FLEXIBLE USE OF SYMBOLIC TOOLS FOR PROBLEM SOLVING, GENERALIZATION, AND EXPLANATION

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We provide evidence that student representations can serve different purposes in the context of classroom problem solving. A strategy used expressly to solve a problem might be represented in one way, and in another way when the problem is generalized or extended, and yet in another way when the solution strategy is explained to peers or a teacher. We discuss the apparent long-term memory implications this has regarding the preferences that students have for their original versus later developed representations, and how these preferences relate to the use of representational flexibility in classroom settings.

Introduction

We discuss the development, in a classroom setting, of flexibility of student mathematical representations. Through analysis of a case study of a middle school girl and several of her peers, as they solve a mathematics problem, we suggest that student presentations develop greater flexibility in response to questioning or a need to better explain logical and mathematical aspects of procedures and representations to peers and teachers. Further, we suggest that these developing representations are often of different types, showing characteristics that reflect the circumstances under which they were developed. As a result, students often revert to earlier or less refined representations in novel settings, even when more sophisticated representations have been developed for other purposes.

We argue that as students solve problems, and engage in conversations with peers and teachers about their solution procedures, they often, quite naturally make comparisons and reflect on their own procedures in response to questions, comments or requests for clarifications and justifications. In such situations, a student has, at the very least, two cognitive problems to deal with: to solve a particular mathematical problem, and to convince (themselves as well as) peers and teachers that they have a viable procedure or solution. As a result of such discussions, students may be stimulated to create representations that may better reveal their thought process especially when they need to clarify or defend their reasoning. These representations then can become objects of scrutiny in their own right, and may be used in other settings and at other times. Critically, these representations can be modified to apply to new or different problem situations. Thus, the representations are, in effect, encapsulations of reflections on prior memories of acting, and allow a considerable amount of prior thought and action to be unpacked as and when required by the student.

Students reflect on their actions, and create new memories, as a result of many things. In this case we note in particular, challenges exerted upon them to justify or defend their ideas by others in their classrooms. As they reflect on their actions, their representations may evolve into increasingly more sophisticated versions. However, old representations are not simply abandoned - rather, new representations are formulated and reformulated as students modify, adjust, and tinker with existing representations in response to a new problem or questioning by others. This evolution of representations, we will argue, is associated with the growth of understanding. From this perspective, learning with understanding is related to the evolution of flexible representational systems that allow students the possibility of successfully addressing novel problems, utilizing not just their prior knowledge, but their prior representational re-organization of that knowledge, obtained through introspective comparison of prior memories of action.

Theoretical Framework

1.1 Overview

Our basic theoretical perspective is that the evolution of individual student representations is greatly influenced and stimulated by classroom interactions, peer to peer, and between student and teacher. Interactions are commonly in the form of a request to explain a line of reasoning or aspects of a particular representation. As students figure out ways to modify their representations and explanations, two main things become apparent. First, their use of memory becomes more declarative. This is not surprising since students are stimulated to explain by means of words, diagrams, gestures, and calculations, how their reasoning works and what their representations mean. However, the fact that this is not surprising, does not mean that it is unimportant, quite the contrary: we believe that a basic goal of teaching is to help students become more declarative in their reasoning, and in their approaches to problem solving, yet it is a common experience at all levels of mathematical instruction that this is often not achieved with any great degree of success (ref. Stigler and Heibert, 1999). We suggest that classroom interactions, particularly those involving student-to-student or teacher-to-student questioning, plays a critical role in stimulating students to produce representations that are based on declarative memories.

One might imagine that the case study used in this paper could be completely described within Karmiloff-Smith's work, in particular, her representational redescription theory (Karmiloff-Smith, 1995). Indeed there are some important similarities including those that are described in the theoretical framework below. We will also argue that the Karmiloff-Smith notion of representational redescription may not fully account for all of the representational evolution that we note below. We will address these as they arise in the results and discussion sections

1.2 Flexibility in mathematical thought

Flexibility is often described as the capacity to exhibit a variety of novel or invented strategies or retrieve and use a large repertoire of strategies for solving problems (e.g. Heirdsfield and Cooper, 2002; Carey, 1991; Klein & Beishuizen, 1994; Vakali, 1994; Beishuizen et al, 1997; see also Shore, Pelletier & Kaizer, 1990). Flexibility, accordingly, may arise from a rational choice between or among several types of strategies depending on the particular problem at hand (Threlfall, 2002).

Verschaffel, Luwel, Torbeyns and Van Dooren (in press) state that the term flexibility is primarily used in the literature to refer to switching smoothly between different strategies. They use the dual term 'flexibility/adaptity' as an overall term: using 'flexibility' for the use of multiple strategies, and 'adaptivity' for making appropriate strategy choices. Hatano (2003, as cited in Verschaffel et al., in press) describes adaptive expertise as "the ability to apply meaningfully learned procedures flexibly and creatively" and opposes it to routine expertise, i.e. "simply being able to complete school mathematics exercises quickly and accurately without understanding". Verschaffel, et al. (in press) emphasize that the opposition between routine and adaptive expertise in mathematics education applies to mathematical strategies and procedures, as well as other aspects of mathematical expertise, such as representational acts.

Flexibility in mathematical thought is often missing or poorly explained in many definitions of mathematical learning and knowledge. Nonetheless, such flexibility is important for many reasons. For example, Star and Rittle-Johnson (in press) note that:

> "Students who develop flexibility in problem solving are more likely to use or adapt existing strategies when faced with unfamiliar transfer problems and have a greater understanding of domain concepts."(p. 2)

Gray and Tall (1994) emphasize that flexible thinking involves an ability to move between interpreting notation as a process to do something (procedural) *and* as an object to think with and about (conceptual), depending upon the context. Thus, flexibility, as used in this manner, provides a pathway for students to move from the procedural to the conceptual.

Our original path to flexibility came from a study of the literature on human memory systems (ref. Tulving & Craik, 2000), much knowledge of which is of relatively recent origin. Our original formulation of flexible mathematical thinking came from a desire to understand why, on the occasions in which students could recall facts and procedures, they often were not able to utilize them in novel settings.

Our understanding of flexible mathematical thinking may best be described as an ability of learners to perform a reorientation in relation to context, place, or person, and a change in their focus of attention as they encounter novel problem settings (Warner *et al*, 2002). In this paper, we argue that this re-orientation is a feature of long-term declarative knowledge, and it is to this point that we direct our attention in the sections that follow.

Before continuing, readers may note a similarity between our notion of flexible mathematical thought and the idea of transfer, attributed to Thorndike & Woodworth (1901), elaborated by Perkins and Salomon (1992), and related to problem solving by Ormrod, 2004. In some ways, our use of flexibility in mathematical thought is very closely related to the idea of transfer. This is particularly so as we consider how students connect facts and procedures and use them in novel situations. The connection between transfer and flexibility is, however, quite complex and worthy of a study in its own right.

1.3 Representational flexibility and declarative knowledge

Procedural memory facilitates procedural learning, which is characterized as acquisition of specific skills and habits, an important part of mathematical problem solving. An important feature of procedural memory is that the knowledge so acquired is often demonstrated in action, and only through actions similar to those used in the learning of the skill or habit (Cohen 1984; Eichenbaum 2002). One may infer, therefore, that procedural memories tend to be less flexible than conceptually based memories in that they only become manifest in restricted settings closely resembling those in which they were formed (Eichenbaum 2002). This may well be because procedural knowledge is more implicit, and consequently, less explicit: it is procedural and not necessarily declarative. Such knowledge is *knowledge-in-action* and the associated memories are those of carrying out certain actions, but not necessarily, memories of reflecting on those actions in order to explain them to someone else.

Flexible mathematical thinking involves transfer of the memories of mathematical actions to another part of the human memory system, one involving relationships between prior memories. This part of the human memory system is referred to as declarative memory, largely because we exhibit it not through actions, *per se*, but through declarations (ref. Squire & Kandel, 2008). In essence, these declarations, which can be speech, drawings, or gestures are representations of thought. Declarative memory facilitates relational learning, which is characterized as acquisition of relationships between recalled facts or episodes. Declarative memory involves the comparison of recollections, and a potent effect of this comparison is the possibility of drawing inferences about recollections. This point cannot be stressed too strongly: declarative memory facilitates thinking and reasoning through the drawing of comparisons. What makes this aspect of declarative memory powerful is the possibility of novel insights into relationships. This feature of declarative memory is what gives rise to representational flexibility – the inferential use of memory in novel situations. We will see, from the examples given below, that students are eminently capable of taking their own experience, turning it into an object of reflection, and representing their thoughts in novel ways that assist them to solve problems in new settings. This type of activity is in sharp contrast to the behavior of students who repeat taught, or learned, actions, such as factoring a quadratic, or carrying out a subtraction, in a fixed, automatic way, with little or no understanding of why or how the procedure works. Of course, a reader might observe that there seem to be many students who seem eminently *incapable* of taking their own experience, and using it as we have described to assist them to solve problems in new settings. Our point, in this paper, is that a setting of the classroom in which students are free and encouraged to ask other students questions, extend the problem in ways that encourage sense making, and justify their conjectures and ideas, stimulates this type of flexibility of thought.

