NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

Engaging Whole-Number Knowledge for Rational-Number Learning Using a Computer-Based Tool

Author(s): Robert P. Hunting, Gary Davis and Catherine A. Pearn

Source: *Journal for Research in Mathematics Education*, May, 1996, Vol. 27, No. 3 (May, 1996), pp. 354-379

Published by: National Council of Teachers of Mathematics

Stable URL: https://www.jstor.org/stable/749369

REFERENCES
Linked references are available on JSTOR for this article:
https://www.jstor.org/stable/749369?seq=1&cid=pdf-reference#references_tab_contents
You may need to log in to JSTOR to access the linked references.

JSTOR is a not-for-profit service that helps scholars, researchers, and students discover, use, and build upon a wide range of content in a trusted digital archive. We use information technology and tools to increase productivity and facilitate new forms of scholarship. For more information about JSTOR, please contact support@jstor.org.

Your use of the JSTOR archive indicates your acceptance of the Terms & Conditions of Use, available at https://about.jstor.org/terms

National Council of Teachers of Mathematics is collaborating with JSTOR to digitize, preserve and extend access to *Journal for Research in Mathematics Education*

JSTOR

This content downloaded from
71.232.212.173 on Sun, 18 Oct 2020 18:51:25 UTC
All use subject to https://about.jstor.org/terms

Journal for Research in Mathematics Education
1996, Vol. 27, No. 3, 354–379

ENGAGING WHOLE-NUMBER KNOWLEDGE FOR RATIONAL-NUMBER LEARNING USING A COMPUTER-BASED TOOL

ROBERT P. HUNTING, *La Trobe University*
GARY DAVIS, *University of Southampton*
CATHERINE A. PEARN, *La Trobe University*

We present data from a two-year teaching experiment involving 8- and 9-year-old children. The purpose of the teaching experiment was to investigate children's fraction learning and the role their whole-number knowledge might play in it. A major source for the children's experiences and experiments was an operator-like computer program called Copycat. In particular, we focus on the accomplishments of Elliot and Shannon as they solved fraction comparison problems. We identify and discuss three cognitive schemes that Elliot and Shannon used. These were an Equal Outputs scheme, limited in effectiveness to unit fraction comparisons; an Equal Inputs scheme that activated strategies for determining a common multiple; and a Scaling scheme in which fractions were scaled up or down using ratios. The major focus of the discussion is the role whole-number knowledge played in the application of these schemes. This study demonstrates an interdependence between development of rational-number knowledge and whole-number knowledge. Rational-number tasks in operator settings can help stimulate and extend children's whole-number knowledge. Facility with whole-number relationships enables students to solve fraction comparison problems.

Behr, Harel, Post, and Lesh (1992) have noted that considerable mathematics education research has been conducted on topics closely related to the area of rational numbers and proportional reasoning, as for example, whole-number arithmetic (Carpenter & Moser, 1982; Steffe, Cobb, & von Glasersfeld, 1988), algebra (Wagner & Kieran, 1989), and problem solving (Charles & Silver, 1989). They further observed that although research on these parallel topics has been progressing simultaneously, developmental interdependencies between topics are not clear.

> We know that certain ideas about fractions, ratios, or proportions are developing at the same time as some concepts related to (a) whole number multiplication and "composite units" (Steffe, von Glasersfeld, Richards, & Cobb, 1983), (b) exponentiation and "unit splitting operations" (Confrey, 1988), or (c) rates or "intensive measures" (Hart, 1981; Kaput, 1985; Schwartz, 1988). Yet, in general little is known about developmental interrelationships. (Behr et al., 1992, p. 299)

Efforts aimed at understanding the informal mathematics knowledge that children bring to formal schooling have been productive in the area of whole numbers

We are thankful for the cooperation and assistance of the principal, Kevin Maguire, and staff of the Bulleen Primary School, Melbourne, Victoria. An earlier version of this paper was presented at the Annual Meeting of the American Educational Research Association, New Orleans, April 1994. The research reported in this paper was supported by a grant from the Australian Research Council.

This content downloaded from
71.232.212.173 on Sun, 18 Oct 2020 18:51:25 UTC
All use subject to https://about.jstor.org/terms

(Carpenter, 1985; Fuson, 1988) and, more recently, rational-number learning (Hunting & Davis, 1991; Lamon, 1993a; Mack, 1993; Resnick & Singer, 1993; Streefland, 1991). Carpenter, Fennema, and Romberg (1993) observed that students' understanding of some of the basic principles underlying rational numbers could be tapped if they are given realistic problems that do not emphasize symbolic manipulations. This study reports on the accomplishments of two 9-year-old children who participated in a year-long teaching experiment. These children learned about fractions using a computer program called Copycat. This computer tool was a major source of the children's experiences and experiments. It emphasized fractions as operators and was designed to draw on existing whole-number knowledge. The study's aim was to investigate what basic fraction knowledge might be learned in a context where children's knowledge of whole-number relationships, including numerical sequences, was needed. The whole-number knowledge needed to succeed with the computer-based fraction tasks drew on preschool and early school experiences foundational to formal symbolic manipulation. In that sense it was informal knowledge. We will detail several cognitive schemes developed by these children as they learned to solve fraction-comparison problems. A surprising outcome of the study was the degree of interdependence observed between the development of whole-number concepts and relationships and basic fraction knowledge.

By the time fraction instruction commences, children have considerable knowledge of whole numbers and how they work. It has been observed that children's whole-number schemes interfere with the acquisition of fraction knowledge (Behr, Wachsmuth, Post, & Lesh, 1984; Gray, 1993; Hunting, 1986; Streefland, 1984, 1991; Tirosh, Wilson, Graeber, & Fischbein, 1993). Recently attention has turned to investigating ways in which the whole numbers might assist, and indeed form a basis for, developing concepts of rational numbers (Carraher, 1993; Steffe & Olive, 1993). Fractions have many interpretations, including that of measure, quotient, ratio, and operator (Kieren, 1976, 1988, 1993). The interpretation embodied in the computer tool we have been using is that of fractions as operators.

Children's schemes for working with fractions as operators have been examined by Behr, Harel, Post, and Lesh (1993), Davis (1991), Hunting, Davis, and Bigelow (1991), Kieren and Nelson (1978), Kieren and Southwell (1979), and Southwell (1984). Behr et al. (1993) have argued that the elementary school curriculum is deficient in failing to include the range of concepts relating to multiplicative structures that are necessary for learning in middle grades. Operator models of fractions in didactic settings have also been studied or advocated by Dienes (1967), Braunfeld and Wolfe (1966), Freudenthal (1983), Hilton (1983), Pickert (1968), and Usiskin (1979).

Our approach to studying children's mathematics is based on detailed observations of children's actions and behavior indicative of scheme reorganization. The word scheme means a mental coordination of actions. A scheme is goal-directed and consists of three parts (Steffe, Cobb, & von Glasersfeld, 1988). First is the child's recognition of an experiential situation as one that has been experienced before. Second is the activity the child has learned to associate with the situation. Third is the result the child has come to expect of the activity. Piaget (1970, 1972) considered a scheme

This content downloaded from 71.232.212.173 on Sun, 18 Oct 2020 18:51:25 UTC
All use subject to https://about.jstor.org/terms

to be a general structure of a set of actions that conserves itself by repetition, consolidates itself by exercise, and applies itself to situations that vary because of contextual modification.

