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Tensor products and entropic varieties

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Although tensor products exist in any variety, they are well behaved only in entropic varieties. These are varieties in which every operation is a homomorphism; equivalently, they are the varieties $\mathbf{K}$ for which the hom-set $\mathbf{K}(A, B)$ is a subalgebra of $B^{|A|}$ for all $A, B \in \mathbf{K}$.

Tensor product is defined in section 1. In section 2 we show that $F\mathbf{K}(1)$ is a unit for $\otimes$ precisely when all unary term functions are homomorphisms. In this case a binary operation, $*$, which distributes over all other operations may be defined on $F\mathbf{K}(1)$ so that $\langle F\mathbf{K}(1); *, x_1\rangle$ is a commutative monoid with the free generator x_1 as unit: this corresponds to the fact that the free abelian group on one generator, $\mathbb{Z}$, carries a natural commutative ring structure. We aim, in section 3, to show that tensor products behave as nicely as we would wish exactly when we restrict ourselves to entropic varieties. As a corollary of the main theorem of this section if follows that $\langle \mathbf{K}, \otimes, F\mathbf{K}(1)\rangle$ is a closed category, which leads us naturally into a discussion in section 4 of monoids and semigroups (with respect to $\otimes$) in an entropic variety. We prove, for example, that the tensor product of two commutative monoids of $\mathbf{K}$ is equipped with a natural monoid structure which makes it the coproduct in the category of commutative monoids. Congruences on tensor products are considered briefly in section 5. In section 6 we apply our results to the entropic varieties $\mathbf{S}$ of join-semilattices and $\mathbf{S}_0$ of join-semilattices with zero. Indeed it was the authors' desire to give a more general universal-algebraic presentation of the results of G. A. Fraser [11], on semilattice tensor products of distributive lattices, which led them to write this paper.

The present paper is a slightly condensed version of B. A. Davey and G. Davis [4].

1. The tensor product

Our definition of tensor product is the obvious analogue of the usual definition of tensor product of abelian groups. Throughout the paper $\mathbf{K}$ will denote a fixed

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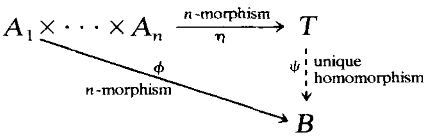
variety, and all algebras and all morphisms we consider will be **K**-algebras and **K**-morphisms.

Given algebras $A_1, \ldots, A_n$ and B, a map $\phi: A_1 \times \cdots \times A_n \to B$ is an *n-morphism* if for each i the map

$$a_i \to \langle a_1, \ldots, a_i, \ldots, a_n \rangle \phi : A \to B$$

is a homomorphism for each choice of the remaining variables.

A pair $\langle T, \eta \rangle$ is a *tensor product* of $\langle A_1, \ldots, A_n \rangle$ if T is an algebra of **K**, $\eta : A_1 \times \cdots \times A_n \to T$ is an n-morphism, and for each n-morphism $\phi : A_1 \times \cdots \times A_n \to B$ there is a unique homomorphism $\psi: T \to B$ for which $\phi = \eta\psi$.



Since tensor product is defined by a universal mapping property, it is unique up to isomorphism. As usual, we denote the algebra T by $A_1 \otimes \cdots \otimes A_n$, and denote the element $\langle a_1, \ldots, a_n \rangle \eta$ of $A_1 \otimes \cdots \otimes A_n$ by $a_1 \otimes \cdots \otimes a_n$. We often omit reference to the universal n-morphism and simply refer to $A_1 \otimes \cdots \otimes A_n$ as the tensor product of $\langle A_1, \ldots, A_n \rangle$.

It is easily seen that tensor products exist in any variety. If $A_1, \ldots, A_n \in \mathbf{K}$ then let $F\mathbf{K}(A_1 \times \cdots \times A_n)$ denote the free algebra in **K** on the set $A_1 \times \cdots \times A_n$ and let Θ be the smallest congruence on $F\mathbf{K}(A_1 \times \cdots \times A_n)$ which identifies

$$\langle a_1, \ldots, f(a_{i1}, \ldots, a_{im}), \ldots, a_n \rangle$$

with

$$f(\langle a_1, \ldots, a_{i1}, \ldots, a_n \rangle, \ldots, \langle a_1, \ldots, a_{im}, \ldots, a_n \rangle)$$

for all possible choices of elements and operations. It is easily seen that $F\mathbf{K}(A_1 \times \cdots \times A_n)/\Theta$ has the required universal mapping property. One can also give a proof of the existence of the tensor product using Freyd's representability theorem. For if $A_1, \ldots, A_n \in \mathbf{K}$ and $K_n : \mathbf{K} \to \mathbf{Set}$ is the functor defined as follows: $K_n(B)$ is the set of n-morphisms from $A_1 \times \cdots \times A_n$ to B, and if $\phi: B \to C$ is a homomorphism then $K_n(\phi): K_n(B) \to K_n(C)$ is composition with ϕ; then the tensor product is nothing more than a representing object for K_n. The existence of the tensor product follows from the easily established fact that K_n is continuous and the solution set condition of the representability theorem holds.

Note that, as usual, the uniqueness of the fill-in map, ψ, in the definition of tensor product is equivalent to the fact that $A_1 \otimes \cdots \otimes A_n$ is generated by the set of all elements of the form $a_1 \otimes \cdots \otimes_n$ where $a_i \in A_i$. It should be noted that if 0 and 0′ are nullary operations in the type of **K**, then, since both $-\otimes 0$ and $0' \otimes -$ are homomorphisms, 0 equals 0′ in $A \otimes B$.

Since we are interested in determining when a variety **K** is a monoidal category with respect to tensor product, we record here the way in which tensor product defines a multifunctor on a power of **K**. Suppose $A_1, \ldots, A_n$, $B_1, \ldots, B_n \in \mathbf{K}$ and for each i we have a homomorphism $\phi_i : A_i \to B_i$. Then the map $\langle a_1, \ldots, a_n \rangle \to a_1\phi_1 \otimes \cdots \otimes a_n\phi_n : A_1 \times \cdots \times A_n \to B_1 \otimes \cdots \otimes B_n$ is clearly an n-morphism, so there is a unique homomorphism, $\phi_1 \otimes \cdots \otimes \phi_n : A_1 \otimes \cdots \otimes A_n \to B_1 \otimes \cdots \otimes B_n$ satisfying $(a_1 \otimes \cdots \otimes a_n)(\phi_1 \otimes \cdots \otimes \phi_n) = a_1\phi_1 \otimes \cdots \otimes a_n\phi_n$. We now define the *(n-fold) tensor product functor* $T_n : \mathbf{K}^n \to \mathbf{K}$ by $T(A_1, \ldots, A_n) = A_1 \otimes \cdots \otimes A_n$ and $T(\phi_1, \ldots, \phi_n) = \phi_1 \otimes \cdots \otimes \phi_n$. Note the following interchange law for composition: if $\phi_i : A_i \to B_i$ and $\psi_i : B_i \to C_i$, then

$$(\phi_1 \otimes \cdots \otimes \phi_n)(\psi_1 \otimes \cdots \otimes \psi_n) = \phi_1\psi_1 \otimes \cdots \otimes \phi_n\psi_n.$$

For each $A \in \mathbf{K}$ we define a functor $A \otimes - : \mathbf{K} \to \mathbf{K}$ which sends an algebra $B \in \mathbf{K}$ to $A \otimes B$, and a morphism $\phi : B \to C$ to $id_A \otimes \phi : A \otimes B \to A \otimes C$. The functor $- \otimes A : \mathbf{K} \to \mathbf{K}$ is defined analogously.

2. The free algebra on one generator as a unit

In this section we are concerned with the proposition that the free algebra on one generator, $F\mathbf{K}(1)$, behaves as a unit for tensor product. This is an entirely reasonable requirement; for example, $\mathbb{Z}$ acts as a unit for tensor products in the category **Ab** of abelian groups, and, of course, $\mathbb{Z}$ is the free abelian group on one generator.

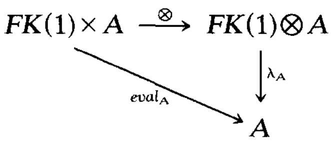
The proofs of the propositions below are quite straightforward and are omitted.

The evaluation map, $eval_A : F\mathbf{K}(1) \times A \to A$, is defined by $\langle p(x_1), a \rangle eval_A = p(a)$, where x_1 is the free generator of $F\mathbf{K}(1)$.