A potent example of declarative memory and declarative knowledge is the representation of Pascal's triangle as a branching road system, constructed by a grade 6 student, David (Warner *et al*, 2002). His representation was dynamic – that is, involving movement in time – even though we see in figure 1, below, only the static drawing.

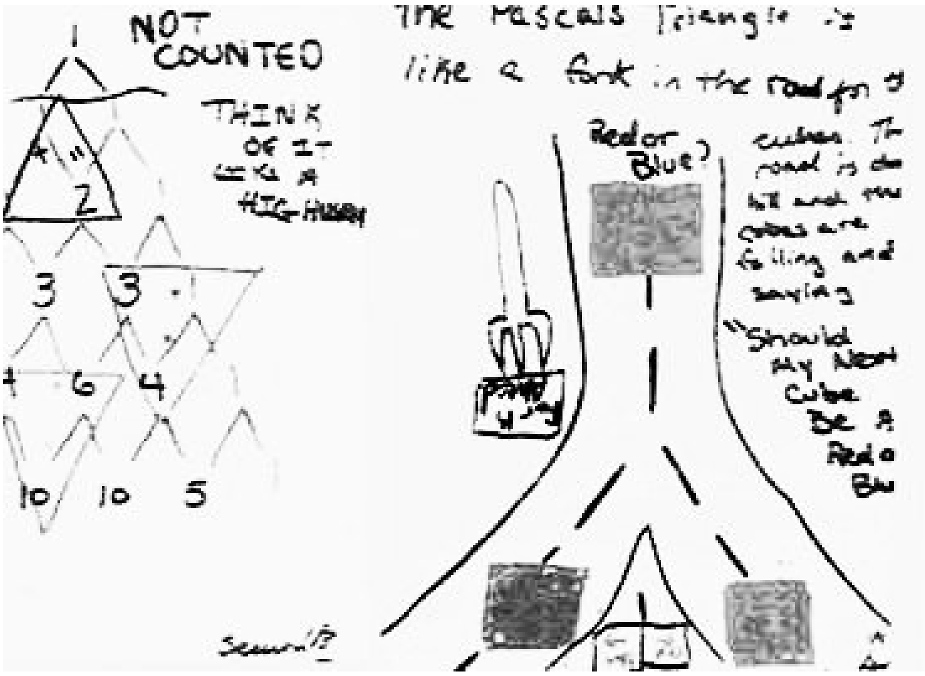

Figure 1. David's dynamic representation of Pascal's triangle

David represented Pascal's triangle as a process in which one came to forks in the road and dropped blocks – a certain color if the fork was to the left, another color if it was to the right. This dynamic representation linked Pascal's triangle, which he heard about from another student, with building and counting block towers. David's representation of Pascal's triangle was dynamic and memorable enough that he could use it in another situation that reminded him of the block tower building. He took his dynamic Pascal's triangle representation and modified it so as to get a contextually dependent model that allowed him to systemically count objects consisting of linked cubes of two different colors.

It is generally believed, though by no means certain, that declarative memories have their basis in episodic memories (Eichenbaum 2002; Tulving 1985). The latter are memories of episodes such as shaking hands with someone, being in a classroom at a particular time on a particular day, or recalling what the teacher wore on a particular day. However, it seems that the dynamic representation David built of Pascal's triangle was not simply a combination of related prior episodic memories. There appeared to be a genuinely creative synthesis in David's constructed representation. Out of an analogy, or metaphor, of Pascal's triangle with a road system, David built a dynamic symbolic representation of Pascal's triangle that made sense to him in terms of his own prior memories. Then, when he had that dynamic symbolic tool he was able to utilize it, and draw new conclusions, in situations that were different, but which stimulated its recall by being sufficiently similar. This, for us, is *representational flexibility* in a nutshell.

1.4 Representational redescription

On the face of it, the changes in representations we describe in this paper might seem to be well described by appropriate use of Karmiloff-Smith's representational redescription (RR) theory (Karmiloff-Smith, 1994, 1995). Representational redescription is a hypothesized process in which cognitive information becomes progressively and explicitly available to a person:

> : " ….representational redescription, turns IMPLICIT information embedded in special-purpose procedures into EXPLICIT knowledge but which is not yet available to conscious verbal report." (Karmiloff-Smith, 1994; author emphasis).

Using the ideas and findings of the neuroscience of memory systems (Tulving & Craik, 2000), we might reformulate this as: representational redescription is a process in which relatively implicit memories become progressively reformulated as explicit memories. Declarative memory – the memory exhibited in talking about, drawing, or otherwise representing a prior event or recalled thing – is, by its nature, explicit. It is recall at the higher levels of the theory of representational redescription. RR is, however, a specific theory, and not a generic term for representations that change in description: it is a theory of development useful for providing insights into adult representational structure through an understanding of how children's representations change over time (Karmiloff-Smith, 1995). The representations we deal with here are at the higher levels of Karmiloff-Smith's theory: they are for the most part explicit, presented by students in full consciousness, and capable of being discussed and modified consciously: this places them, at a minimum, at Karmiloff-Smith's E3 level of representational redescription (Karmiloff-Smith, 1995). The representational modifications we report on are generally brought about by the stimulus to explain thinking to other students or teachers, and there is no strong sense in which we are dealing with a developmental issue as we might see with younger children. The process we see and describe has some similarities to representational redescription as described by Karmiloff-Smith. However, Karmiloff-Smith's use of that term indicates that we are describing a somewhat different phenomenon: the students in our study are driven, at least so we believe based upon our analysis, by exogenous demands to reformulate their representations. As they do so, their representations usually become more explicit and more likely to be reflected on consciously and verbally. The stimulus from other students is critical. In our observations, we are seeing something somewhat different, or certainly in addition to the developmental issue addressed by representational redescription. Certainly the underlying phenomena of representational change seems quite similar especially when considering that Karmiloff-Smith's work also addresses the issue of conscious versus. unconscious access to strategies and representations. However, our emphasis, involves changes in representation, and a developing awareness of the nature of those representations through participation in a social network, in which questioning from other students provides a critical stimulus to reflect on exiting representations and modify them in response to that questioning. This is somewhat different to the main emphasis of representational redescription:

> "The RR model is fundamentally a hypothesis about the specifically human capacity to enrich itself from within by exploiting knowledge it has already stored, not by just exploiting the environment. (Karmiloff-Smith, 1995, p.192)

Certainly internal changes take place when students begin to alter their representations. However, the process we observe is not wholly internally driven: it is, rather, largely driven by social need and social demands of other students, and teachers, and in this sense the observations and analysis we provide here add to an understanding of representational flexibility at a level beyond the highest levels of the representational redescription theory. Our data provides a clue as to what stimulates students to greater representational flexibility when they are already at a high level according to representational redescription theory.

2. Overall plan and evaluation of data

Our basic plan was to document and analyze the evolution of student representations, and to analyze where and when we saw evidence of representational flexibility. In particular we made note of places in space and time where students:

- Modified existing representations.
- Asked questions that seemed to contribute to a modification or change in representation
- Responded to other students' or the teacher's requests for explanations.
- Posed or shared extensions to the problem.

Where one or more of these episodes occurred, we analyzed whether the student's actions or responses were largely procedural, and the representations were more or less a record of working, or appeared to be more flexible in that they were created to explain and apply to a more general problem situation.

Additionally, we analyzed student responses for evidence of knowledge why something was true, and distinct from knowledge that it was true.

Instances of explanation why or justification were taken as evidence that students were adopting a more flexible approach to the problem solution, by seeing a rationale for a more general answer to the problem.

3. Methods

The study took place over a six-month period. It was the result of a professional development project[1] whose goal was to help teachers use instructional methods that helped students learn mathematics with a deeper level of understanding (see Schorr, Warner, Gearhart and Samuels, 2007, for a more complete description of the project). In short, district teachers participated in workshops and attended courses with university faculty and research specialists, and they received follow-up visits in their classrooms by the university partners. These visits were designed to help teachers plan, implement, and reflect on classroom activities in their own classrooms. In the case of this study, such visits occurred approximately once a week as part of the teacher's regularly scheduled 8th grade math class. Each class was approximately 50 minutes in duration.

During the course of the time period in which this study took place, many different tasks were introduced to the students. In most instances, but particularly when the University partners visited the classroom, the students worked in groups of 3-5 (arranged by the teacher). Generally speaking, the university researcher (UR) and the classroom teacher jointly taught the class. Consistent with the goals of the project, the students were encouraged to talk about ideas, record the ideas; make conjectures, question each other, discuss disagreements, justify and defend solutions; generalize and extend their ideas, and revisit ideas over time.