Research in this paradigm in the domain of whole-number acquisition has generated a coherent theory of learning (Steffe, 1988, 1993; Steffe, Cobb, & von Glasersfeld, 1988; Steffe & von Glasersfeld, 1985; Steffe, von Glasersfeld, Richards, & Cobb, 1983). In part the theory is significant because it hypothesizes specific ongoing fundamental changes in the mental functioning of children as they build mathematical knowledge, through the guidance of researchers and teachers, as well as through interactive communications between their peers and teachers (Cobb, Wood, & Yackel, 1990; Cobb, Perlwitz, & Underwood, 1994).

METHOD

Ten children were selected for the teaching experiment that commenced in May, 1992. These children were all the Year 3 part of a mixed Year 2/Year 3 class in a state-funded primary school consisting of predominantly middle-class children in Melbourne's eastern suburbs. Their ability range in mathematics was wide. When the study began, Elliot and Shannon, the two students we report on in this study, were 8 years 4 months and 8 years 11 months old, respectively. They were considered by their class teacher to be of average ability overall, but in the light of detailed initial interviews and their responses during the teaching sessions, we found them to be mathematically able for their ages.

The methodology used was that of the constructivist teaching experiment (Cobb & Steffe, 1983; Hunting, 1983; Steffe, 1984) incorporating adaptations of the clinical interview (Davis & Hunting, 1991; Hunting, 1983; Opper, 1977). In this methodology a predetermined curriculum is not followed. Rather, the activities of a session are decided on the basis of observations and interpretation of children's behavior in prior sessions. Although one of us had previously conducted a teaching experiment with older children (Hunting, 1986; Hunting & Korbosky, 1990), the Copycat computer program was a radically different source of fractions experiences, compared with those used in that study. Our general goal at the outset of the experiment was to have the children investigate the behavior of the Copycat set to the fraction $1/2$ and progress to explorations of other unit fractions such as $1/3$ and $1/4$. Our decision to introduce comparison tasks was based on our assessments of the children's progress at that time. We conducted teaching sessions twice weekly for periods of 3 weeks. Each session lasted approximately 20–25 minutes. In a typical morning we would conduct three to four sessions—each session working with groups of two or three children. The children were grouped initially according to performance on a set of interview tasks administered prior to the teaching sessions. The teaching groups were changed over time in response to differential changes in progress and to vary the effect of social dynamics between children. For example, we conducted a number of sessions with Shannon alone because he was not only more advanced than the other children but also highly competitive and dominant. The children continued to participate in ongoing

This content downloaded from
71.232.212.173 on Sun, 18 Oct 2020 18:51:25 UTC
All use subject to https://about.jstor.org/terms

classroom mathematics instruction. However, we requested that no instruction on the topic of fractions be given for the duration of the experiment. A summary of activities from the 1992 and 1993 teaching sessions involving Elliot and Shannon can be found in Tables 1 and 2. All sessions were video-taped for later analysis.

Table 1
Summary of Teaching Activities for 1992

Date	Summary of teaching activities for 1992
1. May 29	First session was an introductory one, using SuperPaint, sharing designated numbers of pieces of chocolate to determine the number of people, when given the number of pieces per person.
2. June 2	Children were introduced to the Copycat program with the fractions hidden. They were shown how the buttons on the machine worked and allowed to experiment with various inputs.
3. June 5	Copycat was used this time with the fraction revealed. The following fractions were used: 2 for 3 and 3 for 4 using various inputs. Using a 3 for 6 machine they realized that although 3 for 6 was equivalent to 1 for 2 they would need to be able to divide the inputs by 6.
4. June 12	A worksheet was given that required the types of machines that would work on an input of 20 balls to be listed. Experiments using the computer were encouraged but not required.
5. June 16	Review of 1 for 2 then 3 for 2. Shannon realized fairly quickly that he could halve the input number and add it to itself. He went on to try 2 for 1, 4 for 3, and 6 for 10.
6. June 19	Children completed worksheets containing three columns—Number In, Number Out, and Fraction. Two entries were given and one was to be found. The Copycat program could be used for confirmation or assistance. Worksheets involved fractions $^1/_2$ and $^1/_3$. Shannon was given a sheet of mixed problems with more difficult fractions.
7. June 23	Copycat program was used. Session commenced with a review of the previous session and completion of worksheets. Two groups were given a sheet asking for the output given the input of 6 balls, then 15 balls, using the fractions $^1/_3$, $^2/_3$, $^3/_3$, $^4/_3$ … . Another group investigated problems involving $^2/_1$, $^1/_2$, $^1/_1$, and $^2/_2$.
8. Sept. 1	Copycat used with hidden fractions $^2/_3$, $^3/_4$, and $^2/_4$. Experiments were conducted to discover the fractions.
9. Sept. 4	All students were given the same worksheet that listed the fractions $^1/_4$, $^1/_2$, $^1/_3$, $^2/_3$, $^3/_4$ horizontally, and inputs of 2, 3, 4, 5, 6, 7, 8 vertically. All except Shannon used the machine to either prove predictions or solve the problem.
10. Sept. 8	Different problems were planned for three groups. Alisha and Elliot worked on sheets as before but the fractions were fifths and inputs ranged from two to 15.
11. Sept. 11	Brief exploratory session using the Copycat and SuperPaint. The key problem was: "Would the $^2/_3$ Copycat machine work if we put 30 chocolates on the In tray?" Shannon and Elliot attempted a worksheet involving comparison of pairs of fractions.
12. Sept. 15	Discussed comparison worksheet. Copycat used to confirm answers.
13. Sept. 18	Both SuperPaint and Copycat programs were used. Elliot and Shannon worked on problems of comparing fraction pairs.
14. Oct. 13	A large cardboard voting booth was modified to simulate a Copycat machine and the children had to design problems for each other using fractions given by the teacher. One child was stationed inside the booth and operated on the inputs. Problems arose when children put out incorrect outputs.
15. Oct. 16	Inputs and outputs from the previous session were recorded on a board and discussion took place. worksheets were given in which various input-output pairs were displayed, and the appropriate fraction required. The sheets encompassed a range of difficulty.

(table continued)

This content downloaded from
71.232.212.173 on Sun, 18 Oct 2020 18:51:25 UTC
All use subject to https://about.jstor.org/terms

Table 1—continued
Summary of Teaching Activities for 1992

Date	Summary of teaching activities for 1992
16. Oct. 20	The SuperPaint program was used to make rectangles $3/2$ times larger and $3/4$ times larger than a given rectangle. Alisha and Sukey, together, and Elliot by himself did this.
17. Oct. 23	Work on Copycat set to the fraction $5/10$, which was hidden. The first input was 10. All thought the Copycat was a $1/2$ machine. Other inputs attempted did not produce expected results.
18. Oct. 27	Following on from Session 17, the children in the various groups were asked to predict numbers that would make a $5/10$ machine work. There was some discussion about other machines that would work like a $1/2$ machine.
19. Oct. 30	Children in the various groups were asked to explain what they thought was happening inside the Copycat, and to explain how a new machine could be built.