2.1. PROPOSITION. *The following are equivalent:*

(i) *$eval_A$ is a bimorphism for each $A \in \mathbf{K}$;*

(ii) *for each $A \in \mathbf{K}$ there is an isomorphism $\lambda_A : F\mathbf{K}(1) \otimes A \simeq A$ for which the* *diagram below commutes:*



(iii) *if p is a unary term and q is an n-ary term, for any n, then the following is an identity of* $\mathbf{K}$:

$$p(q(x_1, \ldots, x_n)) = q(p(x_1), \ldots, p(x_n));$$

(iv) *if p is a unary term and f is an n-ary operation for any n, then the following is an identity of* $\mathbf{K}$:

$$p(f(x_1, \ldots, x_n)) = f(p(x_1), \ldots, p(x_n)). \quad \square$$

Note that Condition (iv) of this result can be restated as "*Unary term functions are homomorphisms.*"

A particularly interesting case of the above result occurs when we let A equal $F\mathbf{K}(1)$: we obtain conditions for $F\mathbf{K}(1) \otimes F\mathbf{K}(1)$ to be isomorphic to $F\mathbf{K}(1)$ in a natural way. The following result generalizes the fact that the free abelian group on one generator, $\mathbb{Z}$, carries a natural structure as a commutative ring with unity.

2.2. PROPOSITION. *The following are equivalent:*

(i) *there is an isomorphism* $\psi : F\mathbf{K}(1) \otimes F\mathbf{K}(1) \to F\mathbf{K}(1)$ *such that* $(x_1 \otimes x_1)\psi = x_1$;

(ii) *if p and q are unary terms, then $p(q(x)) = q(p(x))$ is an identity of* $\mathbf{K}$;

(iii) *a binary operation $*$ may be defined on $F\mathbf{K}(1)$ such that*

(a) $a * f(b_1, \ldots, b_n) = f(a * b_1, \ldots, a * b_n)$ *for all* $a, b_1, \ldots, b_n \in F\mathbf{K}(1)$ *and all operations f,*

(b) $a * b = b * a$ *for all* $a, b \in F\mathbf{K}(1)$,

(c) $x_1 * a = a$ *for all* $a \in F\mathbf{K}(1)$,

(d) $(a * b) * c = a * (b * c)$ *for all* $a, b, c \in F\mathbf{K}(1)$;

(iv) *a binary operation $*$ may be defined on $F\mathbf{K}(1)$ such that* (a), (b) *and* (c)′ *hold, where*

(c)′ $x_1 * x_1 = x_1$. $\square$

We refer to the equivalent conditions of this proposition by the phrase: *unary terms commute in* $\mathbf{K}$.

3. Entropic varieties

We say that the variety $\mathbf{K}$ is *entropic* if whenever $A \in \mathbf{K}$, f is an m-ary operation, g is an n-ary operation, and $a_1, \ldots, a_n \in A^m$, we have

$$f(g(a_1, \ldots, a_n)) = g(f(a_1), \ldots, f(a_n)); \tag{*}$$

that is, if every operation is a homomorphism. Alternatively, we insist that for all m-ary operations f and all n-ary operations g, $\mathbf{K}$ satisfies the identity

$$f(g(x_{11}, \ldots, x_{n1}), \ldots, g(x_{1m}, \ldots, x_{nm})) = g(f(x_{11}, \ldots, x_{1n}), \ldots, f(x_{n1}, \ldots, x_{nm})) \tag{**}$$

Note, in particular, that if 0 and $0'$ are nullary operations in an entropic variety $\mathbf{K}$, then $0 = 0'$ is an identity of $\mathbf{K}$ and $\{0\}$ is a one-element subalgebra for all $A \in \mathbf{K}$.

There seems to be no agreement upon the apropriate name for these varieties. For example, L. Klukovits [17], W. D. Neumann [24] and F. E. J. Linton [19] refer to them as *commutative* varieties, while Linton makes the remark that *distributive* may be a better name. These varieties are special cases of what Linton calls *autonomous* categories. We choose to follow T. Evans [6, 7, 8] and refer to such varieties as *entropic.*

The characterization of entropic varieties in terms of hom-sets is well known; see for example, T. Evans [7], F. E. J. Linton [19], L. Klukovits [15] or B. A. Davey [3]. For $A, B \in \mathbf{K}$ we write $\mathbf{K}(A, B)$ for the set of $\mathbf{K}$-morphisms from A to B. Then $\mathbf{K}$ is entropic if and only if $\mathbf{K}(A, B)$ is a subalgebra of B^A for all $A, B \in \mathbf{K}$. Our next theorem says that tensor products are well behaved precisely in entropic varieties. To some extent this was already known to Linton [19] who showed, in his terminology, that a coherent lifted hom-functor for $\mathbf{K}$ has a left adjoint (which is necessarily tensor product) precisely when $\mathbf{K}$ is entropic. Our objective is to relate this to the distributivity of tensor product over coproducts and to the multiplicative behavior of tensor products with respect to free algebras.

3.1. THEOREM. *The following are equivalent:*

(i) $\mathbf{K}$ *is entropic;*

(ii) $\mathbf{K}(A, B)$ *is a subalgebra of* B^A *for all* $A, B \in \mathbf{K}$;

(iii) *unary terms commute in* $\mathbf{K}$ *and for all* $B \in \mathbf{K}$ *the functor* $-\otimes B : \mathbf{K} \to \mathbf{K}$ *has a right adjoint* $H_B : \mathbf{K} \to \mathbf{K}$;

(iv) *unary terms commute in* $\mathbf{K}$ *and the functor* $-\otimes B : \mathbf{K} \to \mathbf{K}$ *preserves colimits;*

(v) *unary terms commute in* $\mathbf{K}$ *and for all* $B \in \mathbf{K}$, *the functor* $-\otimes B : \mathbf{K} \to \mathbf{K}$

preserves finite coproducts, i.e. for all $A_1, A_2 \in \mathbf{K}$, *the natural map from* $(A_1 \otimes B) \coprod (A_2 \otimes B)$ *to* $(A_1 \coprod A_2) \otimes B$ *is an isomorphism;*

(vi) *for all* $m, n \in \mathbb{N}$, *there is an isomorphism* $\phi_{m,n} : F\mathbf{K}(m) \otimes F\mathbf{K}(n) \simeq F\mathbf{K}(mn)$ *such that if* $x_1, \ldots, x_m$ *are the free generators of* $F\mathbf{K}(m)$ *and* $y_1, \ldots, y_n$ *are the free generators of* $F\mathbf{K}(n)$, *then the collection of all* $(x_i \otimes y_j)\phi_{m,n}$ *is the free generating set of* $F\mathbf{K}(mn)$.

Proof. The equivalence of (i) and (ii) is straightforward.

(ii) $\Rightarrow$ (iii). Assume that (ii) holds. Since (ii) implies (i), $\mathbf{K}$ is an entropic variety, and a simple induction shows that unary terms commute. It is easily seen that there is a natural bijection between $\mathbf{K}(A \otimes B, C)$ and $\mathbf{K}(A, \mathbf{K}(B, C))$, whence $H_B = \mathbf{K}(B, -)$ is the required right adjoint to $- \otimes B$.

(iii) $\Rightarrow$ (iv). This follows from the fact that left adjoint functors preserve colimits.

(iv) $\Rightarrow$ (v). This is trivial.

(v) $\Rightarrow$ (vi). Since unary terms commute there is, by Proposition 2.2, an isomorphism $\phi_{1,1} : F\mathbf{K}(1) \otimes F\mathbf{K}(1) \to F\mathbf{K}(1)$ with $(x_1 \otimes x_1)\phi_{1,1} = x_1$. Recall that $F\mathbf{K}(k+1)$ is the coproduct of $F\mathbf{K}(k)$ and $F\mathbf{K}(1)$. An easy induction now establishes the implication.

(vi) $\Rightarrow$ (i). Assume that there is an isomorphism $\phi_{m,n} : F\mathbf{K}(m) \otimes F\mathbf{K}(n) \to F\mathbf{K}(mn)$ such that the $(x_i \otimes y_j)\phi_{m,n}$ are the free generators of $F\mathbf{K}(mn)$. Then $F\mathbf{K}(m) \otimes F\mathbf{K}(n)$ is freely generated by the elements $x_{ij} = x_i \otimes y_j$. The identity (**) for entropic varieties follows from the fact that $\otimes$ is a bimorphism. □

By *epimorphism* we mean onto homomorphism, and *projectivity* is defined with respect to epimorphisms rather than with respect to epics.

3.2. COROLLARY. *Let* $\mathbf{K}$ *be an entropic variety. If* P *and* Q *are projective in* $\mathbf{K}$, *then so is* $P \otimes Q$.