The setting for this study was an eighth grade class in an urban school district. The class consisted of 32 students, with equal numbers of boys and girls. The class was considered to be average in terms of academic achievement. In this paper, we focus on the representational flexibility of 3 students. The first two, Aiesha and Bianca, worked together as a member of a group, along with Brittney and Edgar. The third student, Dominique, was one of the students who questioned them during their presentation to the class, along with other students (eg. Shaniqua), about their representations.

Problem Tasks: The students initially explored a task entitled the "Handshake Problem" (adapted from a similar task that appeared in NCTM 2000), along with extensions of the task, and two additional sessions took place, six months later, in which the students explored a problem similar in structure, ("Yakia's Slumber Party").

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The Handshake problem is as follows: *John is having a Halloween party. Every person shakes hands with each person at the party once. Twenty-eight handshakes take place. How many people are at the party? Convince us*.

This task was chosen because it provided an opportunity for students to engage in mathematical discovery and problem solving. The problem provided opportunities for students to find solutions with differing degrees of sophistication and knowledge. Further, it provided an opportunity for teachers to consider how students interact with each other as they solve problems, and what the teacher's role could be in this process.

The students also explored two extensions of the problem:
Part 2: There are 11 people at the Thanksgiving party. Each person shakes hands with every person at the party once. How many handshakes take place? Convince us.
Part 3: There are 101 people at the Holiday party. Each person shakes hands with every person at the party once. How many handshakes take place? Convince us.

These extensions of the problem, which occurred over time (at least one week apart), provided an additional opportunity for students to build a solution that could be generalized to a larger class of problems involving similar structures. In this case, an example of a generalized solution could be $\binom{n}{2} = \frac{n(n-1)}{2}$.

Six months after the initial problem was presented, the class was given the following problem (referred to as the Yakia's Slumber Party Problem),

On the night of Yakia's slumber party there was a terrible storm. The main road washed out, so Yakia and her 14 girlfriends (15 girls altogether) decided to have a "Phone party" instead. The idea was for each friend to talk to every friend at the party on the telephone. With all of the 15 friends taking part in the phone party, what was the fewest number of calls that could be made so that every person talks to every person in the phone party?

This report involves 10 classroom sessions over the course of 6 months (8 focusing on the Handshake problem and several extensions, posed by the teacher and students, and 2 focusing on the Yakia slumber party problem and extensions, posed by the students). During each class session, two cameras captured different views of the group work, class presentations and associated audience interaction. In addition, careful field notes were taken after each session. Student artifacts were also collected as part of the data set.

Descriptive summaries were written for all ten sessions. Instances that involved modifying existing representations, asking questions that seemed to contribute to a modification or change in representation, responding to other students' or the teacher's requests for explanations, posing or sharing extensions to the problem, were selected for deeper analysis. These were then transcribed as needed. Such instances were identified as an "episode" in the present study.

4. Results

We begin our analysis by documenting how Aiesha, a 13 year old girl solved the Handshake problem, as posed above. We will make the case that her representations evolved over the course of the sessions, as will be seen below, as a consequence of many things, most notably her interactions with peers. Throughout the analysis, we include instances of her peer's solutions and representations as well.

4.1 Aiesha's initial representation

Aiesha's first action was to actually shake hands with her group members. From there, she and her peers spoke about people shaking hands without being tied to the action of actually shaking hands. Aiesha and her peers then discussed how they would count the handshakes, addressing the question: *when two people shake hands, is that one handshake or two?* We believe that this provides at least some evidence that their thinking was based on episodic memories of actually shaking hands. This simple point is not a trivial one. There is debate in the memory research community whether declarative representations are necessarily based on episodic memories (Eichenbaum, 2002). As obvious as this might seem at first encounter, to a reflective mind it is not at all apparent that episodic memories form a basis for later declarative memories. Here, however, we see Aiesha and her friends acting out a scene in which they literally carry out handshakes, and then use that as a basis for discussion.

Aiesha's first representation (see figure 2) involved a "picture" which was, it would seem, her way to represent, via short hand, a drawing of people shaking hands without actually drawing people. In this case, the circles were stand-ins for people, and the lines joining them a depiction of handshakes. Again, this representation seems to be based on recalled episodes of people shaking hands. There is nothing in this representation of Aiesha's to suggest that it was anything other than an inscription, a record of her work on the problem, much as a student might carry out a calculation on paper.

After drawing this representation, Aiesha tried several different possibilities for the number of people present at the party, along with an associated multiplication number sentence for each. This approach appeared to be based upon her making an 'educated' guess. Her representation in figure 2 was based upon her assumption that there might be 14 people at the party. She then explained, in response to a question by the UR, why she multiplied 14 by 13, for example, how each of the 14 people would shake hands with the other 13 people, drawing 13 loops from the first circle, which represented a single person at the party, to each of the remaining 13 circles, as in fig. 2. When she multiplied 14 by 13, the result was 182 handshakes, which was clearly too many handshakes for the problem scenario. She also tried the same approach imagining a 7 person party, a 9 person party, and an 8 person party.

During this episode, Aiesha went from the action of actually shaking hands to imagining people shaking hands. Next she constructed a pictorial representation of this, using circles to represent people and loops to represent handshakes. Finally, she wrote an associated numerical inscription, which in this case was 8x7. In effect, Aiesha had created an image – a representation—of a possible solution process.

Aiesha used this pictorial representation to construct the idea of multiplying the total number of people by one less than the total number of people to arrive at the number of handshakes. She provided a reason for her number sentence by explaining that each of the people or “circles” must shake hands with the remaining number of people. At this point, we infer that she had an image of the rest of the handshakes taking place and no longer needed to draw the rest of the loops to show it. Every time she multiplied, however, she arrived at double the number of actual handshakes in the solution, and consequently, she concluded that there was no answer to the problem.

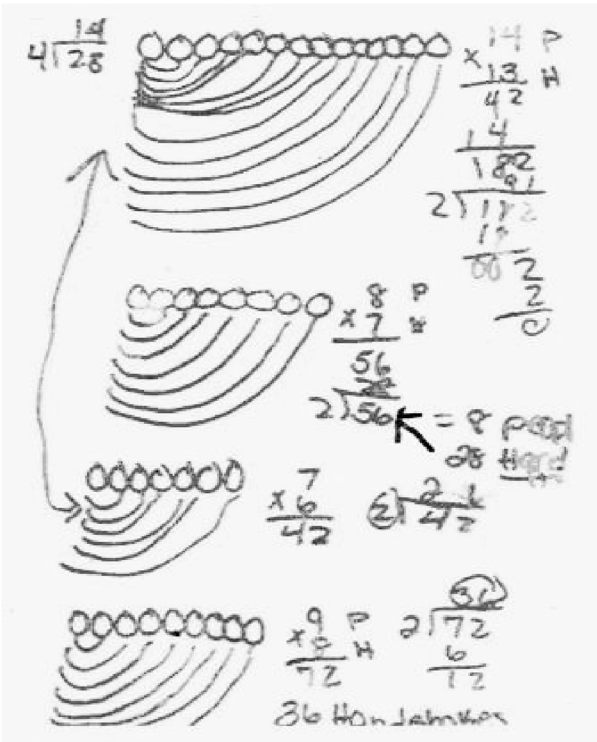

Figure 2. Aiesha’s initial strategy for finding the number of people for 28 handshakes.

At this point, another student in her group, Edgar, arrived at an answer of 8 people at the party, using his own method. When Edgar shared his solution with her, Aiesha noticed that she could get the same answer if she divided her answer (56) for an 8 person party by two. At this point, she proceeded to divide all of her products by 2 and concluded that her solution for the 8 person party gave her 28 handshakes. Aiesha noted that she did not know why it worked, only that it worked.

What we see in this episode is apparently a record of calculation, based on a pictorial schema whose basis lay in a recalled episode of actually shaking hands. We see a schematic representation of handshakes, and a systematic attempt at counting them, without a realization that she was counting each handshake twice. The modification Aiesha made to this representation was prompted, we believe, by Edgar's answer and was largely procedural – dividing the supposed number of handshakes by 2 (along with her other products) to get an answer that agreed with Edgar's for the number of people at the party. This is just one piece of evidence regarding the role of peer-to-peer interaction in Aiesha's representational evolution.

4.2 An extension to the problem and a challenge from a peer: Aiesha is confronted with a need to extend her representation

Two weeks later, the students were presented with an extension of the original task involving 11 people. When Aiesha was introduced to this extension, she was able to spontaneously recall and use her original representation involving circles and loops (below). She drew eleven circles, connecting the first circle to the other ten circles with loops (see figure 3). Then she multiplied eleven by ten.