Table 2
Summary of Teaching Activities for 1993

Date	Summary of teaching activities 1993
20. May 11	Individual interviews were conducted. Shannon was asked what numbers would make $2/4$ and $1/3$ machines work. He was asked to compare $2/4$ and $2/6$. Elliot was asked similar questions.
21. May 14	Alisha, Elliot, and Sukey were given a worksheet that asked them to compare pairs of fractions (see Figure 2). Shannon had a session by himself with more challenging comparisons.
22. May 18	Fractions on flash cards were given to Elliot and Sukey to compare. Shannon (by himself) was asked about $1/2 + 1/3$, and $1/2 + 1/4$. He was also given sets of fraction flashcards to arrange in order.
23. May 21	Alisha, Elliot, and Sukey were given pairs of fractions on flashcards to compare. They used Copycat to check. Shannon was asked to reconsider his responses to $1/2 + 1/3$ and $1/3 + 1/4$.
24. May 25	Alisha, Elliot, and Sukey used SuperPaint with a range of square units consisting of various partitions. Comparisons such as $1/3$ and $1/4$, $2/3$ and $3/4$, $1/2$ and $3/7$ were considered. Shannon (by himself) was asked to compare fractions like $3/5$ and $5/8$ using the SuperPaint objects.
25. June 18	Alisha, Elliot, and Sukey were given 7 sets of four fractions on flash cards to compare. The last set was $7/4$, $3/6$, $10/5$, $8/8$. Shannon (by himself) was given two sets of four fractions to compare. He was asked to find a fraction between $7/4$ and $8/8$.
26. June 22	Alisha, Elliot, and Sukey were given the question given to Shannon on June 18: Find a fraction between $7/4$ and $8/8$. Shannon (by himself) reviewed fraction between $7/4$ and $8/8$.
27. August 6	Shannon (by himself) was asked to find fractions between $1/2$ and $1/3$, and between $2/4$ and $2/6$. Elliot and Sukey discussed fractions between $5/4$ and $8/8$. They sorted a set of fractions.
28. August 25	Elliot and Shannon together work on problems of finding fractions between $1/2$ and $2/5$, and between $2/5$ and $1/3$.

The Computer Tool

The Copycat is an operator-like, computer-based learning tool developed in Hypertalk 2.0 for use on Apple Macintosh computers. Experiments performed may be directed at determining what fraction is responsible for observed numerical inputs

This content downloaded from
71.232.212.173 on Sun, 18 Oct 2020 18:51:25 UTC
All use subject to https://about.jstor.org/terms

and corresponding outputs, or at determining the numerical value of inputs or outputs for selected fractions governing the Copycat's behavior.

On the Copycat itself are three "buttons"; each can be activated with a mouse-click when the cursor is positioned over it (see Figure 1). The arrow buttons control the number of counters placed on the "in-tray" (left side of graphic). Counters are added or subtracted one at a time. The Go button activates a script that determines what the Copycat will do. If the number of counters placed on the in-tray is divisible by the fraction denominator, *d*, and the Go button is clicked, an observer sees a group of *d* counters removed, one at a time, from the in-tray; simultaneously a distinctive sound is heard. Next, counters appear one at a time in the output tray until the number of counters in the tray equals the number in the numerator of the selected fraction. A new sound accompanies the appearing counter. This process continues until all the counters in the in-tray have been used up. Finally, to the accompaniment of applause, all the counters reappear in the input tray; and the windows, above the words "In" and "Out," display the number of input and output counters. If the number of input counters is not divisible by *d*, the Copycat "explodes" to appropriate noises. Other buttons—visible and invisible—can be used to reset the machine after each experiment and to activate a screen to cover the selected fraction. The fraction can be selected by choosing from a restricted set of numerators and denominators from the menu bar at the top of the display (not visible in Figure 1). Possible numerators are numbers one through eight; possible denominators are numbers one through six, eight, and ten.

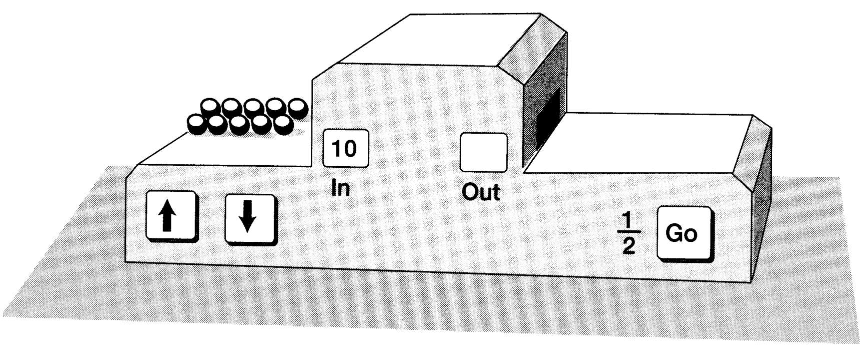

Figure 1. Graphical features of the Copycat.

The Copycat allows children to explore relationships governing inputs and outputs of discrete items. If the selected fraction is hidden, the children could be asked to verbalize a rule or reason for the behavior of the Copycat. If the selected fraction is visible the children might be asked to predict inputs given outputs or vice versa. Sometimes, with the selected fraction hidden, a specific input and output will be given and the children encouraged to search for the appropriate fraction by selecting a numerator and a denominator from a menu.

This content downloaded from
71.232.212.173 on Sun, 18 Oct 2020 18:51:25 UTC
All use subject to https://about.jstor.org/terms

The Initial Interviews

All children in the study were interviewed individually 3 weeks prior to commencement of the teaching sessions. The set of interview tasks included partitioning of noncontinuous items, basic fraction knowledge, verbal counting, counting composite units, quantification of arrays, and ratios. Shannon and Elliot were remarkably similar in their performance on these tasks.

The partitioning tasks. These involved distribution of items equally between dolls: 12, 28, and 56 items among four dolls, 19 among three, and a task in which 20 items were to be shared equally in as many different ways as possible. For the task of distributing 28 items among four dolls, Shannon showed his facility with whole numbers by allocating seven items to the first doll, seven to the second doll, and so on, indicating that he used numerical operations to anticipate the result. For 56 dolls he placed 14 items in front of the first doll, saying he was "doing it by fours." Further questioning indicated Shannon first considered the results of giving 10 items to each doll. This would have left 16 items, which when divided by four, meant 4 more items would be needed for each doll. On the final partitioning task Shannon was able to determine fairly easily, using 20 items to manipulate, the numbers of dolls that could share the items equally to be 5, 4, 2, 1, 10, and 20.

Elliot, like Shannon, also initially gave 10 items to each doll in the 56-item task. However, this result was likely an estimate. He gave one to each of the first two dolls, but stopped to straighten the lines of items. He then placed another item in front of the second doll, continuing by placing items one at a time in order. At the end of the last complete cycle, when Elliot had three items left, he began to check. He was observed to count each share and make adjustments in a nonsystematic trial-and-error fashion. For the task of determining the different number of possible shares that could be made with 20 items, Elliot made five groups of 4, split each of the five groups to find ten groups of 2, and then split each of the ten groups to yield twenty groups of 1. His final successful result was four groups of 5. He unsuccessfully attempted to make seven equal groups.

Counting tasks. Shannon and Elliot were fluent in counting composite units where cumulative totals were to be determined as rows of objects were progressively uncovered. Composite units of two, three..., ten objects were displayed. Verbal counting skills were also assessed. Both Shannon and Elliot were able to count forwards and backwards by twos, threes, fours, fives, sixes, and tens, beginning at any number. Elliot displayed verbal behavior reminiscent of a tables singsong. He was slower with nines, where he segmented each unit into four and five.

Array tasks. For the first task, rectangular arrays of small square regions were displayed with instructions to determine their number. Shannon used the strategy of relating multiplication facts with the relevant number of items in each row and column, or added or subtracted a row or column from a previously computed display. Elliot was observed to do likewise. For the second task a large base array (10×12) was displayed. The interviewer placed different-sized rectangular pieces of card over

This content downloaded from
71.232.212.173 on Sun, 18 Oct 2020 18:51:25 UTC
All use subject to https://about.jstor.org/terms

the base array to cover certain subarrays. The problem was to calculate how many items were covered. Both children were quite fluent with this task type. They seemed able to visualize and count each nonvisible row in the covered array.

