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HOW DOES FLEXIBLE MATHEMATICAL THINKING CONTRIBUTE TO THE GROWTH OF UNDERSTANDING?

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We examine aspects of flexible mathematical thinking in middle school students and its contribution to the growth of their mathematical understanding. We inductively analyzed data collected from video-episodes of an interactive, problem-solving based after-school mathematics class. Flexibility is essential in such classrooms since students must apply knowledge rather than simply retrieve it in its original context. The study provides links between flexibility and an existing theory of mathematical development to give researchers and teachers a better theoretical grasp of what happens in interactive classrooms.

IMPORTANCE OF FLEXIBLE KNOWLEDGE & THINKING

In the last decade, the focus in mathematics classrooms has changed from computational ability to higher-level problem-solving skills (National Research Council, 1989), as advocated in the National Council of Teachers of Mathematics Standards (2000). An important component of problem-solving is durability of knowledge, which is the ability to recall and use facts, skills, procedures and ideas. However, this does not imply its use in contexts other than those in which they were constructed. Flexibility is essential if students are to apply knowledge to new contexts, and is therefore a key ingredient in problem-solving. The goal of this study is to identify acts of flexible thinking, trace the events that surround these acts, analyze the mathematical knowledge that the students build, and understand how these acts, events and knowledge intertwine.

THEORETICAL FRAMEWORK

In psychology the term "flexibility" is often used without being defined (e.g. Jausovec, 1994), or is investigated via specific problems that are thought to require flexibility for their solution (e.g. Kaizer & Shore, 1995). Studies in mathematics education generally treat flexibility as the capacity to exhibit a variety of invented strategies or a large repertoire of strategies for solving problems (e.g. Heirdsfield, 2002). More specific uses of the term are offered in educational psychology and mathematics education. In particular, Spiro & Jehng (1990) define cognitive flexibility as the ability to spontaneously restructure one's knowledge, in adaptive response to radically changing situational demands. In mathematics education, Krutetskii characterizes flexible thinking as reversibility of thought (Krutetskii, 1969). Gray and Tall (1994) characterize flexible thinking in terms of an ability to move between interpreting notation as an instruction to do something (procedural) and as an object to think with and about (conceptual). Our conceptualization of flexible mathematical thought extends the literature reviewed above, based on the first author's classroom experience and inductive analysis of the data in the present study. It may be summarized by stating that a student thinks flexibly when they exhibit some or all of the following behaviors (Warner, Coppolo & Davis, 2002): interpretation of their own or someone else's idea (e.g. through questioning it and thus showing it to be valid or invalid; through using, reorganizing or building on it); use of the same idea in different contexts; sensible raising of hypothetical problem situations based on an existing problem: creating "What if…?" scenarios; use of multiple representations for the same idea; connecting representations.

METHODS/ANALYSIS

Ten sixth and seventh grade students, of mixed abilities, volunteered to participate for twelve weeks in an after-school mathematics class, in which students worked on combinatorial tasks (such as, Building Towers - finding the number of possible 4 tall towers of building blocks, choosing from 2 colors of block; Martino & Maher, 1999). Three to five students worked in each group and presented, discussed, and argued about, their solutions. Additional time was provided for students to further discuss, rewrite and reorganize their ideas. The teacher/researcher (first author) encouraged the students to discuss disagreements and differences, question and justify solutions, revisit ideas over time, generalize and extend ideas. A month later, individual and group interviews were conducted with half the students to examine the durability of their ideas.