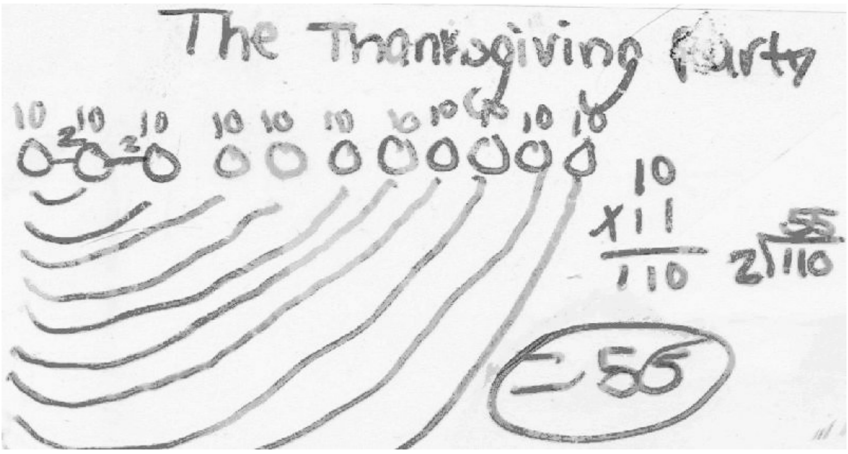

Figure 3. Aiesha's initial strategy for finding the number of handshakes when 11 people are at the party.

At first Aiesha did not recall the reconciliation with Edgar's answer for an 8 person party in which she divided her previous answer by two, however as is indicated in figure 3, she ultimately realized that she needed to divide by two. We describe that process below.

3.3. Aiesha's modifications of her representations in response to the questions of her peers and researcher

After Aiesha drew the 11 circles, with the 10 loops (see above), along with the multiplication of 11 by 10, she recalled that she needed to divide the product of the two numbers, but was not sure what to divide it by. At this point, the UR questioned her about the method she used for solving the original task—thereby eliciting in Aiesha memories of the previous problem solving session in which she solved a related problem. In response, Aiesha re-explained her number sentence for an 8 person party, and remembered that she should divide the product by 2. That dialogue appears below:

> UR: What were you dividing one hundred and ten by here? [*pointing to a division number sentence that Aiesha wrote, with 110 under a bracket.*] I remember you did that last time, too (see figure 3). [*Aiesha tried to divide 110 by eleven, using her calculator, then by ten*.]
> Aiesha: That's not it.
> UR: What did you do when you solved for an eight person party?
> Aiesha: Eight times seven, then it gave me fifty-six, then I divided... Oh, two into fifty-six. So, it's supposed to be one hundred and ten divided by two [*She entered this into her calculator*]. It's fifty-five. [*She wrote 110 divided by 2 on figure 3*].

Through questioning by the UR, Aiesha gradually recalled her procedure for the 8 person party, relating it to the 11 person party. Again, Aiesha seemed to be operating procedurally, with a method that worked in a previous situation—and a memory stimulated by the UR.

The UR then asked Aiesha to explain *why* the method worked:

> UR: I'm curious about this [*pointing to Aiesha's number sentence at the top of figure 3*], right over here. Why is this working? It worked again. Why? Lets think, originally, why were you multiplying eleven by ten?
> Aiesha: Because, all right, there's eleven people, and one person shakes ten people's hands. So, instead of drawing all of those squiggly lines, I multiplied ten times eleven. So this part should be ten, ten, ten, ten, ten, like that [*pointing to the circles on the top of figure 3*].
> UR: So, you are saying that each of these people [*pointing to the circles*] would shake ten hands? Why don't you write that? [*Aiesha writes 10 above each circle – see figure 3*].

However, despite the UR's rather persistent prompting, Aiesha was not able to provide a convincing reason as to why she divided by two. It was not until several *other* students questioned her that Aiesha came up with a reason, as will be described below.

Brittney, Aiesha's group partner, questioned how Aiesha's picture – circles with loops, could be useful in finding the final solution. She didn't understand how such a depiction could possibly account for all of the handshakes, especially since all of the loops extended from only the first circle.

Brittney: When you did the demonstration you did up there [*referring to Aiesha presenting figure 2 to the class]* I didn't get it.
Aiesha: What do you mean?
Brittney: All of these lines [*pointing to the loops on figure 2*]. What about these people [*pointing to all of the circles on the right*]?
Aiesha: I'm going to show you all. I am multiplying [*writing 11 x 10 = 110*].
Brittney: This person [*pointing to the first circle on the left*] is shaking hands with all of these people [*pointing to the circles on the right*], and this is all of his shakes. Right, and how many handshakes is right here [*pointing to the first circle to the left*]?
Aiesha: Ten.
Brittney: And then this one [*pointing to the second circle*] is going to be nine, right?
Aiesha: And then eight, seven, six, five, four, three, two, one.

Seemingly, to address Brittney's concerns, Aiesha looked carefully at Brittney's representation, and actually used it to construct a new type of representation, involving a chart with letters that could represent the people at the party (figure 4). Aiesha, in effect, took advantage of the representation that Brittney had originally built (see figure 5) and modified it to address Brittney's concerns. Using this newly developed representation, Aiesha was able to explain to Brittney that her original reasoning (with circles and loops) was correct. It is not clear, however, if Brittney was fully convinced.

Figure 4. Aiesha's chart for finding the number of handshakes for an 11 person party.

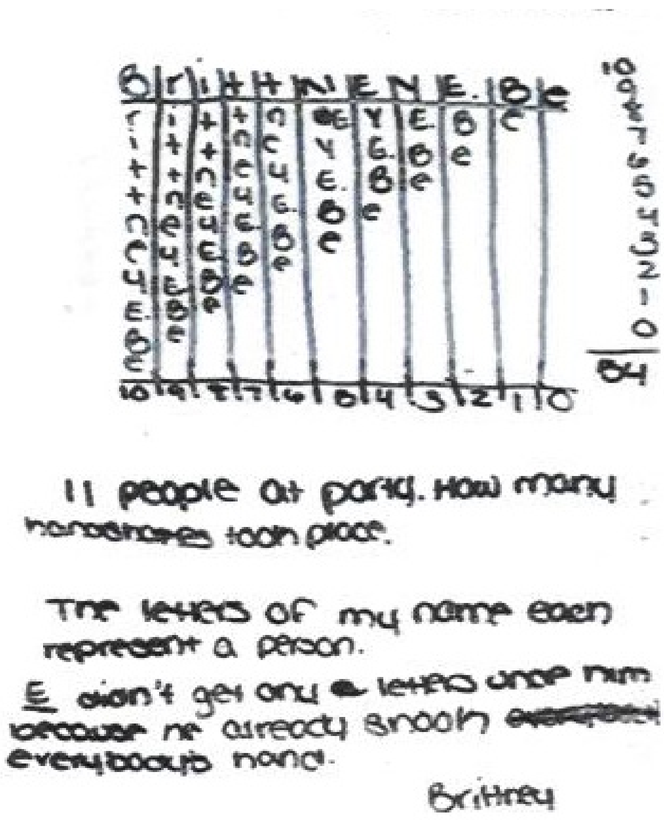

Figure 5. Brittney's chart for finding the number of handshakes for an 11 person party.

Note the critical importance of Brittany's challenge to Aiesha. Aiesha's change in representation, we believe, indicates a certain level of flexible thought in that she (re)structured her knowledge to generate a representation that was useful to her, while simultaneously addressing the understanding of Brittany. This is a particularly important point. Aiesha's recourse to her first solution method *may* have been based on procedural memory – her memory of what she had previously done in solving this type of problem, and recalled as a process of solution in action – her diagram may have been simply her inscription of the solution process. Her solution was carried out in action, and the context is not much different from that in which she established her original method of solution. We are not claiming that her knowledge was procedural and not declarative, only that her representation may have been simply her method in action of solving the problems, and without further evidence we have no way of knowing. In response to Brittney's questioning however, she made a representational change in indicating how she knew what she knew, not simply that she was capable of solving the problem: she moved from demonstrating *how* to do something, and *that* she could solve the problem by producing a representation, to *why* her method worked by producing a new representation. We infer that Aiesha reoriented her thinking to accommodate Brittney's question and changed her focus of attention from showing that she could do the problem to demonstrating why her method was valid. By our definition, therefore, Aiesha demonstrated the beginnings of flexible mathematical thinking.

The role of Brittney's questioning in at least to some extent, stimulating Aiesha to reflect on her prior actions and represent them declaratively, not simply to repeat them, should not be underestimated. It would seem that this type of questioning on the part of one student to another plays an important role in prompting Aiesha to explain that her reasoning was in fact correct. In the process of the explanation, Aiesha moved convincingly from demonstrating possible procedural knowledge to demonstrating clear declarative knowledge, and changed her focus of attention from *showing that* to *explaining why*.

By explaining that each letter at the top of each column "shakes hands" with every letter in that column, we can see that Aiesha could represent the handshakes, and was no longer tied to the action of showing each handshake. This indicates flexible mathematical thought in that she was able to construct a new representation, which ultimately was more useful to her when justifying her ideas, as will be shown below. This constitutes declarative knowledge, and the evidence for such knowledge comes from the student's representations. These representations, in turn, become flexible cognitive tools that can be recalled and used in situations that differ from those in which the original problem was posed.