Ratio task. A question typical of the sequence asked was, "There are six dolls and I want to give five counters for each doll. How many counters will there be?" Counters were available, and six dolls were placed in a line across the table. Shannon gave the answer to these questions without reference to counters or physical contact. Elliot used multiplication facts for simple tasks, but he also used serial group counting, as evidenced by his head movements.

Fraction tasks. Three types of fraction tasks were given. A square arrangement of Cuisenaire rods adapted from a task used by Saenz-Ludlow (1992, 1994) showed rods arranged as in Figure 2.

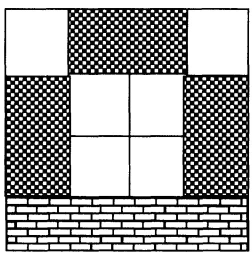

Figure 2. Unit pattern made from Cuisenaire rods.

The children were asked to show one half, one quarter, and one eighth of the unit pattern. Shannon responded appropriately for the fractions one half and one fourth. For one eighth of the unit pattern he divided the rods into two equal parts. Elliot's first attempt to show $1/2$ was unsuccessful, but on a second attempt he divided the rods appropriately. The second task involved coins with denominations of 5, 10, 20, and 50 cents. Fractions of one dollar were required, including $1/2$, $1/5$, $1/4$, and $1/10$. Shannon did not know the fractions $1/5$ and $1/10$ in this context. Elliot was successful with these fractions. Small chocolate Easter eggs were used to pose problems about an Easter egg hunt for the third task type. Fractions involved were $1/2$, $1/3$, and $3/5$. Shannon did not succeed with $1/3$ or $3/5$. Elliot succeeded with $1/2$ only. Overall Shannon was able to respond appropriately to tasks involving the fractions $1/2$ and $1/4$. Elliot was successful with tasks involving $1/2$.

NOVEL COMPARISON SCHEMES

In the context of comparison tasks given during September 1992, evidence was found for two schemes for dealing with given tasks based on the operations of the Copycat. In the Equal Outputs scheme, different inputs were compared when similar outputs were

This content downloaded from
71.232.212.173 on Sun, 18 Oct 2020 18:51:25 UTC
All use subject to https://about.jstor.org/terms

achieved. For example, a child might imagine the situation where two items placed on the in-tray of a $1/2$ machine produces one item on the out-tray, and four items placed on the in-tray of a $1/4$ machine also produces one item on the out-tray. The Equal Outputs scheme was reinforced by the way the Copycat program worked as it dealt with appropriate inputs. When fractions to be investigated were unit fractions, the Copycat algorithm showed repeated outputs of one item for every group of items corresponding with the denominator, and provided the Copycat would operate on the given input, there would always be an output of one no matter what the input number (or denominator) happened to be. In the simplest cases, involving unit fractions, inferences about the relative sizes of fractions are based on an inverse relationship between size of fraction and size of denominator. As one child explained when arguing why $1/2$ was larger than $1/3$: "If you put three in then only one will come out, but if you put two in then one will come out, but two is smaller than three so it makes it bigger." That two being smaller than three should lead to the conclusion that $1/2$ was larger suggests that this child was drawing on other experiences of quantities and numerical relationships, such as the presence of a part-whole scheme in which a unit partitioned into two gives larger parts than a similar unit partitioned into three (Behr et al., 1984).

A different scheme, called the Equal Inputs scheme, was also used when two fractions were to be compared. In this case the child searched for some number that would make the Copycat work when either of the fractions was selected. A comparison of numbers of output items would then be made. There is a direct relationship between the size of the fraction and the number of output items for that fraction; the larger the number of output items, the greater the fraction. For example, to compare $1/2$ and $1/3$, a child might select six for an input number. For a $1/2$ machine, three items would be output; for a $1/3$ machine, two items would be output.

A third scheme used by Shannon later in 1993, called a Scaling scheme,was not unique to use of the computer program but was used powerfully in a number of comparison tasks presented. Fractions were scaled up or down, using addition or doubling and halving of whole numbers and of fractions. For example, to find a fraction between $2/5$ and $1/3$, Shannon tested

$$\frac{1\frac{3}{4}}{5}$$

by adding $1^3/4$ three times to verify that five-and-one-quarter fifteenths was greater than five fifteenths.

RESULTS FROM THE TEACHING SESSIONS

Results are reported from the 12th teaching session in 1992, when simple pairwise comparison tasks were introduced, and from sessions in 1993, where more challenging comparison tasks were given that involved ordering sets of fractions on flash cards.

The First Set of Comparison Tasks (September 1992)

The first comparison tasks were worksheets consisting of pairs of fractions with the instructions "Use the Copycat to work out which of the two fractions is larger. Write a sentence to explain your reason" (see Figure 3).

This content downloaded from
71.232.212.173 on Sun, 18 Oct 2020 18:51:25 UTC
All use subject to https://about.jstor.org/terms

Shannon (11 September 1992). The first pair of fractions on the first worksheet were 1/2 and 1/3. Shannon considered the task by himself while Elliot used the computer to complete work from a previous session. When Elliot was given the worksheet Shannon moved to the computer. Shannon set the Copycat to 1/2, input two items, clicked the Go button, and observed the Copycat output one item. After setting the machine to 1/3, he input three, clicked the Go button, and observed one item output once again.

Use the Copycat to work out which of the two fractions is larger.
Write a sentence to explain your reason.

$\frac{1}{2}$ $\frac{1}{3}$

$\frac{1}{5}$ $\frac{1}{2}$

$\frac{1}{2}$ $\frac{1}{4}$

Figure 3. Sample worksheet.

T: Okay now, what happened?
Sh: One came out for the one half and one for the one third.
T: Yes, so what does that tell you?
Sh: So, if you put in six you would have two coming out, and if you put in six for the half, you would have three coming out.

Although Shannon notices the equality of outputs as a result of his experiment, without further interaction with the Copycat he switches to the Equal Inputs scheme, choosing six items. However, when the teacher asked Shannon to work out the difference between 1/2 and 1/3, his answer showed that the Equal Inputs scheme was unstable.

Sh: You don't have to use as much bits (items) to make one come for one half as for one third.

One week later Shannon again began an argument based on the Equal Outputs

This content downloaded from
71.232.212.173 on Sun, 18 Oct 2020 18:51:25 UTC
All use subject to https://about.jstor.org/terms

scheme and switched to the Equal Inputs scheme. He had begun a justification for concluding that 1/3 was larger than 1/5:

Sh: There's this—needs, uses up, more bits just to get one with one third. You use up more bits with one third. One third—you only need three to get one bit and with one fifth you have to take up five to get one fifth.

But when the teacher asked: "Why does that make one third larger?" Shannon began to use the Equal Inputs scheme.

Sh: 'Cause every three, like you put in 15, and you get five out.

T: Uh-huh.

Sh: 'Cause you only put in three and leave ... and it gives you one, but if you put in one fifth too, at 15, you only get three out.

Elliot (15 September). To compare 1/2 and 1/3 Elliot had the Copycat set to one third. He input nine and observed three come out. He then reset the machine for 1/2 and input two. He realized the desirability of choosing the same number of items for each fraction. On his worksheet he wrote: "A half would be more because a third takes away more than a half." The teacher asked him to explain.

E: Because, um, so a six, right, then a half of it, that's three, and a third takes away four.

Elliot focused on the differences between corresponding inputs and outputs. His explanation indicates that he used the Equal Inputs scheme. The reference to "a third takes away four" is consistent, for the fraction 1/3, with an input of six leading to an output of two, the resulting difference being four. However, his emphasis on the difference across input and output reflects commonly observed early thinking about ratio that involves additive rather than multiplicative relationships.