Proof. An algebra $P \in \mathbf{K}$ is projective if and only if the set-valued functor $\mathbf{K}(P, -) : \mathbf{K} \to \mathbf{Set}$ maps epimorphisms to surjections; since $\mathbf{K}$ is entropic this is equivalent to requiring the functor $\mathbf{K}(P, -) : \mathbf{K} \to \mathbf{K}$ to preserve epimorphisms. Let $P, Q \in \mathbf{K}$ be projectives and let $\phi : A \to B$ be an epimorphism. Then the induced map $\mathbf{K}(Q, A) \to \mathbf{K}(Q, B)$ is an epimorphism and hence the induced map $\mathbf{K}(P, \mathbf{K}(Q, A)) \to \mathbf{K}(P, \mathbf{K}(Q, B))$ is an epimorphism. By the right adjointness of $\mathbf{K}(-, C)$ to $- \otimes C$, the induced map $\mathbf{K}(P \otimes Q, A) \to \mathbf{K}(P \otimes Q, B)$ is an epimorphism. □

We close this section with a discussion of the associativity of tensor product. It is easily seen that in any variety there is an isomorphism $A \otimes B \simeq B \otimes A$ natural in both A and B. Thus the bifunctor $T_2 : \mathbf{K}^2 \to \mathbf{K}$ is commutative; and it is tempting to believe that it is also associative. That in general T_2 is not associative was pointed out for the variety of semigroups by P. A. Grillet [14], but it was mistakenly assumed to be always associative in F. E. J. Linton [19].

We say that tensor product is *associative* if for each $A, B, C \in \mathbf{K}$ there is an isomorphism

$$\tau_{A,B,C} : A \otimes (B \otimes C) \to (A \otimes B) \otimes C$$

which is natural in A, B and C; that is

$$\tau : T_2(id_{\mathbf{K}} \times T_2) \to T_2(T_2 \times id_{\mathbf{K}}),$$

where $id_{\mathbf{K}}$ is the identity functor on $\mathbf{K}$.

The following result follows from Corollary 1 to Proposition 1 in E. Nelson [23].

3.3. PROPOSITION. *Tensor product is associative in an entropic variety.* □

A direct proof of this result is sketched below.

Suppose that $A, B, C \in \mathbf{K}$ and $\phi : A \times B \to C$ is a map. Then ϕ is a *right sesquimorphism* if there is a generating set Σ for A such that:

(i) the map $a \mapsto \langle a, b \rangle \phi : A \to C$ is a homomorphism for each $b \in B$, and

(ii) the map $b \mapsto \langle a, b \rangle \phi : B \to C$ is a homomorphism for each $a \in \Sigma$.

We define *left sesquimorphism* dually and use the term *sesquimorphism* to refer to either. The map $eval_A : F\mathbf{K}(1) \times A \to A$ of Proposition 1.1, and more generally the map $eval_A^n : F\mathbf{K}(n) \times A^n \to A$ given by $\langle p, a \rangle eval_A^n = p(a)$, is a right sesquimorphism. It is straightforward to show that $\mathbf{K}$ is entropic if and only if every right (left) sesquimorphism is a bimorphism: necessity is a simple calculation while sufficiency relies on the fact that $eval_A^n$ is a right sesquimorphism.

In order to show that $A \otimes (B \otimes C)$ and $(A \otimes B) \otimes C$ are naturally isomorphic it suffices to show that each is naturally isomorphic to the 3-fold tensor product $A \otimes B \otimes C$. The definitions guarantee the existence of a unique homomorphism $\beta : A \otimes B \otimes C \to A \otimes (B \otimes C)$ such that $(a \otimes b \otimes c)\beta = a \otimes (b \otimes c)$, and a diagram chase using the fact that sesquimorphisms are bimorphisms establishes the existence of a unique homomorphism $\alpha : A \otimes (B \otimes C) \to A \otimes B \otimes C$ such that $a \otimes (b \otimes c)\alpha = a \otimes b \otimes c$. Clearly α is inverse to β, whence $A \otimes (B \otimes C)$ and $A \otimes B \otimes C$ are naturally isomorphic, as required.

The authors know of no non-entropic variety in which tensor product is associative.

The following corollary, whose proof is analogous to the proof for abelian groups (see [20; p. 159, p. 180]), is the basis of the next section.

3.4. COROLLARY. *If* $\mathbf{K}$ *is an entropic variety, then* $\langle \mathbf{K}, \otimes, F\mathbf{K}(1)\rangle$ *is a closed category; i.e. a symmetric monoidal category, with unit* $F\mathbf{K}(1)$, *in which each functor* $-\otimes B$ *has a specified right adjoint, namely* $\mathbf{K}(B, -)$. □

Assume $\mathbf{K}$ is entropic. Since $\langle \mathbf{K}, \otimes, F\mathbf{K}(1)\rangle$ is monoidal, the coherence results of MacLane [20; pp. 161–166] establish the general associative law for $\otimes$ (up to a natural isomorphism). In fact we have a little more: if $A_1, \ldots, A_n \in \mathbf{K}$, then any well-formed bracketing of $A_1 \otimes \cdots \otimes A_n$ using the binary tensor product is naturally isomorphic to the n-fold tensor product $A_1 \otimes \cdots \otimes A_n$. For example, in the proof of 4.2 we shall use the fact that $(A \otimes B) \otimes (C \otimes D)$ is naturally isomorphic to $A \otimes B \otimes C \otimes D$.

4. Monoids in an entropic variety

Throughout this section $\mathbf{K}$ denotes an entropic variety and our attention focuses on the closed category $\langle \mathbf{K}, \otimes, F\mathbf{K}(1)\rangle$.

We recall that a *monoid* in $\mathbf{K}$ is an algebra $M \in \mathbf{K}$ with two homomorphisms, $m: M \otimes M \to M$ and $i: F\mathbf{K}(1) \to M$, for which the diagrams below commute:

$$\begin{array}{ccccc} M\otimes(M\otimes M) & \xrightarrow{\tau} & (M\otimes M)\otimes M & \xrightarrow{m\times id_M} & M\otimes M \\ {\scriptstyle id_M\otimes m}\downarrow & & & & \downarrow{\scriptstyle m} \\ M\otimes M & & \xrightarrow{\qquad\quad m \qquad\quad} & & M \end{array}$$

$$\begin{array}{ccccc} F\mathbf{K}(1)\otimes M & \xrightarrow{i\otimes id_M} & M\otimes M & \xleftarrow{id_M\otimes i} & M\otimes F\mathbf{K}(1) \\ & {\scriptstyle\lambda_M}\searrow & \downarrow{\scriptstyle m} & \swarrow{\scriptstyle\rho_M} & \\ & & M & & \end{array}$$

where τ is the natural isomorphism guaranteed by Proposition 3.4, $\lambda_M: F\mathbf{K}(1)\otimes M \to M$ is the homomorphism induced by the bimorphism $eval_M: F\mathbf{K}(1)\times M \to M$, and $\rho_M: M\otimes F\mathbf{K}(1) \to M$ is the homomorphism induced by the bimorphism $eval_M: M\times F\mathbf{K}(1) \to M$.

If $(M; m, i)$ and $(M'; m', i')$ are monoids in $\mathbf{K}$ then a map $\phi: M \to M'$ is a *monoid-morphism* if it is a **K**-morphism such that the diagrams below commute:

$$\begin{array}{ccc} M\otimes M & \xrightarrow{\phi\otimes\phi} & M'\otimes M' \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle m'} \\ M & \xrightarrow[\phi]{} & M' \end{array} \qquad \begin{array}{ccc} M & \xrightarrow{\phi} & M' \\ & {\scriptstyle i}\nwarrow \quad \nearrow{\scriptstyle i'} & \\ & F\mathbf{K}(1) & \end{array}$$

We are interested in the equational character of the category $\mathbf{Mon_K}$ of monoids of **K**.

If F is the set of operation symbols for **K**, then we denote the set $F\dot{\cup}\{*, 1\}$ by F^1_*, where $*$ is a binary operation symbol and 1 is a nullary operation symbol. The proof of the following result is omitted.

4.1. PROPOSITION. *Let* $\mathbf{K}^1_*$ *denote the class of all algebras* $\langle M; F^1_*\rangle$ *for which* $\langle M; F\rangle \in \mathbf{K}$ *and the following identities hold*:

(i) $y * f(x_1, \ldots, x_n) = f(y * x_1, \ldots, y * x_n)$, *for all* $f \in F$;

(ii) $f(x_1, \ldots, x_n) * y = f(x_1 * y, \ldots, x_n * y)$, *for all* $f \in F$;

(iii) $x * (y * z) = (x * y) * z$;

(iv) $1 * x = x = x * 1$.