As will be seen below, Aiesha and her group members were soon able to also use this chart to show why division by two would work. Student to student questioning played a critical role in this process as well:

> Bianca: This person [*pointing to the second circle*] won't shake ten people's hands. But it says every person at the party shakes hands once.

As Bianca questioned Aiesha's idea (ten handshakes for each of the 11 people). Aiesha filled in her chart as in figure 6. Ethan then noted: Everyone's not going to shake everyone's hands two times.

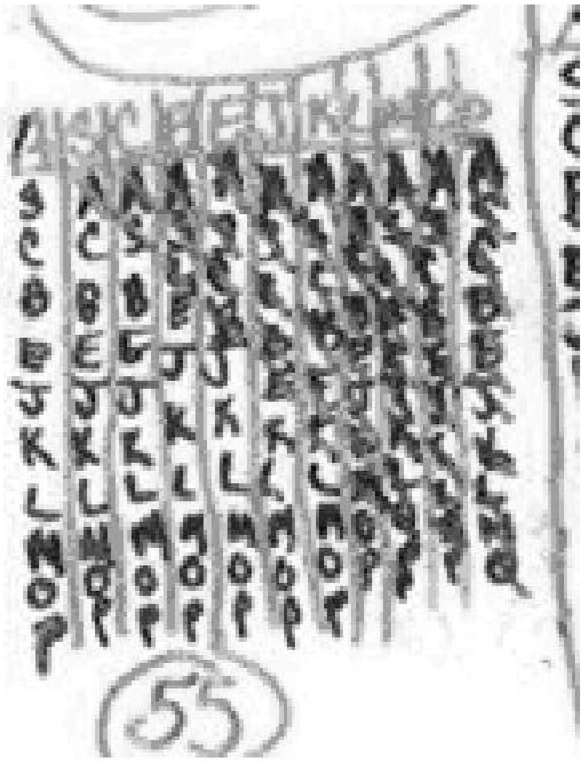

Figure 6. Aiesha's modification of an existing representation.

The other students realized that if they continued the chart to show each of the 11 people shaking hands with all other people, each person would in fact be shaking hands twice. Therefore, they crossed off half the represented handshakes. This prompted Aiesha to draw the horizontal line on the bottom of figure 4, which resulted in the construction of the chart (figure 6) as a way to explain why they needed to divide their answers by 2. The students' questions contributed to the connections that Aiesha made between her picture representation (figure 3) and letter chart representation (figure 4), which enabled her to build on these ideas, to create her new chart (figure 6). These questions acted as a catalyst, which enabled her to build on her chart representation, which we believe, demonstrated some level of representational flexibility. In this case, Aiesha began to link her number sentence, picture representation and chart representation to justify her division by two.

Aiesha's representation on figure 6 is far from spontaneous. It resulted largely from interactions with the other students and the UR, and discussion about how handshakes were counted. Aiesha constructed the representation in figure 6 as a declarative statement that she knew how to calculate the number of handshakes, and why division by 2 was necessary. Her representation seems, in other words, to be a declaration of *why* something is true, not just *that* it is. In this sense, her representation seems to serve a different purpose than her first representation (figure 2).

3.4. An extension to the problem: A need for more symbolic representations

In the following episode, which took place a few weeks into the investigation (4th session of working on this problem), Aiesha, in response to a series of questions posed by the UR (in the beginning of this session when her group prepared for their presentation) about whether or not the method would work in instances where there were there were 200 people, 500 people, etc. ...and how they could be sure it would always work, repeated the question to the class focusing on a 500 people party. This extension to the problem, seemingly provided the motivation for Aiesha to move to a new representation, as well as a more sophisticated symbolic notation and generalizing.

> Aiesha: I wanted to know that, what if there was around say five hundred people at the party? Would you be able to do that same table with…?
> Brittney: It would take me like an hour to do it but I would do it.
> Aiesha: So, would you rather save time, than just keep doing that with five hundred people?

By (re)posing and beginning to solve this hypothetical situation based on the existing problem, Aiesha seemingly motivated the class to explore the task more deeply.

Aiesha began to solve the problem by explaining that for a 500 person party, she would have 500 letters across and 499 letters in each column. Then, she said she would cross out half of the letters on the chart. She never drew the entire table: rather she extended her previous method and multiplied 500 by 499 and divided the product by 2 (See fig. 7).

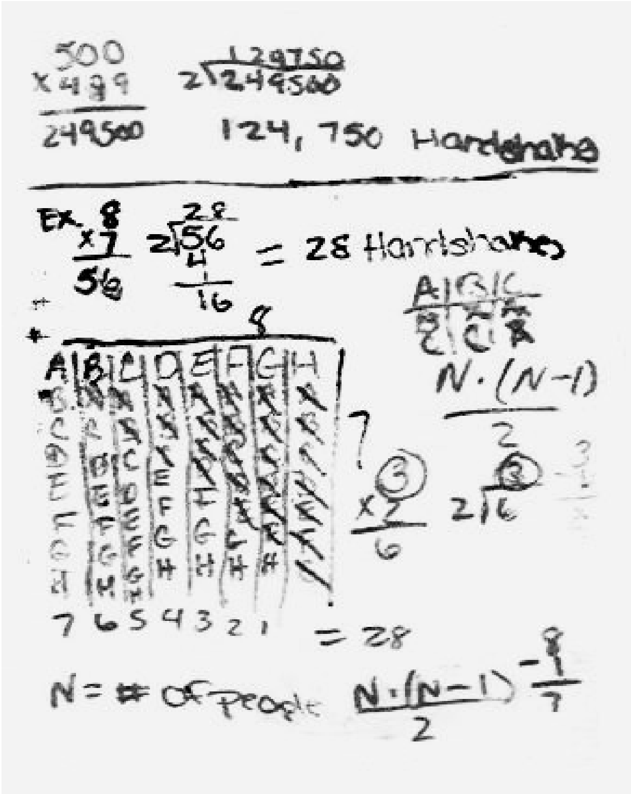

Figure 7. Linking representations.

This is a critical point in Aiesha's thinking and in her flexible use of representations. We can now see clearly that Aiesha's more abstract representation of the problem solution allowed her to focus on a critical feature of her actions – producing 500 columns, 499 rows, and crossing out half the entries. But she did not carry out this sequence of actions – she only imagined the possibility of doing so—a rather critical point to note. She did not need to actually *do* the calculation, only *imagine* how it could be done. Her representation, which demonstrated her declarative knowledge of the problem's solution, became a tool for thinking about and thinking through. A critical point to notice is that Aiesha's representation, which is by now thoroughly declarative and not simply a record of her work as she solved a problem, has become an object in itself. This object can be observed by her, and others, to have certain properties. The representation, therefore, becomes a cognitive tool to further assist a student's thinking. The very existence of this cognitive tool - this flexible representation - resulted from a dynamic of questioning and answering that took place in the classroom. Without such a prior dynamic it is not at all clear that Aiesha would have spontaneously represented her solution process in such a flexible and potent form.

After finding her solution, Aiesha and other members of her group attempted to generate a generalized symbolic representation that could work with any number of people. For this, they reverted to the original problem involving 28 handshakes. Aiesha drew a chart for an eight person party (see figure 7) and constructed a number sentence along with a formula using both words and standard algebraic notation. Originally, her symbolic notation entailed several errors involving the placement of parentheses, but these were quickly corrected by the students themselves. Ultimately, she was able to come up with the formula $n(n-1)/2$. The process by which this occurred will be described below.

3.5. Linking Representations

We note that several questions, mostly from other students, prompted Aiesha to link representations. A student questioned Aiesha and in turn, Aiesha showed that her own idea was valid, using multiple representations to solve the problem.

> Aiesha: n equals the number of people at the party. What I did was n times, well, we're going to do n times n minus one, n minus one in parentheses [*tracing the parentheses with her marker*]. First what we have to do, eight, there's eight people, we have to take minus one, so there's seven *[writing 8-1 = 7*]. So, n times n minus one, then you divide that by two. You would multiply eight by seven, then you would divide that whole answer by two.
>
> Shaniqua: [*Shaniqua raised her hand during Aiesha's presentation and Aiesha called on her to speak*.] I disagree with something. She said that there was five hundred people at the party and each of those people shake hands with four hundred and ninety nine people's hands [*initially directing the comment to the UR*]. That's not true because if you do that, then you're saying each person shook… [*then directing the comment to Aiesha*].

As Shaniqua questioned Aiesha's idea, she decided to use a common heuristic—setting up a simpler problem:

> Shaniqua: … OK, lets say there is three people at the party…
> Aiesha: Yeah.

Shaniqua provided a reason for thinking that this idea is invalid, by describing the strategy that produces double handshakes:

> Shaniqua: And you are saying that every one of these three people are shaking the same three people's hands. They are shaking the same people's hands.
> Aiesha: Do you want to see how that works with three people?
> Shaniqua: Yeah.
> Aiesha: What you do is, I'm going to take this three…
> Shaniqua: All right.