Elliot (18 September). Three days later Elliot and Shannon were continuing with a worksheet from the session of 15 September. Elliot announced to the teacher that 1/2 was larger than 1/5.

E: So each five, one comes out, for each two—half comes out—put in 10, then it's going to come out two (pointing to symbol 1/5 on the worksheet), that's going to come out five (pointing to the symbol 1/2 on the worksheet).

Elliot described the behavior of the Copycat. Although he knew that an input of five items would result in success for a 1/5 machine, and likewise an input of two items for a 1/2 machine, he also knew that an input of ten items would have different consequences for each of these fractions. In the task following, involving comparison of 1/3 and 1/5, Elliot knew three input items for a 1/3 machine would result in a single output, and that five input items for a 1/5 machine would also output one item. However, for this task he was unable to use the Equal Outputs scheme to reach a conclusion.

Later in the same session Elliot reconsidered the fractions 1/3 and 1/5 at the computer. He engaged in discussion with two of the investigators:

E: They're the same. If you put in six, two will come out; if you put in ten, two will come out.

T1: Uh-huh. But up here (pointing to previous task involving 1/2 and 1/4), you said you put four in for this and four in for that.

E: (Sits silently for a few seconds.)

This content downloaded from
71.232.212.173 on Sun, 18 Oct 2020 18:51:25 UTC
All use subject to https://about.jstor.org/terms

T1: You put the same in for each. Is there a number that this fraction will work on and this fraction (pointing) will work on?

E: No.

T1: Uh-huh.

T2: Are there some numbers you could experiment with?

E: (Puts five into $^1/_5$ machine—machine works).

T1: Okay, five makes that work, but will five make a $^1/_3$ machine work?

E: No.

T1: All right, is there another number that can make a $^1/_5$ machine work?

E: (Resets Copycat to $^1/_3$. Inputs five, clicks on Go. The Copycat explodes.)

T2: What other numbers can you try, Elliot?

T1: What numbers make the $^1/_3$ machine work?

E: Nine.

T1: Mmm. What other numbers?

E: Six.

T1: Yeah, what's going to come out for six?

E: Two.

T1: Okay, what other numbers make the $^1/_3$ machine work?

E: Twelve.

T1: What's going to come out?

E: Um, four.

T1: Uh-huh. All right. The question is, is there any number that makes the $^1/_3$ machine work, and also makes the $^1/_5$ machine work?

Elliot did not respond to this question. Shannon was invited to take a turn at the computer.

T1: Explain it to him (Elliot moves over). Big voice so everyone can hear.

Sh: (Machine is set to $^1/_3$. Inputs 15.)

E: Oh, yeah.

Sh: (Clicks mouse on Go button. Five items are output.)

E: Oh, yeah, three will come out of the fifth machine.

Elliot was able to use his whole-number knowledge to anticipate the output number for an input of 15 when the machine was set for $^1/_5$. But he needed assistance to consider 15 as a possibility. In contrast, Shannon showed that he could envisage other possibilities as input numbers:

Sh: Could have 30, too.

T: Okay, so. All right? Mmm. Good.

E: Thirty could work as well.

T: How do you know that?

Sh: Yeah, and 45, and 60, and 75 and 80 (puts hand up to face), not 80.

Elliot was less successful than Shannon at coordinating whole-number sequences to solve fraction comparison tasks. In the initial interviews Elliot was successful with counting tasks involving numerical sequences but demonstrated rote-like responses. His lack of depth of understanding prevented him from succeeding with the second task.

This content downloaded from
71.232.212.173 on Sun, 18 Oct 2020 18:51:25 UTC
All use subject to https://about.jstor.org/terms

During these sessions in September 1992 we observed Shannon and Elliot use two basic schemes to deal with comparison problems involving two fractions. The Equal Outputs scheme was based on having the Copycat respond to different numbers of input items. It is a more fundamental scheme because it does not require the coordination of numerical sequences between different fractions. On several occasions Shannon switched to the Equal Inputs scheme to respond to given tasks. Elliot also used this scheme. The Equal Inputs scheme requires a sound knowledge of whole number relationships—principally, strategies to find numbers divisible by each fraction's denominator. A question of interest is just how a successful student might determine the "right" number when applying the Equal Inputs scheme.

The Second Set of Comparison Tasks (May, June, and August 1993)

During teaching sessions conducted in May, June, and August 1993, a different type of comparison task was given to the children. Sets of fraction flash cards were administered, and the children were asked to place the cards in order from largest to smallest. Initially these cards were administered one at a time, so that the teacher had control over which card to reveal next, according to the previous cards placed out. Subsequently a set of five or six cards together would be given for the children to sort. As the children gained in proficiency, a more challenging task was given: to find a fraction between two given fractions. In these teaching sessions the Copycat was available. However, Shannon and Elliot rarely used it.

Shannon (11 May 1993). The first 1993 session with Shannon was an individual interview. After some preliminary discussion about what numbers would make a Copycat go that was set to the fractions 1/3, 2/3, and 4/3, Shannon was asked to compare the fractions 2/4 and 2/3. The Copycat was available on the table. The teacher wrote the two fractions on a sheet of paper in front of Shannon and asked, "What's the difference between those two fractions? Is one larger than the other, or are they both the same, or what? If one is larger, how much bigger is it than the other?" Shannon responded, saying that 2/4 was larger.

Sh: (Points to 2/4) That one takes more in than the other one.

T: Which one's that?

Sh: Two fourths.

T: Mmm.

Sh: That one takes in four blocks (pointing to 2/4), and that one takes in three (pointing to 2/3).

Shannon maintained that 2/4 was one larger than 2/3. For the Equal Outputs scheme to work Shannon had to remember that the greater an input number, the smaller the fraction. But on this occasion Shannon did not seem to be aware of the need to make this compensation. At this point the observer-researcher (represented by O in the transcript) interjected by asking, "What number could you put in that would ... that you could use with the 2/4 machine and the 2/3 machine?" Shannon's almost instant reply was 12.

O: Why did you choose 12?

This content downloaded from
71.232.212.173 on Sun, 18 Oct 2020 18:51:25 UTC
All use subject to https://about.jstor.org/terms

Sh: Because four threes are 12 and three fours are 12.

O: What will come out of which one?

Sh: Six will come out of $2/4$. Eight will come out of the $2/3$.

T: Ah. Okeydoke. But you told me before that $2/4$ was the largest fraction.

Sh: (Points to $2/4$ on the sheet of paper) Only that it looks like it.

Elliot was given the same comparison task in a separate interview. He also said $2/4$ was larger. When asked to check his answer with the Copycat, he tried to input six items. This number failed for $2/4$. It was suggested that he make a list of inputs and outputs that made each of the machines work. This he did in a systematic fashion by adding on lots of three (in the case of $2/3$), based on previously identified inputs, and lots of four (in the case of $2/4$). But he was not able to detach himself from the detail of each list he had constructed in order to search for a common input from each list.

Elliot (14 May 1993). Insight was gained into how Elliot determined the number of inputs in a discussion about what he had written when comparing $1/2$ and $1/4$. He said that $1/2$ was larger "because if you put 12 in six comes out and on the other one ($1/4$) three comes out.

T: Is there a smaller number you could use for both machines?

E: Eight.

T: Is there a smaller number?

E: No.

T: Can you think of a larger number that you could use with both of them?

E: Than 12?

T: Mm.

E: Um, let's see. Let's see, um, 16, 20, 24, 28, 32, 36,...

T: What numbers are they?

E: Just add on four. 'Cause two goes into four.