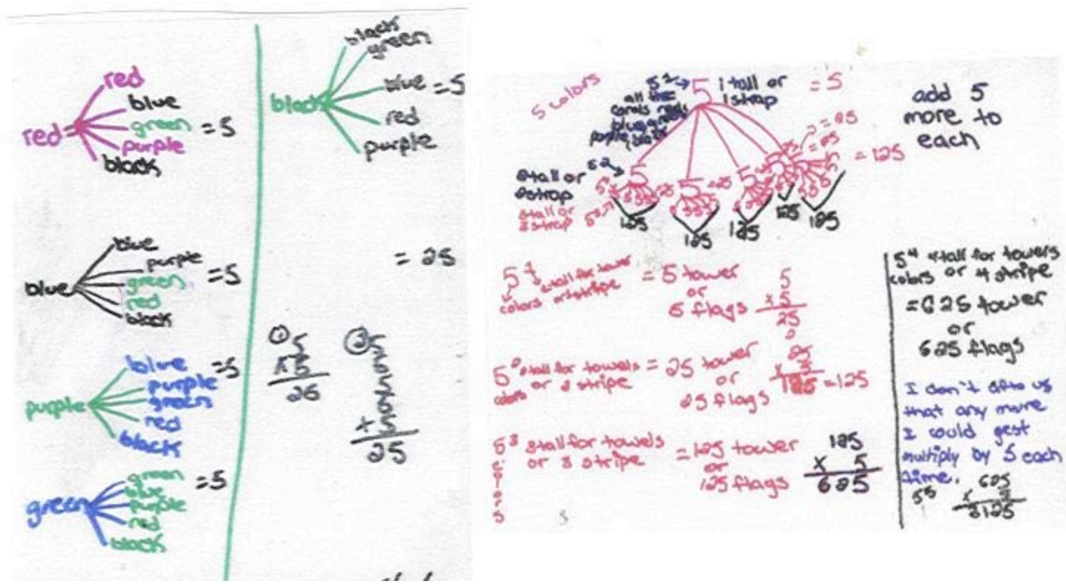
This study focuses on six of the hundred and fifty videotapes generated in this manner, as the students investigate a student-posed task, called the Castle problem. We coded transcripts to identify critical events (Powell, Francisco & Maher, 2001), determined by acts of flexibility. The analysis led beyond our characterization of flexibility (Warner, Coppolo & Davis, 2002) to a recognition of links between acts of flexibility and movement between layers in the Pirie/Kieren model for the growth of mathematical understanding (Pirie & Kieren, 1994). For example, questioning someone else's idea often resulted in a change to a new representation for the same idea, which moved the student from *image making* (doing something to get an initial idea of a concept) to *image having* (at which point the image is no longer tied to an action). In the following events, we examine the mathematical development of one student, Anna, by linking her acts of flexibility to movement through the layers of the Pirie/Kieren model. Anna is placed in a modified class based on her below average performance on standardized tests.

RESULTS

The students investigated six tasks, in depth, during the six month period of this study, two of which Anna connect to "The Castle Problem" (posed by Jessica): *If you're at the beach and there are 3 different kinds of buckets in front of you (small, medium, large). How many different ways can you build castles if there are a total of 10 castles?* The Flag problem (posed by Amanda) asks for all possible flags with two stripes, choosing from 5 colors. David extends it by also providing a choice of 5 shapes for the corner. Some of Anna's work for the Flag Problem, to which we refer later, is shown below.

The first day: understanding the task

In this first section, Anna uses a previous idea (see figure 1a below) in a new context while she is working in the *image making* layer. David begins by reading Jessica's written version of the Castle problem.



(a). (b).

Figure 1. Anna's work on the Flag problem

David If you're at the beach, and there's three different kinds of buckets, small, medium and large, in front of you, how many different ways there are [sic] [*David laughs*] to make them if there are only a total of 10 buckets.

Anna So, they're saying that there's only 10 of them.

T/R So [looking at a student's written version of the problem] you have ten castles in total and you have a choice of a small, medium or large bucket.

Anna This is hard.

T/R So, basically you want to know how many possibilities there are.

Anna immediately begins writing, and although she says that this is a difficult problem, she immediately comes up with a correct strategy and begins to solve. (See figure 2a)

Anna Small you can have…it's almost…small, medium…and large. [*drawing a tree with three branches coming out of each of the three sizes*]

When Anna uses the words "it's almost", we infer that she is *folding back* (Pirie & Kieren, 1994) to make a connection with a previous context. This facilitates her *image-making* in this context.