Aiesha began to justify her idea with symbols:

> Aiesha: And I'm going to do this formula. So you have n times n minus two [*writing n times (n-1) as she presented this to the class-see figure 7*] over two. So, if you go to three people…

Her choice of words was questioned and she corrected herself:

> Student: You said minus one.
> Aiesha: I mean minus one, divided by two.

Now, she moved to an explanation using numbers:

Aiesha: So, if you go to three people, times two, it gives you six (*writing 3x2=6*). Two divided by six is three (*writing 6/2=3*). How many handshakes took place? It's three [*see figure 7*].

A student questioned Aiesha and prompted her into linking representations – symbols, numbers, and words:

Student: What's the n?
Aiesha: All right, the n equals the number of people, n, three people, right. What you have to do first is 3 minus one, it gives you two. Then you have to do three times two, and it gives you six. You divide two into six and it gives you three. That's how many handshakes.
Student: Oh.
UR: What would the chart look like?

Aiesha drew the chart for a three person party with three letters going across and two letters in each column on figure 7. She initially made a mistake by adding an extra letter in one column. She realized something was wrong during her explanation and reorganized her idea. She ended with the three by two chart on the right side of figure 7.

UR: So you have a three by two?
Aiesha: Yeah, I crossed off three of them, an A, an A and a B. It still gives you a remainder of three people, three handshakes.

Another student questioned Aiesha, which prompted her to link the action of shaking hands to the chart representation:

Dominique: Why do you use the number two to divide?
Aiesha: All right, I use two because look, when two people (*shaking Bianca's hand*), it gives you two handshakes (*pointing to her and Bianca*), but normally….
Student: One
Student: Two handshakes?
Edgar: One for each person.
Aiesha: And normally it would be…
UR: Lets say you were A and she was B, where would it be in the diagram?
Bianca: B would shake A's hand and A would shake B's hand.
UR: So, how many handshakes are there?
Aiesha: One, two.
UR: And in this problem do you want to count both of those handshakes?
Aiesha: Well, not really, we want to count them, then we want to divide…
UR: Where would they be in the diagram?

She linked the action of Bianca and herself shaking hands to the chart:

Aiesha: So, it's me and Bianca and it's A and S [*drawing the chart on the bottom of figure 8*]. A shakes that person's hand [*writing an S*

under the A on figure 8], so then it's like that [writing an A under the S]. But you don't need that, so it's like that [crossing out the A].
Student: I don't get it.

Figure 8. Linking more representations.

Aiesha used her picture for an eleven person party and traced lines connecting two circles (see figure 3), writing a 2 above each line, to show that each line counts as two handshakes on her picture representation (circles and loops). In doing this, she linked the picture representation to the A & S chart and the action of shaking hands.

Aiesha: Because, all right, these two people here are shaking hands, right [*drawing a line connecting the first circle to the second circle on her picture representation, writing a 2 above it – see figure 8*] and that equals two handshakes. And those two shake hands [*drawing a line connecting the second circle to the third circle, writing a two above it*], that's two handshakes. So, you're going to keep on writing two, so you are going to divide by 2 because you don't need that extra handshake [*tracing the division symbol at the top of figure 8*]. You only shake hands one time.
U/R: On the chart over here (*11 by 10 chart on figure 8*), why did you cross off so many?
Aiesha: They already shook hands with that person already.
John: Is two people considered one handshake or two handshakes?
Aiesha: I say it's considered two handshakes, but after you divide by two it is one handshake.

We suggest that Aiesha's representations of the handshake problem have now become very flexible. She was able to explain the division by 2 with her chart representation (crossing off half of the letters in her chart) and with her picture representation (drawing lines between the circles, writing a 2 above each line and explaining that each line represents two handshakes, which represents one handshake repeated). She moved from a pictorial representation to calculation to algebraic reasoning with ease. She also demonstrated that she can deal with party situations with arbitrary numbers of people or handshakes, and is able to explain with fluency why it is that the product n(n-1) is divided by 2. Her ability to recall these representations and to use them flexibly is connected to her operating in the formalizing layer of the Pirie-Kieren model.

3.6. Solving a Structurally Similar Task Six Months Later

Six months later, Aiesha's class was asked to investigate a task that was structurally similar to the handshake problem: *Yakia's slumber party* (see Methods section above).

Within a few minutes, Aiesha and her group members (which included Bianca and Shaniqua) utilized most of the representations (picture, chart, numbers, words and symbols) they constructed six months earlier, and also used another way to represent their generalization symbolically. Interestingly, many students in the class also used the formula Aiesha presented six months earlier for this new task - see figures 9 through 14, below. We believe that this provides evidence that these students had long-term memory of Aiesha's formula.

3.6.1. Aiesha's Approach

The first representation that Aiesha retrieved six months later was her picture representation using circles and loops (see figure 9). It appears that this picture representation helped her retrieve the symbolic notation, which she wrote immediately after drawing the circles and loops (see the top right hand corner of figure 9).

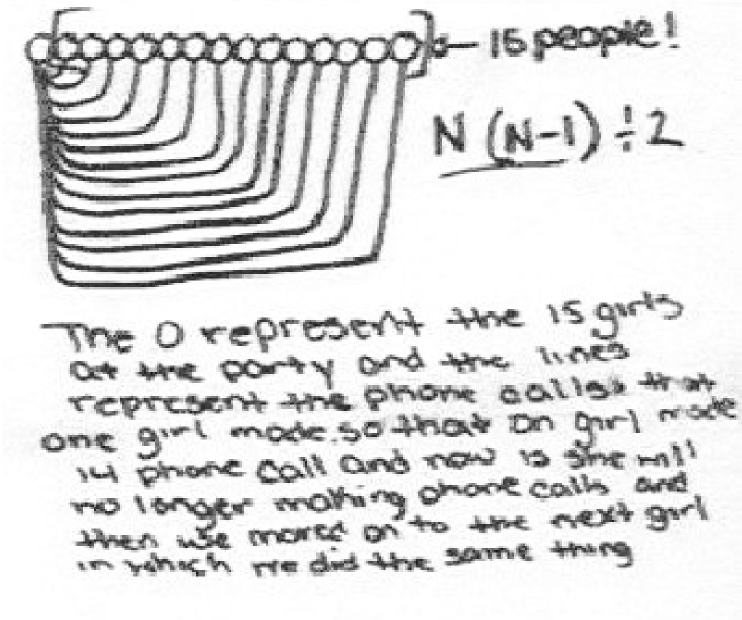

Figure 9. Aiesha reconstructed her original picture representation, then the symbolic notation

What is different in this representation to her initial representation (figure 2) is that now she has a formula with the picture, and a verbal explanation of her reasoning. This representation is not simply a record of working: it has become a declarative represented tool to explain her reasoning, as well as to display a general formula. As such it is a highly distilled and flexible representation.

Next, Aiesha wrote the number sentence and elaborated on it by explaining how to solve it (figure 10).

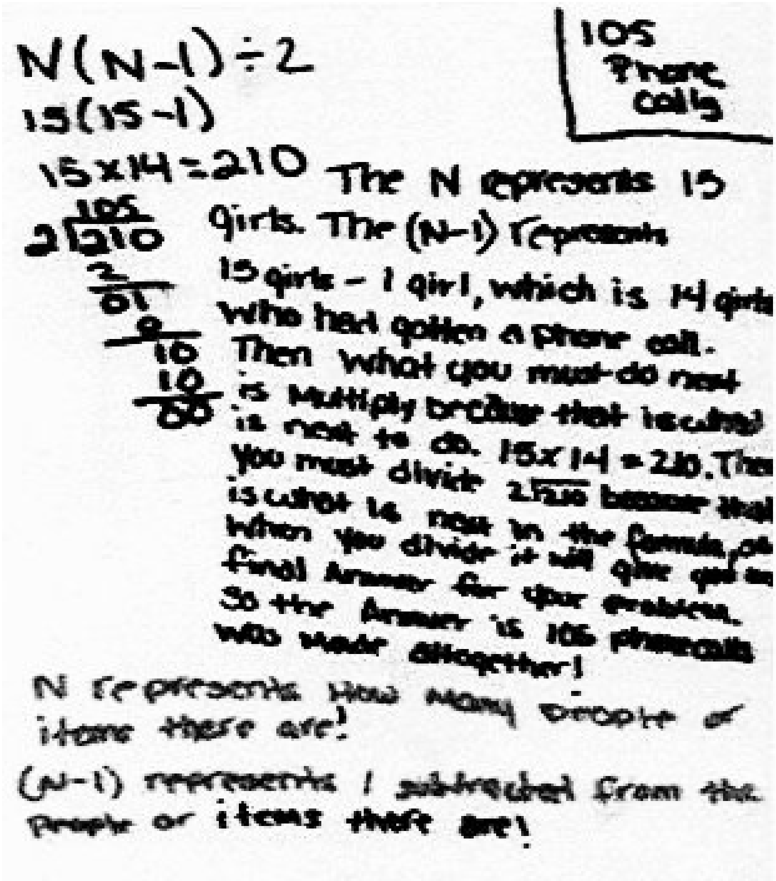

Figure 10. Linking the symbolic notation to numbers and words

Aiesha then set up a hypothetical situation (*what if there were 5 girls?*), and used her symbolic notation to express the generalization. Then she solved with her number sentence (figure 11). Notice that she drew a chart representation (similar to the one she constructed to explain her number sentence to her peers months earlier) last, most likely to explain the number sentence and symbolic notation to the other students when she later presented this to the class. Recall that she also set up a hypothetical situation months earlier, which provided her with a reason to imagine the chart at that time. Aiesha also stated that this is similar to the handshake problem.