Elliot understood that whatever input numbers worked for $1/4$ would also work for $1/2$. The fact that he was able to generate a list of multiples of four indicates his familiarity with this sequence. We do not know why he chose 12 to begin with. We could speculate that Elliot knew on the basis of experience that 12 worked as an input number for many fractions, and he therefore decided to choose the most likely candidate. Shannon was also able to exhibit many possibilities for input numbers. When comparing $3/4$ and $6/8$ he said, "If you did 40, they would be both the same." After being asked whether there would be a smaller number, Shannon offered 24, 16, and 8 as candidates.

Elliot (18 May 1993). On this occasion the flexibility of the Equal Inputs scheme was put to the test, because more than two fractions were to be compared. The teacher introduced fractions on flash cards one by one and asked Elliot and Sukey to place them in order from largest to smallest on the desk top in front of them. Elliot was ebullient and dominant. Sukey was, by nature and temperament, quiet and passive. The order of presentation was as follows: $1/2$, $1/4$, $1/5$, $1/3$, $2/6$, $1/10$, $5/10$, $3/6$, $2/3$. Elliot had noticed the relationship between size of fraction and denominator. In his words, "If the number down the bottom is bigger then it would probably be the smaller

This content downloaded from
71.232.212.173 on Sun, 18 Oct 2020 18:51:25 UTC
All use subject to https://about.jstor.org/terms

one." This relationship helped Elliot and Sukey to sort the first three flash cards. The teacher then introduced 1/3. After considering the situation a minute, Elliot placed the 1/3 card between 1/4 and 1/5.

T: All right, give us a reason.

E: Because if you put 12 in, on this one three will come out (pointing to 1/4), and if you put 12 in that one (pointing to 1/3) four will come out.

T: What happens with this one (points to 1/5) when you put 12 in?

Su: It's not dividing.

E: It won't work.

T: Uh-huh. Mmm. What do you think, Sukey. Is he right?

Su: (Nods head.)

T: Okay, so you're telling me this is the largest fraction, one half (points to 1/2 and both nod), this is the next largest (points to 1/4 and both nod), this is the next largest (points to 1/3). You told me that this fraction (1/3) is smaller than this fraction (1/4).

Su: (Silence.)

E: (Silence.)

T: One third is smaller than one fourth.

E: Oh, I forgot to turn them around (switches order to 1/2, 1/3, 1/4, 1/5) two, three, four, five.

Elliot used the Equal Inputs scheme to determine the order of 1/3 and 1/4. But the relationship between 1/4 and 1/5 was untested by this means. Rather, the pattern of the increasing order of denominators seemed to satisfy these children. Elliot was able to confirm that 1/3 and 2/6 were equivalent on the basis of the number 12 as an input. He decided that 1/10 was smaller than 1/5 by applying 10 in the Equal Inputs scheme. From this point on in the session, Elliot was able to make pairwise comparisons using different numbers as inputs. For example, he sorted 1/10 and 3/6, having chosen 30; determined that 3/6 and 5/10 were equivalent, also using 30; and ordered 1/4 and 3/6, using 12. As he considered the order of 3/6, Elliot said,

> Is it bigger than this one (5/10)? Let's see, 10, 5, 12, 6,... 30, 30, I think that one's larger, let's see (whispers), 15 will come out. They're the same, because if you put 30, 15 will come out (pointing to 3/6) and if you put 30, 15 will come out as well (pointing to 5/10).

His utterances 10, 5, 12, 6,... 30, 30 are a possible record of his search for a satisfactory input that would work for both fractions. It is likely that he mentally tested 10 as input in a 5/10 machine, giving five, then tested 12 as a possible input in a 3/6 machine. How he settled on 30 is not known. By the time the sequence had been determined, Elliot's Equal Inputs scheme had been extended into a more encompassing scheme in which the Equal Inputs scheme had reached a degree of adaptability so that it was applied repeatedly, often requiring different numbers for each pairwise comparison in order to make a decision. A key hurdle for each comparison was the choice of an appropriate input number. An indication of one way Elliot knew how to approach this problem was observed 3 days later.

Elliot (21 May 1993). Elliot, Alisha, and Sukey worked together sorting out flash cards with fractions such as 2/4, 2/6, 3/4, 2/3, 1/6, and 1/8 written on them. In considering the order of the fractions 3/4 and 2/3, Elliot was observed to say,

This content downloaded from
71.232.212.173 on Sun, 18 Oct 2020 18:51:25 UTC
All use subject to https://about.jstor.org/terms

What's the number three goes into and four? Let's see, three times four is 12, so ... (looks at $2/3$) 12 goes in, eight will come out. What about the other (picks up flash card with $3/4$), if you put 12 in, nine will come out (places $3/4$ ahead of $2/3$ on the table).

Elliot found that by multiplying denominators he could determine a number that would serve him as he implemented the Equal Inputs scheme. This reasoning did not become routine for determining a common input number for Elliot. Immediately after this moment he used 20 to compare $3/4$ and $2/10$ and 10 to compare $1/2$ and $2/10$. Elliot's approach was much more an intuitive trial-and-error test, as his approach to comparing $2/6$ and $1/5$ showed:

E: Two sixths. Six goes into 20, no 30.

It is fair to claim that Elliot tried 20 to begin with because it was a multiple of five but had to reconsider when he found 20 was not a multiple of six. It is consistent with previous behavior that Elliot possibly counted on by fives until a number he knew to be a multiple of six was reached.

Shannon (22 June 1993). During an earlier teaching session on 18 June, Shannon was asked to compare the following set of fraction flash cards: $7/4$, $8/8$, $10/5$, and $6/3$. He was able to do this task without any difficulty. The teacher then presented a blank flash card and asked Shannon to write a fraction that would go between $7/4$ and $8/8$. He wrote

$$\frac{7\frac{1}{2}}{8}.$$

At the next session, on 22 June, after reviewing the set of fractions that were presented on 18 June, Shannon was once again asked to find a fraction between $7/4$ and $8/8$.

Sh: There's two.
T: Just one will do for a start.
Sh: (Writes $1^1/4$.)
T: What is $1^1/4$ as a fraction without a whole number?
Sh: (Writes $6/4$.)
T: Is there a fraction between $6/4$ and $8/8$ (puts down blank card) ?
Sh: (Writes $5/4$.)
T: Is there a fraction between $5/4$ and $8/8$?
Sh: (Writes $1^5/8$.)
T: How many eighths is that?
Sh: That's one (changes to $1^1/8$).
T: How many eighths is $1^1/8$?
Sh: Nine eighths.
T: Nine eighths. Good. That's good. So let's see. Nine eighths is larger than $8/8$, but is it smaller than $5/4$? Can you explain how it is?
Sh: This would be $10/8$, because you have to double the top number and you have to double the bottom number.
T: That's good. Very good. So what goes there (points to space between $8/8$ and $9/8$)? Is there a fraction between those two (places down another blank card)?
Sh: (Writes $1^1/16$.)

This content downloaded from
71.232.212.173 on Sun, 18 Oct 2020 18:51:25 UTC
All use subject to https://about.jstor.org/terms

T: Okay, that's very good, that's very good.
O: Has Mum been helping you?
Sh: No.
O: Has Nathalia (older sister) been helping you?
Sh: No.

Shannon's first two responses indicate that he knew two solutions to the problem—$\frac{5}{4}$ and $\frac{6}{4}$. He also showed that he knew $\frac{8}{8}$ was equivalent to one, by writing $\frac{5}{4}$ as $1\frac{1}{4}$ and by renaming $1\frac{1}{8}$ as $\frac{9}{8}$. His subsequent responses of $1\frac{1}{8}$ and $1\frac{1}{16}$ indicate a knowledge of the relative magnitudes of quarters, eighths, and sixteenths. His identification of the fraction $\frac{5}{4}$ as $\frac{10}{8}$ was the first indication of what we call a Scaling scheme. Shannon initially scaled up or down, as he did here, by doubling or halving. He later used this scheme with impressive results in other tasks requiring the generation of fractions between given fractions.