Then $\mathbf{K}^1_*$ *and* $\mathbf{Mon_K}$ *are isomorphic categories.* □

We identify the categories $\mathbf{Mon_K}$ and $\mathbf{K}^1_*$. Let $F_* = F\dot{\cup}\{*\}$ and denote by $\mathbf{K}_*$ the variety of algebras $\langle A; F_*\rangle$ which satisfy identities (i), (ii), and (iii) of Proposition 4.1; then $\mathbf{K}_*$ is isomorphic to the category of *semigroups* (*with respect to* $\otimes$) *in* **K**. The subvarieties of $\mathbf{K}^1_*$ and $\mathbf{K}_*$ in which $*$ is commutative are noted by $\mathbf{CK}^1_*$ and $\mathbf{CK}_*$ respectively.

We now prove that in an entropic variety **K** the tensor product of two [commutative] semigroups in **K** carries an obvious [commutative] semigroup structure; and similarly for two monoids in **K**. Moreover, for two commutative monoids the resulting monoid is their coproduct in $\mathbf{CK}^1_*$.

By 4.1 the category $\mathbf{CAb}^1_*$ of commutative monoids in the category **Ab** of abelian groups is isomorphic to the category $\mathbf{ComR}^1$ of commutative rings with unity; hence 4.2 generalizes the well-known fact that the **Ab**-tensor product of two commutative rings with unit is their $\mathbf{ComR}^1$-coproduct.

4.2. THEOREM. *Let M and N be semigroups in* **K**. *The there is a binary operation* $*$ *on* $M\otimes N$ *satisfying the interchange law*

$$(a\otimes b) * (a'\otimes b') = (a * a')\otimes(b * b'),$$

such that $\langle M\otimes N; *\rangle$ *is a semigroup in* **K**, *and if M and N are monoids in* **K**, *then*

$\langle M\otimes N; *, 1\otimes 1\rangle$ is a monoid in $\mathbf{K}$; if $*$ is commutative on M and N, then it is commutative on $M\otimes N$. Furthermore, if $M, N\in \mathbf{CK}^1_*$ and we define $\sigma_M : M\to M\otimes N$ by $a\sigma_M = a\otimes 1$ and $\sigma_N : N\to M\otimes N$ by $b\sigma_N = 1\otimes b$, then

$$M \xrightarrow{\sigma_M} M\otimes N \xleftarrow{\sigma_N} N$$

is a coproduct diagram in $\mathbf{CK}^1_*$.

Proof. Since $\mathbf{K}$ is entropic, $M\otimes N\otimes M\otimes N$ is naturally isomorphic to $(M\otimes N)\otimes(M\otimes N)$. We define a map $\phi : M\times N\times M\times N\to M\otimes N$ by $\langle a, b, a', b'\rangle\phi = (a * a')\otimes(b * b')$. Since $*$ distributes over the operations of $\mathbf{K}$, ϕ is a 4-morphism, so there is a homomorphism $\psi : M\otimes N\otimes M\otimes N\to M\otimes N$ satisfying $(a\otimes b\otimes a'\otimes b')\psi = (a * a')\otimes(b * b')$. The homomorphism ψ and the natural isomorphism between $M\otimes N\otimes M\otimes N$ and $(M\otimes N)\otimes(M\otimes N)$ yield a bimorphism $* : (M\otimes N)\times(M\otimes N)\to M\otimes N$ satisfying $(a\otimes b) * (a'\otimes b') = (a * a')\otimes(b * b')$. To verify the first half of the theorem is now a straightforward calculation.

Let $M, N\in \mathbf{CK}^1_*$. Clearly the maps σ_M and σ_N are $\mathbf{K}$-homomorphisms. By the definition of the nullary operation $1 = 1\otimes 1$ in $M\otimes N$, both σ_M and σ_N preserve 1. Further, $(a * a')\sigma_M = (a * a')\otimes 1 = (a * a')\otimes(1 * 1) = (a\otimes 1) * (a'\otimes 1) = a\sigma_M * a'\sigma_M$, so σ_M, and similarly σ_N, is a monoid morphism.

Now let $\tau_1\in \mathbf{K}^1_*(M, A)$ and $\tau_2\in \mathbf{K}^1_*(N, A)$ where $A\in \mathbf{CK}^1_*$. The map $\tau : M\times N\to A$ given by $\langle a, b\rangle\tau = a\tau_1 * b\tau_2$ is a bimorphism since $*$ distributes over the operations of $\mathbf{K}$. Thus τ induces a $\mathbf{K}$-morphism $\bar\tau : M\otimes N\to A$ satisfying $(a\otimes b)\bar\tau = a\tau_1 * b\tau_2$. For $a\in M$ we have

$$a\sigma_M\bar\tau = (a\otimes 1)\bar\tau = a\tau_1 * 1\tau_2 = a\tau_1 * 1 = a\tau_1,$$

since $\tau_1\in \mathbf{K}^1_*(M, A)$. Thus $\sigma_M\bar\tau = \tau_1$, and similarly $\sigma_N\bar\tau = \tau_2$.

We must now see that $\bar\tau$ is a $\mathbf{K}^1_*$-morphism. Firstly $(1\otimes 1)\bar\tau = 1\tau_1 * 1\tau_2 = 1 * 1 = 1$. Secondly, if $a = p(a_1\otimes b_1, \ldots, a_m\otimes b_m)\in M\otimes N$ and $b = q(a'_1\otimes b'_1, \ldots, a'_n\otimes b'_n)\in M\otimes N$ then, with $t_{ij} = (a_i\otimes b_i) * (a'_j\otimes b'_j)$, we have

$$\begin{aligned}(a * b)\bar\tau &= (p(a_1\otimes b_1, \ldots, a_m\otimes b_m) * q(a'_1\otimes b'_1, \ldots, a'_n\otimes b'_n))\bar\tau\\ &= p(q(t_{11}, \ldots, t_{1n}), \ldots, q(t_{m1}, \ldots, t_{mn}))\bar\tau\\ &= p(q(t_{11}\bar\tau, \ldots, t_{1n}\bar\tau, \ldots, q(t_{m1}\bar\tau, \ldots, t_{mn}\bar\tau)),\end{aligned}$$

and

$$\begin{aligned} t_{ij}\bar{\tau} &= ((a_i \otimes b_i) * (a'_j \otimes b'_j))\bar{\tau} = ((a_i * a'_j) \otimes (b_i * b'_j))\bar{\tau} \\ &= (a_i * a'_j)\tau_1 * (b_i * b'_j)\tau_2 = (a_i\tau_1 * a'_j\tau_1) * (b_i\tau_2 * b'_j\tau_2) \\ &= (a_i\tau_1 * b_i\tau_2) * (a'_j\tau_1 * b'_j\tau_2) = (a_i \otimes b_i)\bar{\tau} * (a'_j \otimes b'_j)\bar{\tau}, \end{aligned}$$

so $(a * b)\bar{\tau} = a\bar{\tau} * b\bar{\tau}$; whence $\bar{\tau}$ is a $\mathbf{K}^1_*$-morphism.

Finally, suppose $\psi \in \mathbf{K}^1_*(M \otimes N, A)$ and $\sigma_M\psi = \tau_1$ and $\sigma_N\psi = \tau_2$. Then $(a \otimes 1)\psi = a = a\tau_1$ for all $a \in M$, and $(1 \otimes b)\psi = b\sigma_M\psi = a\tau_2$ for all $b \in N$; thus $(a \otimes b)\psi = ((a \otimes 1) * (1 \otimes b))\psi = (a \otimes 1)\psi * (1 \otimes b)\psi = a\tau_1 * b\tau_2 = (a \otimes b)\bar{\tau}$ for all $\langle a, b\rangle \in A \times B$. Thus ψ and $\bar{\tau}$ agree on a generating set for $M \otimes N$, and so $\psi = \bar{\tau}$. □

Although the previous theorem describes coproducts in $\mathbf{CK}^1_*$, it tells us nothing about coproducts in $\mathbf{CK}_*$. Nevertheless, if we restrict our attention to the subvariety $\mathbf{IK}_*$ of $\mathbf{CK}_*$ determined by the idempotent law $x * x = x$, and the corresponding subvariety $\mathbf{IK}^1_*$ of $\mathbf{CK}^1_*$, then the two coproducts are closely related.