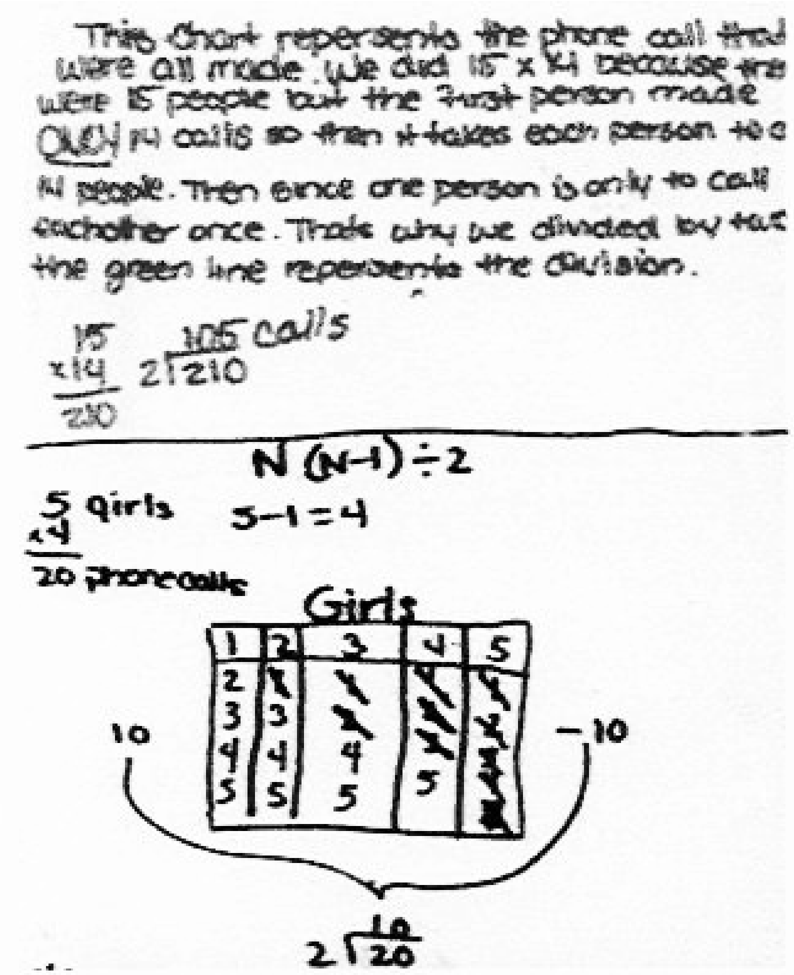

Figure 11. Linking the chart to words and symbols & setting up a hypothetical situation

Aiesha's representations have now reached a high level of flexibility. They combine pictures, formulas, and elaborated sentences to describe what the pictures and formulas represent.

3.6.2. Bianca's retrieval six months later

After reading this new problem six months later, Bianca, a member of Aiesha's group, immediately retrieved the chart representation to solve the problem. This was the representation she was asked to explain in relation to the division by 2 as her group presented their general solution (figure 12) six months earlier. Recall that she was the one who stood next to Aiesha as she presented the information captured in figure 7 above; she was the one who Aiesha shook hands with during the presentation and she was the one who helped Aiesha justify her symbolic notation with the chart and the action of shaking hands to address students' (in the audience) questions.

Figure12. Bianca retrieving the chart representation first, then symbolic notation

Bianca also solved Aiesha’s hypothetical situation (what if there were 5 girls?) using the chart representation (see fig. 13), without the diagonal, which was the first thing Bianca retrieved in figure 12. This chart representation led to her retrieval of the number sentence that helped her solve the problem. It appears that for Bianca, the rectangular array without the diagonal is the representation that makes sense to her. It seems to be the representation she uses to solve the problem as opposed to Aiesha, who uses this representation as a tool for explaining to her peers. This makes sense because Bianca was actively involved in building on Aiesha’s chart when she and her group members realized that the people at the party would shake hands twice.

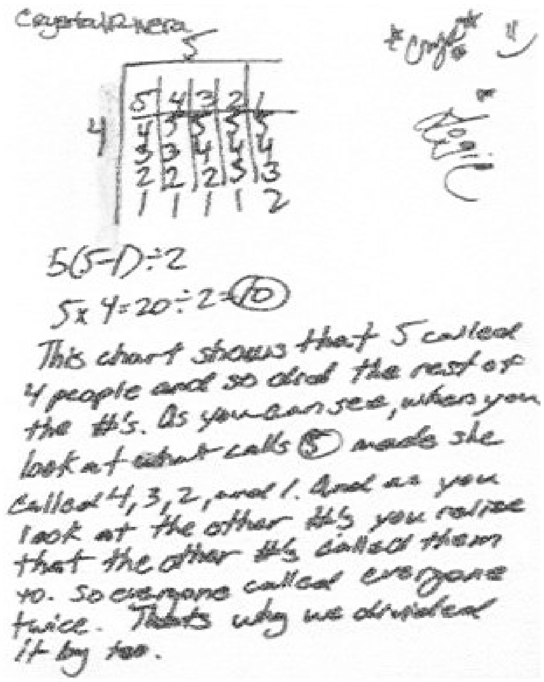

Figure 13. Bianca solved Aiesha's hypothetical situation with a chart

3.6.3. Dominique's retrieval six months later (a student in the audience asking questions)

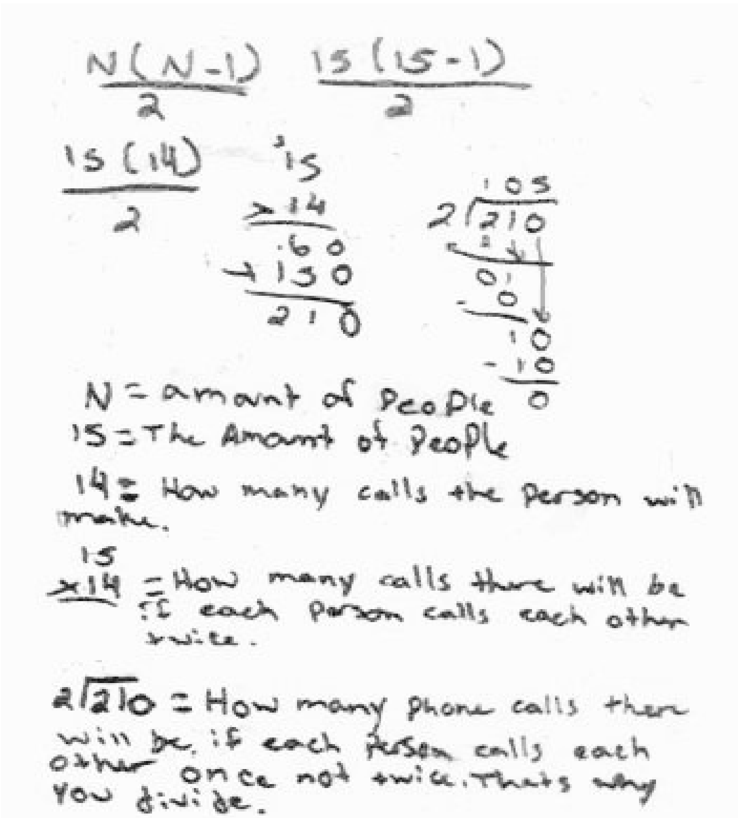

Figure 14. Student work from Dominique, one of the students who questioned their ideas

What is important to us is how, six months later, the students used these symbolic representations. A procedural use of the formula would result from simply plugging in numbers to a formula as a process carried out in action. An alternative would be that they use the formula as a recalled idea in a different setting, with enough similarities to the original setting in which the formula was first represented to stimulate recall of the idea. The students' writing indicates clearly that they are not behaving simply procedurally – they are indicating, declaratively, that they have explicit interpretations of the recalled formula and that they can re-interpret the formula in this new setting. Thus, the formula has become a flexible representational cognitive tool that students can use in novel settings to solve problems, similar, but not identical, in structure, to those they have seen before.