Ideas are questioned and a revision is offered

In this section, Anna uses a previous representation in another context. Being questioned by David deepens her understanding, which enables her to reorganize her ideas, flexibly move to a new representation, and move from *image making* to *image having* (Pirie & Kieren, 1994). Anna continues the next part of her solution (bottom of figure 2a). In the event below, David questions Anna's answer of nine possibilities and her second representation (See the bottom of figure 2a). He offers a suggestion and they continue drawing their trees using his idea.

David Can I ask you a question? How come you did small small, small small and medium? …because then there's three castles. [*David points to her trees.*]

Anna [*connecting the small small, medium medium and large large tree and writing a nine at the end of it]…*but, there are nine possibilities.

David No, there's more than nine. Hold on let me…this would be…let me get another sheet here. [*taking a new sheet*]

At this point Anna becomes excited - it appears that she recognizes her mistake. David begins to show her how to reorganize her idea by drawing the possibilities.

David You would have small small. Then you would have small medium and small large.

David Small, small and that branches three ways. [*He draws three branches coming out of small, small.*]

Anna Yes.

David So it's small small small. [*writing the word small coming out of a branch and drawing all possibilities for Small small, Small medium and Small Large…*]

Anna Medium…Now, would you do Medium, small?

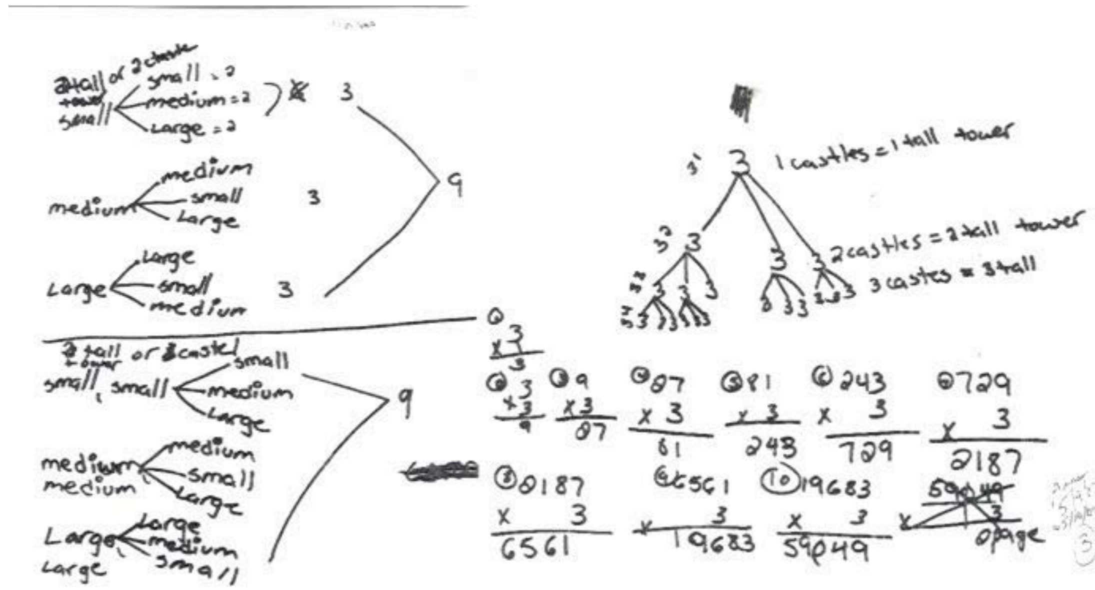
David Yeah. [*He continues with his tree.*]

David questions Anna's idea, points out what she missed and offers a way to revise it. Anna uses David's suggestion and reorganizes her tree. Anna now has a deeper understanding of her representation and how it connects to the task. David and Anna continue to draw their trees, separately, working in the *image making* layer. In the following event, Anna moves beyond this to *image having*. Immediately after David questions Anna again, she corrects her mistake, then flexibly uses an old idea (Figure 1b) to move to a new representation (Figure 2b).