4.3. THEOREM. *Let $A, B \in \mathbf{IK}^1_*$, then the $\mathbf{IK}^1_*$-coproduct of A and B is a $\mathbf{K}_*$-retract of the $\mathbf{IK}_*$-coproduct of A and B. Moreover, the same is true of any subvariety of $\mathbf{IK}_*$ and the corresponding subvariety of $\mathbf{IK}^1_*$.*

Proof. Let $A, B \in \mathbf{IK}^1_*$ and let

$$A \xrightarrow{\sigma_1} A \coprod B \xleftarrow{\sigma_2} B$$

be a coproduct diagram in $\mathbf{IK}_*$. Define a map ρ from $A \coprod B$ to itself by $x\rho = x * (1\sigma_1 * 1\sigma_2)$. By the definition of $\mathbf{K}_*$, and since $*$ is a semilattice operation in $\mathbf{IK}_*$, it follows that ρ is a $\mathbf{K}_*$-homomorphism and $\rho^2 = \rho$. Hence $C = \mathrm{Im}(\rho)$ is a $\mathbf{K}_*$-retract of $A \coprod B$. Let $\tau_1 = \sigma_1\rho$ and $\tau_2 = \sigma_2\rho$; then we claim that

$$A \xrightarrow{\tau_1} C \xleftarrow{\tau_2} B$$

is a coproduct diagram in $\mathbf{IK}^1_*$. For all $x \in C$ we have $x * (1\sigma_1 * 1\sigma_2) = x\rho = x$, and so $1\sigma_1 * 1\sigma_2$ is an identity element for $*$ on C: whence $C \in \mathbf{IK}^1_*$, and τ_1 and τ_2 are

1-preserving and hence are $\mathbf{K}^1_*$-homomorphisms. Let $D \in \mathbf{IK}^1_*$ and let $\alpha_1 : A \to D$ and $\alpha_2 : B \to D$ be $\mathbf{K}^1_*$-homomorphisms. Then there exists a $\mathbf{K}_*$-homomorphism $\bar{\alpha} : A \coprod B \to D$ such that $\sigma_1\bar{\alpha} = \alpha_1$ and $\sigma_2\bar{\alpha} = \alpha_2$. For all $a \in A$, $a\tau_1\bar{\alpha} = a\sigma_1\rho\bar{\alpha} = (a\sigma_1 * 1\sigma_1 * 1\sigma_2)\bar{\alpha} = (a\sigma_1 * 1\sigma_2)\bar{\alpha} = a\sigma_1\bar{\alpha} * 1\sigma_2\bar{\alpha} = a\alpha_1 * 1\alpha_2 = a\alpha_1 * 1 = a\alpha_1$, since $\bar{\alpha}$ preserves $*$ and α_2 preserves 1. Hence $\tau_1\bar{\alpha} = \alpha_1$, and similarly $\tau_2\bar{\alpha} = \alpha_2$. Since $(1\sigma_1 * 1\sigma_2)\bar{\alpha} = (1\sigma_1\bar{\alpha}) * (1\sigma_2\bar{\alpha}) = 1\alpha_1 * 1\alpha_2 = 1 * 1 = 1$, it follows that the restriction of $\bar{\alpha}$ to C is a $\mathbf{K}^1_*$-homomorphism which extends α_1 and α_2. The coproduct $A \coprod B$ is generated by $A\sigma_1 \cup B\sigma_2$ and so $A\tau_1 \cup B\tau_2$ generates C, from which the uniqueness of the fill-in map, $\bar{\alpha}$, follows at once. It is clear that the argument given above applies to any subvariety of $\mathbf{IK}_*$. □

Although tensor product preserves the commutativity of $*$, in general identities involving $*$ which are satisfied on A and B need not be satisfied on $A \otimes B$. When, for example, does it follow that if $*$ is idempotent on A and B, then it is idempotent on $A \otimes B$? This holds trivially if **K** is **Sets**, for then $\otimes$ is just direct product and hence $\otimes$ preserves all identities. If **K** is the variety **Ab**, then Boolean rings are by definition the members of $\mathbf{K}^1_*$ on which $*$ is idempotent; an easy calculation, using the fact that Boolean rings satisfy the identity $x + x = 0$, shows that the abelian-group tensor product of two Boolean rings is again Boolean. As we shall see in section 6, if **K** is the entropic variety of join-semilattices with zero, then $\otimes$ does not preserve the idempotence of $*$; nevertheless the following lemma, when applied to join-semilattices, yields a large class within which the idempotence of $*$ is preserved by $\otimes$.

4.4. THEOREM. (i) *Let $A, B \in \mathbf{K}^1_*$ and assume that $A \otimes B$ satisfies the identities*

$$f(1, x_2, \ldots, x_n) = f(x_1, 1, x_3, \ldots, x_n) = \cdots = f(x_1, \ldots, x_{n-1}, 1) = 1 \cdots (A)$$

for every non-nullary operation $f \in F$. Then if $*$ is idempotent on A and B, it is also idempotent on $A \otimes B$.

(ii) *Let $\mathbf{A}$ be the subvariety of $\mathbf{K}^1_*$ determined by the identities (A), and let $\mathbf{B}$ be the subvariety $\mathbf{A} \cap \mathbf{IK}^1_*$. If $\mathbf{A}$ is closed under $\otimes$, then $\mathbf{B}$ is also closed under $\otimes$ and hence $\otimes$ is coproduct in $\mathbf{B}$.*

Proof. (i) The *rank* of a term p is the number of operation symbols occurring in p, counting repetitions. Let $n \geq 1$ and let p be a term of rank greater than 0 with $x_1, \ldots, x_n$ as its rank-zero subterms. An easy induction on the rank of p proves that if C satisfies the identities (A), then C satisfies

$$p(1, x_2, \ldots, x_n) = \cdots = p(x_1, x_2, \ldots, x_{n-1}, 1) = 1.$$

Now let $A, B \in \mathbf{K}^1_*$, and assume that $*$ is idempotent on A and B, and that $A \otimes B$ satisfies the identities (A). Let $G = \{a \otimes b \mid a \in A; b \in B\}$ be the generating set of $A \otimes B$. Recall that a nullary operation $0 \in F$ is interpreted on $A \otimes B$ via $0 \otimes 0$; hence G contains all the nullary operations on G. For all $a \in A$, $b \in B$ we have $(a \otimes b) * (a \otimes b) = (a * a) \otimes (b * b) = a \otimes b$, and thus $g * g = g$ for all $g \in G$. Every element c of $A \otimes B$ can be expressed in the form $p(g_1, \ldots, g_n)$ where $n \geq 1$ and the term p is of rank greater than zero with $x_1, \ldots, x_n$ as its rank-zero subterms. Hence by the observation above

$$\begin{aligned} g_i * c = g_i * p(g_1, \ldots, g_n) &= p(g_i * g_1, \ldots, g_i^2, \ldots, g_i * g_n) \\ &= p(g_i * g_1, \ldots, g_i, \ldots, g_i * g_n) = g_i * p(g_1, \ldots, 1, \ldots, g_n) = g_i * 1 = g_i. \end{aligned}$$

Thus we conclude

$$c * c = p(g_1, \ldots, g_n) * c = p(g_1 * c, \ldots, g_n * c) = p(g_1, \ldots, g_n) = c.$$

(ii) If, for $A, B \in \mathbf{B}$, the $\mathbf{CK}^1_*$-coproduct of A and B belongs to $\mathbf{B}$, then it is also the $\mathbf{B}$-coproduct of A and B; thus (ii) follows from (i) and Theorem 4.2. □

We remark that the forgetful functor from $\mathbf{Mon}_\mathbf{K}$ into $\mathbf{K}$ has a left adjoint. This follows, for instance, because forgetful functors between any two varieties have left adjoints. S. MacLane [20; Theorem 2, p. 168] gives a description of the free monoid generated by a $\mathbf{K}$-algebra: if $\mathbf{K}$ is entropic and $A \in \mathbf{K}$, then the free monoid generated by A is $\coprod(A^{(n)} \mid n \in \mathbb{N})$ where $A^{(0)} = F\mathbf{K}(1)$ and $A^{(n+1)} = A \otimes A^{(n)}$.

Let M be a monoid in $\mathbf{K}$. Recall that a *left action* of M on an algebra $A \in \mathbf{K}$ is a homomorphism $\nu : M \otimes A \to A$ such that the following diagram commutes:

$$\begin{array}{ccccccc} M \otimes (M \otimes A) & \xrightarrow{\tau} & (M \otimes M) \otimes A & \xrightarrow{m \otimes id_A} & M \otimes A & \xleftarrow{i \otimes id_A} & F\mathbf{K}(1) \otimes A \\ \downarrow{\scriptstyle id_M \otimes \nu} & & & & \downarrow{\scriptstyle \nu} & \swarrow{\scriptstyle \lambda_A} & \\ M \otimes A & & \xrightarrow{\qquad\qquad \nu \qquad\qquad} & & A & & \end{array}$$

where λ_A is the isomorphism of Proposition 2.1. The left actions of M on algebras in $\mathbf{K}$ form a category, ${}_M\mathbf{Act}$, whose morphisms are those $\mathbf{K}$-homomorphisms $\phi : A \to B$ such that the following diagram commutes:

$$\begin{array}{ccc} M \otimes A & \xrightarrow{id_M \otimes \phi} & M \otimes B \\ \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle \nu'} \\ A & \xrightarrow{\phi} & B \end{array}$$

If F is the set of operation symbols for $\mathbf{K}$, then we denote by $_MF$ the set $F\cup\{\lambda_m \mid m\in M\}$, where each λ_m is a unary operation symbol. The following result is now obvious.