3.6.4. A view of several other students

Most students in the classroom were able to solve the Yakia problem shortly after reading the problem. Several of them were only able to retrieve the number sentence. These students knew that they needed to multiply 15 by 14 and divide by 2, but could not explain why they were doing so. In these cases, the students did not draw a picture, chart or other representation to explain why their number sentence worked. One might infer that the students were operating procedurally, however the fact that they noticed the structural similarity between the two problems may suggest otherwise. Nonetheless, there is no evidence to suggest that they were not just simply plugging in numbers in a formula as a process carried out in action. Alternatively, it is also possible that at least some of these students had an image that would justify their solution but did not have an opportunity to share it publically. No matter the case, we do not have the data to support one conjecture or another with regards to these students.

4. Conclusion

This paper documents the evolution of what we describe as flexible mathematical thought, and representational flexibility, in several students—with an emphasis on one student, Aiesha. We note in particular several features of this evolution. First, when given the opportunity, students often ask interesting and compelling questions, many times without overt intervention by a teacher or other adult. These questions are most commonly asked in relation to another students' work or reasoning, or in response to teacher questions about their own work, either immediately or delayed.

Second, the questions that the students raise are often a catalyst for them and their peers to refine, explain and re-think their answers, and to construct new and, as Aiesha's case reveals, more flexible representations to answer another student's questions.

Third, tinkering with representations seems to be the norm. Not surprisingly, students rarely construct new representations as if on a blank slate. Rather, they tinkered with and modified their own or other students' representations to meet a need to explain better and more fully. Aiesha's evolving representations provide insight into this: she not only developed more sophisticated and flexible representations, but in the process, she was able to use her peer's work as a model to both help her and her peer understand the problem.

Fourth, representations, flexible or otherwise, are not all of a single type. There are representations that are more or less records of working, representations that explain that something is so, and representations that explain *why* something is so. A student can produce a record of work – an inscription – as they work. This record does not necessarily constitute a form of declarative knowledge: it might simply be what they write as they work. When a student produces a representation to show working to solve a problem – as Aiesha did in her circles and loops representation of the handshake problem – it is our observation, in this case, that this first representation is the one that is consistently recalled in later similar problem settings. This seems to happen despite the fact that she produced different and more elaborate or abstract representations in response to other student questions as the problem session proceeded.

Fifth, students seem to use different types of representations flexibly as the need arises. So, even though a student might construct a sophisticated representation to answer *why* something works as it does, in a related context they might reconstruct a modification of a much simpler representation that shows *how* something is done. This might be explained by regression in the face of a new problem. We feel, however, that there is an element of retrieving a representation for a specific purpose: one is unlikely to retrieve a representation that explains why, when one is simply showing how. For example, Aiesha's representation shown in figure 9 was a modification of her original representation, not of those she produced later, yet it was not a simple regression: she has elaborated on the original representation with formulas and words.

What is it about the first representation that signals it to be first recalled to a student's mind in a novel problem setting? Could it be that we are seeing differing forms of declarative knowledge? For instance, Aiesha's circles and loops representation seems to be her declarative statement *that* she knew how to solve the problem. Her representation is an indication of a type of knowledge- *knowledge that*. When Aiesha was prompted to explain her answer, she produced a different representation. This representation seems to indicate a different form of knowledge – *knowledge why*. Just as *knowledge how* can be exhibited in the doing – carrying out a calculation, for example – *knowledge that* can be exhibited via a declarative representation, and *knowledge why* via a different representation. When Aiesha tackled the variant of the handshake problem – the party problem – she was not, initially, prompted to explain *why*, so her recall was *that* she could solve the problem. Is it possible that we are seeing the functioning of differing memory systems in students' use of representations?

We have seen how these flexible representations, discussed vigorously by students in class, and examined by them for gaps or flaws in reasoning, can become embedded as long-term declarative memories to be used flexibly in related yet novel problem settings. It is these experiences with students that lead us to believe that such classroom debate can be an important catalyst in the development of more flexible representations, and a motive force behind the development of understanding.

What we do not know, from this study is what students would have done if they had been presented with a new task that was superficially similar to the previous one but mathematically not isomorphic and, therefore, requiring a different type of solution. In other words what would happen with mathematically different problems with the same surface story context? This is an issue that would round out the other half of our story on flexibility.

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References

Beishuizen, M., van Putten, C.M. & van Mulken, F. (1997). Mental arithmetic and strategy use with indirect number problems up to one hundred. Learning & Instruction, 7(1), 87-106.
Carey, D.A. (1991). Number sentences: Linking addition and subtraction word problems and symbols. Journal for Research in Mathematics Education, 22 (4), 266-280.
Cohen, N,J. (1984). Preserved learning capacity in amnesia: Evidence for multiple memory systems. In N. Butters & L.R. Squire (Eds.), *The Neuropsychology of memory* (pp. 83-103). New York: Guildford Press,
Eichenbaum, H. (2002). *The Cognitive neuroscience of memory*. Oxford University Press.
Gray, E.M. & Tall, D.O. (1994a.) Duality, ambiguity and flexibility: A proceptual view of simple arithmetic. *The Journal for Research in Mathematics Education, 26* (2), 115-141.
Hatano, G. (2003). Foreword. In A.J. Baroody & A. Dowker (Eds), *The development of arithmetic concepts and skills* (pp. xi-xiii). Mahwah, NJ: Lawrence Erlbaum Associates.
Heirdsfield, A. M., & Cooper, T. J. (2002). Flexibility and inflexibility in accurate mental addition and subtraction: Two case studies. *The Journal of Mathematical Behavior, 21*, 57-74.
Hiebert, J. & Lefevre, P. (1986). Conceptual and procedural knowledge in mathematics: An introductory analysis. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics* (pp. 1-27). Hillsdale, NJ: Lawrence Erlbaum.
Karmiloff-Smith, A. (1994). Precis of Beyond modularity: A developmental perspective on cognitive science. *Behavioral and Brain Sciences*, 17 (4): 693-745.
Karmiloff-Smith, A. (1995). *Beyond modularity: A developmental perspective on cognitive science*. The MIT Press.
Klein & M. Beishuizen (1994). Assessment of flexibility in mental arithmetic. In J.E.H. van Luit (Ed.), Research on Learning and Instruction of Mathematics in Kindergarten and Primary School. Graviant Publishing Company, Esdoornlaan 15, 7004 BA Doetinchem, The Netherlands.
National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics.* Reston, VA: National Council of Teachers of Mathematics.
Ormrod, J. E. (2004). *Human learning*. (4th ed.). Upper Saddle River, NJ, USA: Pearson.
Perkins, D. N., & Salomon, G. (1992). Transfer of learning. International Encyclopedia of Education (2nd ed.). Oxford, UK: Pergamon Press.
Pirie, S.E.B. (1988). Understanding-Instrumental, relational, formal, intuitive…, How can we know?, *For the Learning of Mathematics, 8* (3), 2-6.
Pirie, S. E. B. & Kieren, T. E. (1992a). Watching Sandy's understanding grow. *The Journal of Mathematical Behavior, 11* (3), 243-257.
Pirie, S.E.B. & Kieren, T.E. (1992b). Creating constructivist environments and constructing creative mathematics. *Educational Studies in Mathematics, 23* (5), 505-528.

Pirie, S. E.B. & Kieren, T. E. (1994). Growth in mathematical understanding: How can we characterize it and how can we represent it? *Educational Studies in Mathematics, 26,* 165-190.
Schorr, R.; Warner, L; Geahart, D; Samuels, M. (2007) Teacher development in a large urban district: The impact on students. In Lesh, Kaput & Hamilton (eds.) *Real-world models and modeling as a foundation for future mathematics education*, pp. 431-447. Lawrence Erlbaum Associates, Mahwah, NJ.
Shore, B. M, Pelletier, S. & Kaizer, C. (1990). Metacognition, Giftedness, and Mathematical Thinking. Budapest: European Council for High Ability.
Stigler & Hiebert (1999) *The teaching gap - Best ideas from the world's teachers for improving education in the classroom*. New York: The Free Press.
Squire, L.R. & Kandel, E.R. (2008) *Memory: From mind to molecules*. Roberts & Co.
Star, J.R. & Rittle-Johnson, B. (in press). Flexibility in problem solving: The case of equation solving. Learning and Instruction.
Thorndike, E.L. & Woodworth, R.S. (1901) The influence of improvement in one mental function upon the efficiency of other functions. Psychological Review, 8, 247-261.
Threlfall, J. (2002). Flexible mental calculation. Educational Studies in Mathematics, 50, 29-47.
Tulving, E. (1985) *Elements of Episodic Memory* (Oxford Psychology Series). Oxford University Press.
Tulving, E. & Craik, F.I.M. (2000) The Oxford Handbook of Memory. Oxford University Press.
Verschaffel, L., Luwel, K., Torbeyns, J. & Van Dooren, W. (in press). Conceptualizing, investigating and enhancing adaptive expertise in elementary mathematics education. European Journal of Psychology of Education.
Warner, L.B., Davis, G.E., Alcock, L.J. & Coppolo, J. (2002). Flexible mathematical thinking and multiple representations in middle school mathematics. *Mediterranean Journal for Research in Mathematics Education*, 1(2), 37-61.