David How come you wrote small small, medium medium, medium small, medium small?

Anna What do you mean? Small small, small medium, small large, medium small, medium small, oh. [*Anna fixes the second medium small and writes medium medium, instead.*]

After Anna corrects this mistake, she flexibly changes her representation to a "three tree" (Figure 2b), which moved her from *image making* to *image having*. She no longer needs to write small, medium and large: she has an image of the three sizes and writes a "*3*" to represent them. It appears that being questioned by David pushed her to understand her mistakes which moved her to a deeper level of understanding.



(a) (b)

Figure 2. Anna's changes of representation

Questioning someone's idea and showing that it is invalid

In this section Anna questions David's assertion, shows that it is invalid, and explains why. This necessitates *property noticing* (Pirie & Kieren, 1994). David begins by writing a nine with nine branches coming out of it. Note that Anna questions David's idea and shows it is invalid by providing a reason for using the three in terms of the properties noticed about her images.

T/R Well, what does the nine represent?

Anna Wait, no, first you should do three.

David Wait, let me explain. Nine, sort of like, this would be…are you saying...maybe nine is not the right number?

Anna Yeah, I think three.

David Right. I'm sorry it is three. [*draws a line under his 9 with 9 branches*]

T/R Why three?

Anna Because there's three, there's small, medium and large.

David Right.

Anna So, out of that three you get three more and it's…you get three from small, three from medium and three from large. [*points to the 3's in the 2nd layer of her tree*] Then from that you get three from small, three from medium, three from large. [*points to the 3's in the third layer of her tree*]

Noticing properties by setting up hypothetical situations

In this section, Anna connects the sizes in the Castle problem to the colors and shapes in the Flag problem, by re-explaining her hypothetical situation she created for Flags. She *folds back* (Pirie & Kieren, 1994), then builds on the idea by creating a new hypothetical situation for Castles. When Anna tells us exactly what the tree would look like without drawing it, she also has an image of the castles, choosing from 4 sizes. She builds on her old hypothetical situation by adding another size (XXL), choosing from five sizes, and notices even more properties about the images she has.

T/R So what would this first three represent? [*pointing to Anna's previous representation in which the numeral 3 appeared*]

Anna One castle? [*writing 'one castle' next to the first three*] …and then this one would be two castles [*writing '2 castles' next to the second layer*] …and David's problem would be the same thing except we're talking about size and he's talking about shapes.

T/R Hmm, so how is David's problem the same?

Anna Ok, like you see, I moved on and you can go on with this problem too…like in David's problem I said what if you add another color or add another shape? …In this one, you can say what if you add another size.

T/R Maybe you can work on that, too, what if you did add another size?

Anna It would be four [*tapping her pen next to the three branch*], then four comes from that, [*drawing the four branches with the pen in the air*], that would be four, eight, twelve, sixteen [*tapping her pen on each imaginary spot that the four branches would be in on her paper*]…and then out of those four another four come out. It's the same thing. [*finding her previous work for flags, figure 2b, and pointing to it*] I say five over here [*pointing to the top five*] and then five come out [*pointing to the five branches with her pen*] and for each five you add five more.

T/R Hmmm. So what would that five represent in Castles?

Anna You would need…small, medium, large, extra large, extra extra large.

Anna sets up new hypothetical situations for the existing problem, *folds back* to a previous hypothetical situation she set up for another context, and connects representations for the two problems. In the next line, Anna appears to be saying that "moving on", or setting up hypothetical situations, allows her to move to other representations and discover new things about the problem. Notice that she moves to another representation [*multiplying by 5*] while saying this.

Anna Can I tell you something? Well, when I moved on, I said you times by five each time. If you do that, lets say another person says I want you to do small, medium, large, extra large, so you already have the answer.