4.3. PROPOSITION. *Let $_M\mathbf{K}$ denote the class of all algebras $\langle A; {_MF}\rangle$ for which $\langle A;F\rangle\in\mathbf{K}$ and the following identities hold:*

(i) $\lambda_m(f(x_1,\ldots,x_n))=f(\lambda_m(x_1),\ldots,\lambda_m(x_n))$, *for all $m\in M$ and all $f\in F$;*

(ii) $\lambda_{f(m_1,\ldots,m_n)}(x)=f(\lambda_{m_1}(x),\ldots,\lambda_{m_n}(x))$, *for all $m_1,\ldots,m_n\in M$ and all $f\in F$;*

(iii) $\lambda_m(\lambda_n(x))=\lambda_{m*n}(x)$;

(iv) $\lambda_1(x)=x$.

Then $_M\mathbf{K}$ and $_M\mathbf{Act}$ are isomorphic categories. Moreover, if M is commutative, then $_M\mathbf{K}$ is an entropic variety. □

Of course, we could assume only that M is a semigroup in $\mathbf{K}$, in which case we would drop the triangle from the diagram which defines $_M\mathbf{Act}$ and we would drop identity (iv) from the definition of $_M\mathbf{K}$.

The category $\mathbf{Ab}^1_*$ is just rings with unity, and for each ring $R\in\mathbf{Ab}^1_*$ the class of left R-modules is $_R\mathbf{Ab}$. Just as abelian groups can be identified with modules over the ring $\mathbb{Z}$, in general $\mathbf{K}$ can be identified with the class $_{F\mathbf{K}(1)}\mathbf{Act}$ of actions of the monoid $F\mathbf{K}(1)$; this is an immediate consequence of Propositions 2.1 and 2.2.

As is observed in S. MacLane [20; p 170] a left adjoint to the forgetful functor from $_M\mathbf{Act}$ into $\mathbf{K}$ is provided by the obvious functor $M\otimes - :\mathbf{K}\to {_M\mathbf{Act}}$.

Since $_M\mathbf{K}$ is an entropic variety we can consider tensor products and monoids in $_M\mathbf{K}$. Applied to abelian groups this yields algebras over a ring. Of course the process can be continued to yield a tower of structures.

An important example of a monoid in an entropic variety is the endomorphism algebra, $\mathrm{End}(A)=\mathbf{K}(A,A)$, of an algebra $A\in\mathbf{K}$; we simply let $*$ be composition of functions and let 1 be the identity map.

5. Congruences on tensor products

Let Θ and Φ be elements of the congruence lattices $\mathrm{Con}(A)$ and $\mathrm{Con}(B)$ respectively, and let $\theta:A\to A/\Theta$ and $\phi:B\to B/\Phi$ be the natural maps. Clearly the tensor product map $\theta\otimes\phi$ from $A\otimes B$ to $A/\Theta\otimes B/\Phi$ is onto; hence if we define $\Theta\otimes\Phi$ to be $\ker(\theta\otimes\phi)$, we have

$$(A\otimes B)/(\Theta\otimes\Phi)\cong A/\Theta\otimes B/\Phi.$$

If A is a member of $\mathbf{K}_*$ of $\mathbf{K}^1_*$ then we denote by $\mathrm{Con}^*(A)$ the lattice of congruences on A which also preserve $*$.

5.1. THEOREM. (i) *The congruence* $\Theta \otimes \Phi$ *is generated by the pairs of the form* $\langle a_1 \otimes b_1, a_2 \otimes b_2 \rangle$ *where* $a_1 \equiv a_2(\Theta)$ *and* $b_1 \equiv b_2(\Phi)$.

(ii) *Assume that* $\mathbf{K}$ *is an entropic variety. If* A, B *belong to* $\mathbf{K}_*$ *or* $\mathbf{K}_*^1$ *and* $\Theta \in \mathrm{Con}^*(A)$ *and* $\Phi \in \mathrm{Con}^*(B)$, *then* $\Theta \otimes \Phi \in \mathrm{Con}^*(A \otimes B)$.

Proof. (i) Let Ψ be the congruence on $A \otimes B$ generated by the pairs $\langle a_1 \otimes b_1, a_2 \otimes b_2 \rangle$ with $a_1 \equiv a_2(\Theta)$ and $b_1 \equiv b_2(\Phi)$, and let ψ be the natural map from $A \otimes B$ to $(A \otimes B)/\Psi$. Since $\ker(\theta) = \Theta$ and $\ker(\phi) = \Phi$ it follows that $\Psi \subseteq \ker(\theta \otimes \phi) = \Theta \otimes \Phi$. By the definition of Ψ, we may define a map α from $A/\Theta \times B/\Phi$ to $(A \otimes B)/\Psi$ by $\langle [a]\Theta, [b]\Phi \rangle \alpha = [a \otimes b]\Psi$; trivially α is a bimorphism. Thus there is a homomorphism β from $A/\Theta \otimes B/\Phi$ to $(A \otimes B)/\Psi$ satisfying $([a]\Theta \otimes [b]\Phi)\beta = [a \otimes b]\Psi$. Hence $(\theta \otimes \phi)\beta = \psi$ and so $\Theta \otimes \Phi = \ker(\theta \otimes \phi) \subseteq \ker(\psi) = \Psi$.

(ii) Assume that $\mathbf{K}$ is entropic and let $A, B \in \mathbf{K}_*$. Since $A \otimes B$ is generated by the tensors $a \otimes b$ with $a \in A$ and $b \in b$, and since $*$ distributes over all $\mathbf{K}$-terms, a straightforward calculation using the interchange law for $*$ and $\otimes$ shows that $\theta \otimes \phi$ is $*$-preserving; consequently $\Theta \otimes \Phi = \ker(\theta \otimes \phi)$ is a $*$-congruence. □

5.2. THEOREM. (i) *Let* $\Theta \in \mathrm{Con}(A)$ *and let* $\Phi_i \in \mathrm{Con}(B)$ *for all* $i \in I$, *then* $\Theta \otimes \bigvee(\Phi_i \mid i \in I) = \bigvee(\Theta \otimes \Phi_i \mid i \in I)$.

(ii) *Assume* $\mathbf{K}$ *is an entropic variety and let* A, B *belong to* $\mathbf{K}_*$ *or* $\mathbf{K}_*^1$. *Let* $\Theta \in \mathrm{Con}^*(A)$ *and let* $\Phi_i \in \mathrm{Con}^*(B)$ *for all* $i \in I$, *then* $\Theta \otimes \bigvee(\Phi_i \mid i \in I) = \bigvee(\Theta \otimes \Phi_i \mid i \in I)$.

Proof. (i) Let $\Theta \in \mathrm{Con}(A)$, let $\Phi, \Psi \in \mathrm{Con}(B)$ with $\Phi \leq \Psi$, and let θ, ϕ, and ψ be the associated maps. Since $\Phi \leq \Psi$ there exists a homomorphism γ from A/Φ to A/Ψ such that $\phi\gamma = \psi$. Hence

$$\theta \otimes \psi = (\theta \circ id) \otimes (\phi \circ \gamma) = (\theta \otimes \phi) \circ (id \otimes \gamma),$$

and thus $\Theta \otimes \Phi = \ker(\theta \otimes \phi) \subseteq \ker(\theta \otimes \psi) = \Theta \otimes \Psi$. Consequently tensoring on the left by Θ is order-preserving and thus

$$\Theta \otimes \bigvee(\Phi_i \mid i \in I) \supseteq \bigvee(\Theta \otimes \Phi_i \mid i \in I).$$

By Theorem 5.1(i) in order to prove the reverse inclusion it is sufficient to prove

$$a_1 \equiv a_2(\Theta) \,\&\, b_1 \equiv b_2(\bigvee(\Phi_i \mid i \in I)) \Rightarrow a_1 \otimes b_1 \equiv a_2 \otimes b_2(\bigvee(\Theta \otimes \Phi_i \mid i \in I)).$$

If $b_1 \equiv b_2(\bigvee(\Phi_i \mid i \in I))$ then there exist $i_1, \ldots, i_n \in I$ and elements $c_0, \ldots, c_n \in B$ such that $b_1 = c_0$, $b_2 = c_n$ and $c_{k-1} \equiv c_k(\Phi_{i_k})$ for $1 \leq k \leq n$. Hence if $a_1 \equiv a_2(\Theta)$ we have

$$a_1 \otimes c_{k-1} \equiv a_2 \otimes c_k (\Theta \otimes \Phi_{i_k}), \quad \text{and so} \quad a_1 \otimes b_1 \equiv a_2 \otimes b_2 (\bigvee (\Theta \otimes \Phi_i \mid i \in I)).$$