Elliot and Shannon (25 August 1993). On 6 August, Shannon was asked if there was a fraction between $\frac{1}{2}$ and $\frac{1}{3}$. He required some assistance. Shannon agreed with $\frac{2}{5}$ as the teacher's suggestion, after considering the related problem of finding a fraction between $\frac{2}{4}$ and $\frac{2}{6}$ (see Figure 4). He wasn't sure if there was another fraction different from $\frac{2}{5}$ between $\frac{1}{2}$ and $\frac{1}{3}$. On 25 August Elliot and Shannon worked together. The teacher returned to the question of finding another fraction between $\frac{1}{2}$ and $\frac{1}{3}$.

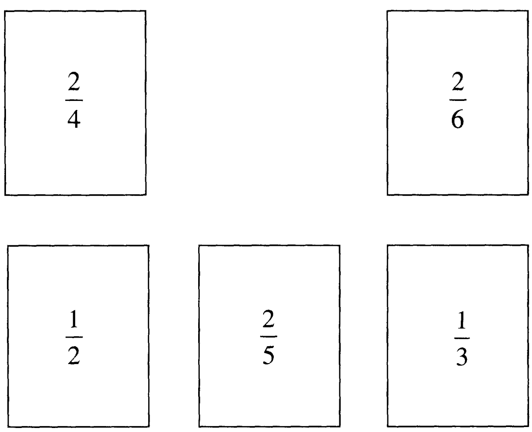

Figure 4. Finding a fraction between one half and one third.

T: Is there another fraction not equivalent to $\frac{2}{5}$ that would go between $\frac{1}{2}$ and $\frac{1}{3}$?
Sh: Two-and-a-quarter fifths.
T: Two-and-a-quarter fifths.
E: That would be four-and-a-half tenths.
T: Keep going.

This content downloaded from
71.232.212.173 on Sun, 18 Oct 2020 18:51:25 UTC
All use subject to https://about.jstor.org/terms

E: That would be nine...
Sh & E: (Simultaneously) nine twentieths.

Discussion followed about on which side of the $^2/_5$ flash card $^9/_{20}$ would lie.

E: No wait, wait (points to the $^2/_5$ card). Four times five is 20, 20 goes in, eight comes out.
Sh: That (points to $^1/_2$ card) would give you 10; that (points to $^9/_{20}$) would give you nine.
E: Yeah, all right, yeah, and that will give you eight (pointing to $^2/_5$), and that will give you s—(points to $^1/_3$). No, that can't go in.

The teacher interrupted to ask if there was another fraction between $^2/_5$ and $^1/_3$. Shannon proposed one-and-three-quarter fifths. By adding $1^3/_4$ three times, Shannon was able to conclude that five-and-one-quarter fifteenths was indeed greater than five fifteenths. This was a further development of a Scaling scheme identified previously. He went on to use this scheme to show

$$\frac{1\frac{3}{4}}{5}$$

to be equivalent to $^7/_{20}$.

T: Can you explain to Elliot how it works?
Sh: 'Cause you're using up four one-and-three-quarter fifths, which is equaling up to seven, and then four plus twenty, so $^7/_{20}$.

The next fraction to be considered was $^3/_{10}$. Shannon's approach was to convert it to fifths. Elliot was able to use the Equal Inputs scheme to locate it behind $^1/_3$. This was a rare occasion when Elliot was quicker to find a solution than Shannon.

T: Where does $^3/_{10}$ go?
E: (Writes it down on a flash card.)
T: Does it go on top of these (indicating $^1/_3$ and $^2/_6$)?
Sh: It's one-and-one-half fifths.
E: It's smaller, Shannon (places $^3/_{10}$ behind $^1/_3$).
Sh: How?
E: Because if you put 30 in this one ($^3/_{10}$) nine will come out, and if you put 30 in this one ($^1/_3$) 10 will come out.

The session began by considering a fraction different from $^2/_5$ between $^1/_2$ and $^1/_3$. These children were able to construct and locate fractions $^7/_{20}$ and $^9/_{20}$ in relation to given fractions and, in addition, locate the fraction $^3/_{10}$ correctly (see Figure 5). This impressive performance was due to the appropriate use of two schemes: a Scaling scheme, whereby fractions were scaled up or down according to need, mainly using doubling or halving of whole numbers and of fractions, and the Equal Inputs scheme, based on the behavior of the Copycat. The Scaling scheme was a tool that Shannon used very effectively. It is not known what experiences were responsible for its genesis, although there were numerous tasks in the teaching experiment that provided opportunities for its exercise and development.

DISCUSSION

The purpose of the teaching experiment was to investigate children's fraction learning and the role their whole-number knowledge might play in it. A major, though

This content downloaded from
71.232.212.173 on Sun, 18 Oct 2020 18:51:25 UTC
All use subject to https://about.jstor.org/terms

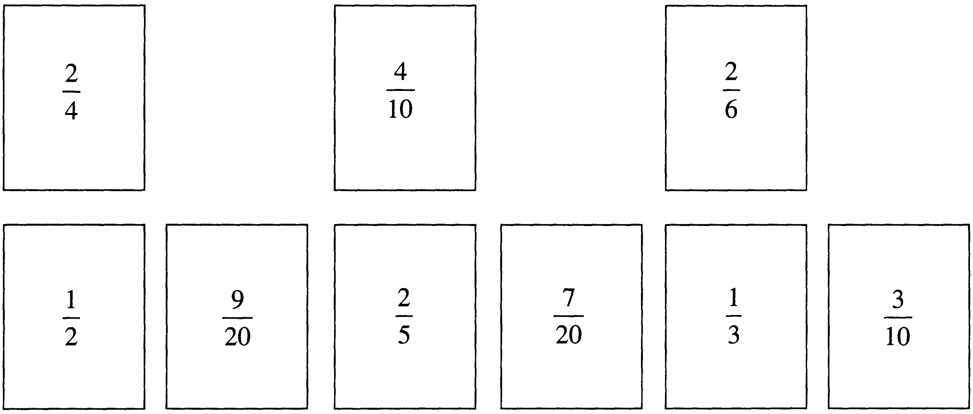

Figure 5. Finding other fractions between one half and one third.

not unique, feature of the teaching sessions was the use of a computer-based learning setting called Copycat. The Copycat program instantiated an "operator" interpretation of rational numbers and was purposely chosen to activate the students' knowledge of whole numbers. Data reported here traces the development, over a 12-month period, of two $8\frac{1}{2}$–$9\frac{1}{2}$-year-old students' deepening understanding of rational numbers as they engaged in problems of fraction comparison and equivalence. Although these students were initially encouraged to use the computer tool to help them solve given tasks (indeed, the computer was available for them to use at all times), they primarily relied on their own knowledge of whole-number relationships to make decisions in relation to the fractions being investigated. In 1992, the comparison tasks consisted of pairs of unit fractions. During 1993, the tasks became more complex, with sets of up to seven or eight fraction flash cards to be ordered. These included equivalents of one half, thirds, fourths, whole-number equivalents such as $\frac{6}{3}$ and $\frac{3}{1}$, and so-called improper fractions such as $\frac{7}{4}$. The most challenging type of task was to find a fraction between two given fractions. Almost all comparison tasks were nonstandard problems that would not be found in conventional elementary school mathematics textbooks. The flash card tasks were challenging, and the students were enthusiastic in their efforts to order them.

