if and only if $\text{supp}(g) \cap \text{supp}(h) = \emptyset$.

THEOREM 8. *Let G be a group with a finite Boolean orthogonality \perp . Then G is \perp -isomorphic with a d^* -group of homeomorphisms of a topological space.*

PROOF. Given $g \in G$ we define $\hat{g}: X \rightarrow X$ by $(Pk)\hat{g} = Pkg$. Then $\hat{G} = \{\hat{g}: f \in G\}$ is a subgroup of the symmetric group on X , and $g \rightarrow \hat{g}$ is an isomorphism onto \hat{G} since the intersection of all prime π -subgroups of G is 1. We now see that $g \perp h$ in G if and only if $\text{supp}(\hat{g}) \cap \text{supp}(\hat{h}) = \emptyset$. Suppose $g \perp h$ in G . If $(Pk)\hat{g} \neq Pk$ then $kgk^{-1} \notin P$. That is, $g \in k^{-1}pk$ and so, since k^{-1} is a prime π -subgroup and $g^{\perp\perp} \cap h^{\perp\perp} \subseteq k^{-1}pk$, we have $h \in k^{-1}pk$. That is, $(Pk)\hat{h} = Pk$ so $\text{supp}(\hat{g}) \subseteq X \setminus \text{supp}(\hat{h})$.

Suppose conversely that $\text{supp}(\hat{g}) \cap \text{supp}(\hat{h}) = \emptyset$. Let P be a prime π -subgroup of G . Then $Pg = P$ or $Ph = p$ so $g^{\perp\perp} \subseteq P$ or $h^{\perp\perp} \subseteq P$. In any case, $g^{\perp\perp} \cap h^{\perp\perp} \subseteq P$. Since the prime π -subgroups of G meet in 1 we have $g^{\perp\perp} \cap h^{\perp\perp} = 1$ and therefore $g \perp h$.

Thus the relation \perp on \hat{G} defined by $\hat{g} \perp \hat{h}$ if and only if $\text{supp}(g) \cap \text{supp}(h) = \emptyset$ is a Boolean orthogonality. Suppose that $g, h \in G$ and $Pk_0 \in \text{supp}(\hat{g}) \cap \text{supp}(\hat{h})$. Then $g, h \in k_0^{-1}Pk_0$. Take $k \in g^{\perp\perp} \cap h^{\perp\perp} \setminus k_0^{-1}Pk_0$, so that $Pk_0 \in \text{supp}(k)$. If $P^1k_1 \in \text{supp}(k)$ then $k \notin k_1^{-1}P^1k_1$ so $g^{\perp\perp} \cap h^{\perp\perp} \not\subseteq k_1^{-1}P^1k_1$. In this case $g, h \notin k_1^{-1}P^1k_1$ so $P^1k_1 \in \text{supp}(g) \cap \text{supp}(h)$. That is, \hat{G} is a d^* -group on X . Clearly each $\hat{g} \in \hat{G}$ is a homeomorphism of X for the d^* -topology.

Transitive representations and examples

We have seen that every group G with a finite Boolean orthogonality \perp is \perp -isomorphic with a d^* -group of homeomorphisms of a topological space.

THEOREM 9. *Let G be a group with a finite Boolean orthogonality \perp . Then G is \perp -isomorphic with a transitive d^* -group \hat{G} of homeomorphisms of a topological space equipped with the $\Sigma_{\hat{G}}$ -topology if and only if G has a prime π -subgroup P such that $\bigcap \{k^{-1}Pk : k \in G\} = 1$.*

PROOF. Suppose that G has a prime π -subgroup P whose conjugates meet in 1. Then by proceeding as in Theorem 8 with the set of all prime π -subgroups of G replaced by conjugates of P we see that G has a transitive representation as a d^* -group of permutations on $X = \bigcup \{G/k^{-1}Pk : k \in G\}$.

Suppose conversely that G is (\perp -isomorphic with) a d^* -group of permutations acting transitively on X . Take $x \in X$. We see that the point stabilizer $G_x = \{g \in G : xg = x\}$ contains a prime π -subgroup of G . The set of π -subgroups of G contained in G_x is inductive so let H be maximal with respect to being a π -subgroup of G contained in G_x . Suppose that $g, h \in H$ but $g^{\perp\perp} \cap h^{\perp\perp} \not\subseteq H$. Then $\langle H \cup \{g\} \rangle$ contains some g' with $xg' \neq x$ and $\langle H \cup \{h\} \rangle$ contains some h' with $xh' \neq x$. Then $g' \in F_1^{\perp\perp} \vee g^{\perp\perp}$ and $h' \in F_2^{\perp\perp} \vee h^{\perp\perp}$, for some finite subsets F_1, F_2 of H , so $(g')^{\perp\perp} \cap (h')^{\perp\perp} \subseteq H$. Then there is a $k \in G$ with $x \in \text{supp}(k) \subseteq \text{supp}(g) \cap \text{supp}(h)$ so $k \in (g')^{\perp\perp} \cap (h')^{\perp\perp} \subseteq H \subseteq G_x$ -a contradiction.

Thus, H is a prime π -subgroup of G and $\bigcap \{k^{-1}Hk : k \in G\} \subseteq \bigcap \{k^{-1}G_xk : k \in G\} \subseteq \bigcap \{k^{-1}G_xk : k \in G\} = 1$.

The motivating class of groups for the study of (finite) Boolean orthogonalities is the class of lattice-ordered groups: a lattice ordered group G has a finite Boolean orthogonality \perp defined by $g \perp h$ in G if and only if $|g| \wedge |h| = 1$ (where $|g| = g \vee g^{-1}$).

Our representation theorem for groups with finite Boolean orthogonalities was modelled on Holland's (1963) representation of lattice-ordered groups as groups of order-preserving permutations of chains.

Another class of examples is provided by the (not necessarily commutative) reduced rings R where we define $g \perp h$ if and only if $gh = 0$. Then g^\perp is just the left annihilator of g in R which, since R is reduced, is a two-sided ideal. It follows then that $g^\perp \subseteq (gh)^\perp$ for all $g, h \in R$, so π -subgroups of R are in fact two-sided ideals of R . Our representation theorem can then be converted to a sheaf-theoretic representation for R . This has been carried out by Davis (1971b) and Stewart (1975).

Another—by Theorem 8 fairly typical—example is $\Gamma(\mathbf{R}^2)$, the homeomorphism group of \mathbf{R}^2 . Indeed, suppose $g, h \in \Gamma(\mathbf{R}^2)$ and $x \in \text{supp}(g) \cap \text{supp}(h) = U$. Find inside U an isometric copy C of the unit open disc O , with $x \in C$. Let $\varphi: O \rightarrow C$ be a homeomorphism from O onto C . Define a map $\psi: \bar{O} \rightarrow \bar{O}$ as follows:

$$z\psi = e^{2\pi i d(z, \partial O)z}, \text{ where } d \text{ is the Euclidean metric and } \partial O \text{ is the boundary of } O.$$

Now define $k: \mathbf{R}^2 \rightarrow \mathbf{R}^2$ by

$$zk = \begin{cases} z & \text{if } z \notin C, \\ z\phi^{-1}\psi\phi & \text{if } z \in C. \end{cases}$$

Then k is a homeomorphism of \mathbf{R}^2 and $x \in \text{supp}(k) \subseteq U$, so $\Gamma(\mathbf{R}^2)$ is a d^* -group and \perp , defined by $g \perp h$ if $\text{supp}(g) \cap \text{supp}(h) = \emptyset$, is a Boolean orthogonality.

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