(ii) Since $\mathrm{Con}^*(A)$ is a complete sublattice of $\mathrm{Con}(A)$ for all $A \in \mathbf{K}_*$, (ii) follows immediately from (i) and Theorem 5.1(ii). □

Let $\mathbf{S}$ be the variety of join-semilattices. By Theorem 5.2, the map Γ from $\mathrm{Con}(A) \times \mathrm{Con}(B)$ to $\mathrm{Con}(A \otimes B)$ defined by $\langle \Theta, \Phi \rangle \Gamma = \Theta \otimes \Phi$ is an $\mathbf{S}$-bimorphism and so induces a join-semilattice homomorphism γ from $\mathrm{Con}(A) \otimes_{\mathbf{S}} \mathrm{Con}(B)$ to $\mathrm{Con}(A \otimes B)$. Similarly, if $\mathbf{K}$ is entropic, then for all A, B belonging to $\mathbf{K}_*$ or $\mathbf{K}_*^1$ there is a join-semilattice homomorphism from $\mathrm{Con}^*(A) \otimes_{\mathbf{S}} \mathrm{Con}^*(B)$ to $\mathrm{Con}^*(A \otimes B)$. In general the map γ is *not* an isomorphism. Consider the variety $\mathbf{Ab}$ and note that for each prime p, $\mathrm{Con}(\mathbb{Z}_p) = \mathrm{Con}^*(\mathbb{Z}_p) = \mathbf{2}$ (the two-element chain). Now let p and q be distinct primes. By Theorem 6.1(ii) of the following section, the $\mathbf{S}$-tensor product of $\mathbf{2}$ with itself is the $\mathbf{Dist}^1$-coproduct of $\mathbf{2}$ with itself, where $\mathbf{Dist}^1$ is the variety of distributive lattices with unit. Hence

$$|\mathrm{Con}(\mathbb{Z}_p) \otimes \mathrm{Con}(\mathbb{Z}_q)| = |\mathrm{Con}^*(\mathbb{Z}_p) \otimes \mathrm{Con}^*(\mathbb{Z}_q)| = 5.$$

Whereas, since $\mathbb{Z}_p \otimes \mathbb{Z}_q$ is trivial, we have

$$|\mathrm{Con}(\mathbb{Z}_p \otimes \mathbb{Z}_q)| = |\mathrm{Con}^*(\mathbb{Z}_p \otimes \mathbb{Z}_q)| = 1.$$

6. Tensor products of join-semilattices

We now turn to the variety $\mathbf{S}$ of join-semilattices and the variety $\mathbf{S}_0$ of join-semilattices with zero. There are several works which provide information on tensor products in these varieties; see, for example, the references given at the end of this paper. Our aim here is to give general universal-algebraic proofs and, where possible, extensions of some of the results of G. A. Fraser [13]. Since each of the varieties $\mathbf{S}$ and $\mathbf{S}_0$ is entropic, all the results of the earlier sections apply. In particular, tensor product is associative in both $\mathbf{S}$ and $\mathbf{S}_0$: associativity in $\mathbf{S}$ is proved in [13] only in the case where the factors are finite distributive lattices.

Let $\mathbf{D}$ be the variety of algebras $\langle A; \vee, * \rangle$ of type $\langle 2, 2 \rangle$ such that

(i) $\langle A; \vee \rangle$ is a join-semilattice,
(ii) $\langle A; * \rangle$ is a commutative semigroup,
(iii) A satisfies $x * (y \vee z) = (x * y) \vee (x * z)$;

let $\mathbf{D}_0$ be the variety of algebras $\langle A; \vee, *, 0\rangle$ such that $\langle A; \vee, *\rangle$ is in $\mathbf{D}$ and 0 is a zero for $\langle A; \vee\rangle$; let $\mathbf{D}^1$ be the variety of algebras $\langle A; \vee, *, 1\rangle$ such that $\langle A, \vee, *\rangle$ is in $\mathbf{D}$ and 1 is an identity element for $\langle A; *\rangle$; and let $\mathbf{D}_0^1$ be the variety of algebras $\langle A; \vee, *, 0, 1\rangle$ such that $\langle A; \vee, *, 0\rangle$ is in $\mathbf{D}_0$ and $\langle A; \vee, *, 1\rangle$ is in $\mathbf{D}^1$. Of course, these are respectively the derived varieties $\mathbf{CS}_*$, $(\mathbf{CS}_0)_*$, $\mathbf{CS}_*^1$, $(\mathbf{CS}_0)_*^1$, and are, in turn, interesting extensions of the variety **Dist** of distributive lattices, $\mathbf{Dist}_0$ of distributive lattices with zero, $\mathbf{Dist}^1$ of distributive lattices with unit, and $\mathbf{Dist}_0^1$ of bounded distributive lattices.

By Theorem 4.2, $\mathbf{D}$ and $\mathbf{D}^1$ are closed under **S**-tensor products, and $\mathbf{D}_0$ and $\mathbf{D}_0^1$ are closed under $\mathbf{S}_0$-tensor products. In the variety $\mathbf{D}^1$ the identities (A) of Theorem 4.4 reduce to $x \vee 1 = 1$; that is, 1 is the largest element. Hence in the subvariety $\mathbf{B}$ of Theorem 4.4 we have

$$x * (x \vee y) = (x * x) \vee (x * y) = x \vee (x * y)$$
$$= (x * 1) \vee (x * y) = x * (1 \vee y) = x * 1 = x.$$

Thus we have both absorption laws, and consequently $\mathbf{B}$ is precisely $\mathbf{Dist}^1$. Similarly, for $\mathbf{D}_0^1$, the subvariety $\mathbf{B}$ is $\mathbf{Dist}_0^1$. Hence, by Theorem 4.4, to prove that $\mathbf{Dist}^1$ is closed under **S**-tensor products it remains to show that if $A, B \in \mathbf{Dist}^1$, then $1 \otimes 1$ is the largest element of $A \otimes B$. Let $a \otimes b$ be a generator of $A \otimes B$. Since both tensoring on the left by a and tensoring on the right by b are join-preserving maps, they are order-preserving and so $a \otimes b \leq 1 \otimes b \leq 1 \otimes 1$. Thus $1 \otimes 1$ dominates all generators, and so dominates any join of generators; whence $1 \otimes 1$ is the largest element of $A \otimes B$. By precisely the same argument, $\mathbf{Dist}_0^1$ is closed under $\mathbf{S}_0$-tensor products.

Now to the varieties **Dist** and $\mathbf{Dist}_0$. In [13] Fraser proves that **Dist** is closed under **S**-tensor products by using a solution to the word problem for the **S**-tensor product of two distributive lattices. In fact, this is an easy consequnce of the closure of $\mathbf{Dist}^1$ under $\otimes$. Indeed, let $A, B \in \mathbf{Dist}$, and let A^1 and B^1 be the lattices obtained by adjoining new units. It is a simple calculation to see that every bimorphism ϕ from $A \times B$ to C can be extended to a bimorphism from $A^1 \times B^1$ to C^1 by defining $x\phi = 1$ for all elements x not in $A \times B$. Now a simple diagram chase, using the universal property of $A^1 \otimes B^1$, shows that the subsemilattice of $A^1 \otimes B^1$ generated by $\{a \otimes b \mid a \in A; b \in B\}$ has the definining universal property of $A \otimes B$. Consequently $A \otimes B$ is embedded as a semilattice into $A^1 \otimes B^1$. Since $(a \otimes b) * (a' \otimes b') = (a * a') \otimes (b * b') = (a \wedge a') \otimes (b \wedge b')$, and since $*$ distributes over $\vee$, it follows that the embedding of $A \otimes B$ into $A^1 \otimes B^1$ preserves $*$, and thus $A \otimes B \in \mathbf{Dist}$. In eactly the same way, $\mathbf{Dist}_0$ is closed under $\mathbf{S}_0$-tensor products.

In summary, we have the following consequences of Theorem 4.2 and Theorem 4.4.

6.1. THEOREM. (i) *If* $\mathbf{K}$ *is* $\mathbf{D}, \mathbf{D}^1, \mathbf{Dist}$, *or* $\mathbf{Dist}^1$ $[\mathbf{D}_0, \mathbf{D}_0^1, \mathbf{Dist}_0$, *or* $\mathbf{Dist}_0^1]$, *then it is closed under* $\mathbf{S}$*-tensor products* $[\mathbf{S}_0$*-tensor products*]; *moreover, for all* $A, B \in \mathbf{K}$, $*$ *is defined on* $A \otimes B$ *via the interchange law*

$$(a\otimes b) * (a'\otimes b') = (a * a')\otimes(b * b'),$$

and where appropriate the identity element for $*$ *is* $1\otimes 1$.