T/R Maybe we can solve that after we are done solving this one. Maybe we could extend the problem. Do you want to do that when we're done solving this one?

Anna It's the same thing except you're adding another size or you're adding another shape or color.

T/R Hmmm, interesting. I'd like you to label them that way if you could.

Anna This one? [*pointing to the 3 labeled 'one castle'*]

T/R How else would you label that besides one castle?

Anna One tall or... (*now connecting Building Towers - See Background section*)

T/R: Go ahead. [*Anna writes '2 castles = 2 tall' next to the second layer of the tree and '3 castles = 3 tall' next to the third layer of the tree*.]

Her talk about "moving on" advanced Anna towards using new representations, namely, multiplying by five and labeling each level with the number tall. Her *property noticing* is now on the level of sophistication at which the properties amount to identifying an isomorphism between the three problems (Castles, Flags and Towers).

Moving from formalizing to observing

We now pass over Anna's move from *property noticing* to *formalizing* (generalizing to create a "for all" statement) in order to highlight her moving beyond this layer to *observing* - noticing properties about "for all" statements. She does this by setting up hypothetical situations for the existing problem (for example, "If I add another size or color to 3 to the n, I get 4 to the n"), by connecting representations (using exponents to label each layer of the tree) and by connecting contexts (labeling each "n" and base with the information that it represents in Castles, Towers and Flags). All of these labels can be seen in Figure 3, which were written by Anna after moving back and forth between the layers over the course of three days.

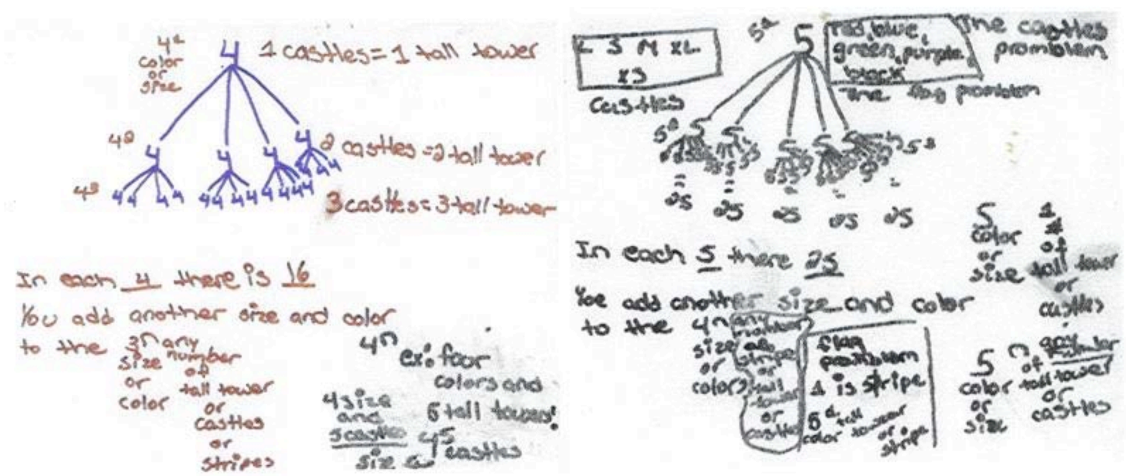


Figure 3. Anna's labeling of diagrams, written at the end of the 3rd day

CONCLUSION

Through a detailed analysis of one student's work on combinatorial problems we have shown that the transitions from one to the next layer in the Pirie/Kieren model of understanding occurred in association with acts of flexible mathematical thinking. We hypothesize that it is the acts of flexible thinking that act as catalysts for transitions in the Pirie/Kieren model for growth of mathematical understanding. We also hypothesize that the success of interactive classrooms is partially due to the opportunities and encouragement they provide for students to engage in flexible thinking. We have also shown that even students who are considered to be of below average ability can develop sophisticated mathematical understanding in an environment in which questioning, debate and discussion are encouraged.

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