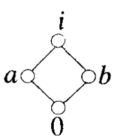
(ii) *If* $\mathbf{K}$ *is* $\mathbf{D}^1$ *or* $\mathbf{Dist}^1$ $[\mathbf{D}_0^1$ *or* $\mathbf{Dist}_0^1]$, *then for all* $A, B \in \mathbf{K}$ *the* $\mathbf{S}$*-tensor product* $[\mathbf{S}_0$*-tensor product*$]$ $A\otimes B$ *is the* $\mathbf{K}$*-coproduct of* A *and* B. □

This result generalizes Theorem 3.3 of [11].

The $\mathbf{S}_0$-tensor product of two finite members of $\mathbf{S}_0$ coincides with their $\mathbf{J}_0$-tensor product, where $\mathbf{J}_0$ is the category of complete-join-semilattices and complete-join-preserving maps. (Note that an object in $\mathbf{J}_0$ has a zero and that a morphism of $\mathbf{J}_0$ is zero-preserving.) Hence for finite members A, B of $\mathbf{S}_0$, their $\mathbf{S}_0$-tensor product $A\otimes B$ is isomorphic to $\mathbf{S}_0(A, B^d)^d$, where X^d denotes the order-theoretic dual of X; see B. Banaschewski and E. Nelson [1], D. Mowatt [21], and Z. Schmuely [26]. The map $a\otimes b: A \to B^d$ is given by

$$x(a\otimes b) = \begin{cases} i & \text{if} \quad x = 0, \\ b & \text{if} \quad x \le a \quad \text{and} \quad x \ne 0, \\ 0 & \text{if} \quad x \not\le a, \end{cases}$$

where i denotes the largest element of B. We can use this representation to show that $\mathbf{S}_0$-tensor product does not in general preserve the idempotence of $*$. To see this, note that if $S \in \mathbf{S}_0$, then, since join distributes over itself, we may let $* = \vee$ and $1 = 0$ on $\mathbf{S}$. Then S becomes a member of $\mathbf{D}_0^1$ on which $*$ is idempotent but which satisfies the identities (A) only if S is trivial. Now let S be the join-semilattice drawn below.



	0	a	b	i
$a\otimes i$	i	i	0	0
$i\otimes b$	i	b	b	b
$(a\otimes i)\vee(i\otimes b)$	i	b	0	0
$i\otimes i$	i	0	0	0

The accompanying table describes certain elements of $\mathbf{S}_0(S, S^d)^d$. As noted earlier, $i\otimes i$ is the largest element of $S\otimes S$, whence $((a\otimes i)\vee(i\otimes b))^2 = [(a\otimes i) * (a\otimes i)]\vee[(a\otimes i) * (i\otimes b)]\vee[(i\otimes b) * (i\otimes b)]$

$$= [(a * a)\otimes(i * i)]\vee[(a * i)\otimes(i * b)]\vee[(i * i)\otimes(b * b)]$$

$$= (a\otimes i)\vee(i\otimes i)\vee(i\otimes b) = i\otimes i.$$

It is clear from the table that $(a\otimes i)\vee(i\otimes b)$ is not equal to $i\otimes i$, and hence is not an idempotent in $S\otimes S$.

S-tensor product behaves well with respect to projectivity in the varieties $\mathbf{S}, \mathbf{D}, \mathbf{D}^1$, **Dist**, and $\mathbf{Dist}^1$. Basically this is because **S** is an idempotent, entropic variety.

A variety K is *idempotent* if for every operation f, **K** satisfies the identity $f(x, x, \ldots, x) = x$, or equivalently, if $F\mathbf{K}(1)$ is trivial; in particular, if **K** is a nontrivial variety, then there are no nullary operations in the type of **K**. The proof of the following lemma is quite straightforward and is omitted.

6.2. LEMMA. (i) *Let* **K** *be an idempotent variety, and let* $A, B \in \mathbf{K}$.

(a) *Every homomorphism out of* $A\times B$ *is a bimorphism; whence there is a natural homomorphism* ρ *from* $A\otimes B$ *to* A *satisfying* $(a\otimes b)\rho = a$.

(b) *For all* $b\in B$ *the natural homomorphism* σ_b *from* A *to* $A\otimes B$ *defined by* $a\sigma_b = a\otimes b$ *satisfies* $\sigma_b\rho = id_A$; *whence* A *is a retract of* $A\otimes B$.

(ii) *Let* **K** *be an idempotent, entropic variety, and let* $A, B\in \mathbf{K}_*$ (*respectively* $\mathbf{K}^1_*$).

(a) *The map* ρ *from* $A\otimes B$ *to* A *is a* $\mathbf{K}_*$-*homomorphism* (*respectively* $\mathbf{K}^1_*$-*homomorphism*).

(b) *If* $*$ *is idempotent on* A *and* B, *then* σ_b *is a* $\mathbf{K}_*$-*homomorphism and* σ_1 *is a* $\mathbf{K}^1_*$-*homomorphism; whence* A *is a* $\mathbf{K}_*$-*retract* (*respectively* $\mathbf{K}^1_*$-*retract*) *of* $A\otimes B$. □

This lemma allows us to strengthen Corollary 3.2.

6.3. THEOREM. *Let* **K** *be an idempotent, entropic variety.*

(i) $A\otimes B$ *is projective in* **K** *if and only if* A *and* B *are projective in* **K**.

(ii) *Let* $A, B\in\mathbf{K}_*$ *and assume that* $*$ *is idempotent on* A *and* B, *then if* $A\otimes B$ *is projective in* $\mathbf{K}_*$, *so are both* A *and* B.

(iii) *Let* $A, B \in \mathbf{K}^1_*$ *and assume that* $*$ *is idempotent on both* A *and* B; *then* $A\otimes B$ *is projective in* $\mathbf{K}^1_*$ *if and only if both* A *and* B *are projective in* $\mathbf{K}^1_*$.

(iv) *In* (i), (ii) and (iii) *the varieties* $\mathbf{K}, \mathbf{K}_*$, *and* $\mathbf{K}^1_*$ *may be replaced by subvarieties* (*of the appropriate type*) *which are closed under* **K**-*tensor products.*

Proof. Since a retract of a projective is projective, (i), (ii), and one direction in (iii) follow from Corollary 3.2 and Lemma 6.2. The other direction in (iii) follows from Theorem 4.2 since the coproduct of two projectives is projective. □

This result may be applied immediately to **S**-tensor products.

6.4. THEOREM. (i) $A\otimes B$ *is projective in* **S** *if and only if both* A *and* B are projective in **S**.

(ii) (a) *If* $A \otimes B$ *is projective in* **D** *or* **Dist,** *then so are* A *and* B; (b) *let* $A, B \in \mathbf{Dist}^1$. *If* A *and* B *are projective in* **Dist,** *then so is* $A \otimes B$.

(iii) $A \otimes B$ *is projective in* $\mathbf{D}^1$ *or* $\mathbf{Dist}^1$ *if and only if both* A *and* B *are projective.*

Proof. Only (ii) (b) requires further proof. Let $A, B \in \mathbf{Dist}^1$. Then by Theorem 6.1(ii), $A \otimes B$ is the $\mathbf{Dist}^1$-coproduct of A and B, and hence, by Theorem 5.3, $A \otimes B$ is a retract of the **Dist**-coproduct of A and B. Since coproducts and retractions preserve projectivity, if A and B are projective in **Dist**, so is $A \otimes B$. □

This result generalizes Corollary 4.2 and Theorem 4.3 of [11] where R. Balbes's characterization of finite projective distributive lattices is used to prove (ii) (b) under the assumption that A and B are finite. Independently H. Lakser [18] has proved that if A and B are nontrivial distributive lattices, then $A \otimes B$ is projective in **Dist** if and only if A and B are projective in **Dist** and both have a greatest element.

It is interesting to note that Theorem 4.3 tells us that if $A, B \in \mathbf{Dist}^1$, then the $\mathbf{Dist}^1$-free-product of A and B is isomorphic to the principal ideal of their **Dist**-free-product generated by $1_A \wedge 1_B$.

One can now proceed to consider ${}_M\mathbf{Act} = {}_M\mathbf{K}$ where **K** is either **S** or $\mathbf{S}_0$ and M belongs to $\mathbf{D}, \mathbf{D}^1, \mathbf{D}_0$, or $\mathbf{D}_0^1$; the case where M is a distributive lattice being of particular interest. Such a program has been begun in the work of Fofanova [9, 10] and Radnev [25], where the objects of ${}_M\mathbf{S}_0$ are referred to as polygons.

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