We have identified three schemes that Elliot and Shannon used effectively to complete the given tasks. The Equal Outputs and Equal Inputs schemes were generalizations of Copycat behavior observed by Elliot and Shannon. The Scaling scheme was an innovation of Shannon's. In the Equal Outputs scheme there is an expectation that different inputs for different fractions will lead to identical outputs. An appropriate conclusion about the relative sizes of two fractions requires a coordination between the sizes of the fractions being compared and either the sizes of the corresponding inputs or the differences between each input and its corresponding

This content downloaded from
71.232.212.173 on Sun, 18 Oct 2020 18:51:25 UTC
All use subject to https://about.jstor.org/terms

output. This amounts to the construction of an inverse relation between fraction size and input. In 1992 evidence for the Equal Outputs scheme was found in the early work of Elliot and Shannon and also in the first session in 1993. The Equal Outputs scheme came to be replaced by the Equal Inputs scheme. The Equal Inputs scheme has as its primary goal the identification and selection of a number that makes the Copycat "Go" for each fraction. What is needed is a number that is a common multiple of the denominators of the two fractions. Observed strategies for selecting an input candidate included remembering and comparing lists of multiples of each denominator. There were numbers, which by nature of their prime factorization, seemed regularly to lead to success—for example, 12, 20, 24, and 30. Numbers such as these were often tried first. Elliot was observed to count with units corresponding to a denominator of interest. In this way he would search for a common multiple. Often he would truncate the counting sequence by not commencing at the beginning. Or he would choose a number he knew to be divisible by one of the denominators and check if the other denominator would divide it. If not, he would proceed further in the number sequence.

It is of interest to ask how the Equal Outputs scheme came to be overtaken by the Equal Inputs scheme. As we have already said, the Equal Outputs scheme was attractive in the earlier stages of the tasks because for comparisons where both fractions had numerators of one, an input corresponding to a denominator would automatically produce equal outputs of precisely one. Also, the process by which the Copycat went about operating on its input—its algorithm—reinforced the notion of one for each number corresponding to the denominator. In the case where composite (nonunit) fractions are to be compared, the choice of input is not a simple matter, and if the denominators are chosen, equal outputs will not result. The Equal Inputs scheme will be more effectively applied in a situation to the extent that the student has a facility with the relevant whole-number sequences. As a student's knowledge of whole-number sequences gathers strength and consolidates, leading to greater efficiency in identifying an input candidate, the Equal Inputs scheme becomes more attractive. Once an appropriate input has been selected, there is a direct relationship between the magnitude of the output and the magnitude of the fraction. The investigators may have implicitly encouraged use of the Equal Inputs scheme by asking students for justifications when the Equal Outputs scheme was used but not when the Equal Inputs scheme was used. Asking for a justification can be interpreted by a student as an indication that they have given an incorrect or unacceptable response. A reason for soliciting justifications upon evidence of the Equal Outputs scheme was that this behavior was something of a novelty for the investigators.

Shannon used a Scaling scheme to find fractions between given fractions. His general approach was to take a known fraction—usually one of the boundary fractions—and create a new fraction by reducing or increasing the numerator a small fractional amount. For example, on 25 August he constructed $9/20$ between $1/2$ and $2/5$ by creating the fraction

$$\frac{2\frac{1}{4}}{5},$$

This content downloaded from
71.232.212.173 on Sun, 18 Oct 2020 18:51:25 UTC
All use subject to https://about.jstor.org/terms

which, with Elliot's assistance he scaled up to

$$\frac{4\frac{1}{2}}{10},$$

then 9/20. It is not known how Shannon came to construct this scheme. There was no discussion related to this kind of possibility in any of the teaching sessions. He was asked by one of the investigators if he had been taught about this at home, which he denied. He did have older sisters with whom he may have discussed tasks from the teaching experiment. However, there were numerous opportunities for Shannon to observe the equivalence of such fractions as 1/2, 2/4, 3/6, and 5/10. He may well have noticed the essential similarity between these symbols and the behavior of the Copycat when set to "behave" in accordance with these fractions. The impressive feature of Shannon's scheme was not that he understood that equivalent fractions were related by a scaleable property but that he could make this knowledge work for him in a powerful way. It is of interest that these students did not have any difficulty writing or interpreting complex expressions such as

$$\frac{2\frac{1}{4}}{5}.$$

Elliot and Shannon were two of the most adept students in this teaching experiment. They had different personalities and approaches to solving the problems we gave them (Davis, Hunting, & Pearn, 1993). But they—Shannon more so—had a superior grasp of whole-number relationships compared with many of the other children in the experiment. Not only did Elliot and Shannon have automatic recall of a plenitude of multiplication and division facts, they also had well-developed thinking strategies for relating numbers and quantities. Of particular importance was their knowledge and application of sequences of composite units—such as sequences of threes, fours, and so on. We claim that the types of tasks we presented during the course of the experiment assisted these children in refining and consolidating their knowledge of whole numbers and their interrelationships.

There are differing views on how rational numbers should be taught. One view holds that a number of interpretations of rational numbers should be taught, beginning from the earliest experiences, to encourage learners to translate between and within representations (Behr et al., 1984; Post, Cramer, Behr, Lesh, & Harel, 1993). Streefland (1991) distinguished three levels—between physical objects requiring division and measurement, visual models and magnitudes, and a formal level of systematic subject matter with its rules and laws. He proposed that instruction integrate work at each of these levels. Another view favors teaching one interpretation in some depth before broadening to introduce other interpretations. Our approach was consistent with the work of Mack (1990, 1993), who suggested that rather than attempting a broad-based approach to developing understanding of rational numbers, a viable alternative may be to develop a particular conception, then expand that conception once students can relate mathematical procedures and their symbols to their informal knowledge. Our project's emphasis on an operator-ratio approach is supported by other recent work of Resnick and Singer (1993) and

This content downloaded from
71.232.212.173 on Sun, 18 Oct 2020 18:51:25 UTC
All use subject to https://about.jstor.org/terms

Lamon (1993a, 1993b), who have investigated the development of children's notions of ratio. Results reported here provide some support for the "one interpretation" approach. Certainly the operator-like Copycat setting allowed these students to develop powerful reasoning that enabled them to solve challenging problems cast in this setting. The platform for this reasoning was whole-number knowledge. We do not claim that operator or ratio embodiments of rational numbers are the only way. Measurement and partition-based approaches using continuous quantities may be equally viable alternatives. The children in this teaching experiment certainly had some measurement- and partition-based understandings of fractions, which we deliberately did not attempt to foster. One possible drawback of the Copycat is its rejection of inputs not divisible by the denominator. For example, one third of seven cannot be done. How students taught with this program would deal with such a task is an open question. This study demonstrates that students can solve challenging tasks involving the order and comparison of rational numbers provided that appropriate whole-number knowledge and strategies are available to them. The operator-like Copycat software was an important tool used to stimulate and harness knowledge of whole-number relationships.

Regardless of whether the "one interpretation" approach is best, data from this study offer an alternative view of the role that whole-number knowledge might play in rational-number learning. We have noted other research that has viewed the influence of whole-number knowledge as debilitating or an interference. A reason for this may be that there are valid physical settings and contexts involving quantities that do not draw on whole-number knowledge in any goal-directed way. Initial Copycat tasks invited children to observe and investigate a considerable number of numerical relationships that came to be associated with fractions. Some numerical relationships would have been known from other contexts; others were refined and extended in the process of investigating a Copycat problem. Later, after 12 sessions, the children began to investigate comparison tasks with a background of whole-number experience that had become purposefully associated with Copycat phenomena. Premature introduction to symbolic notation before adequate opportunities for experiential foundations have been laid courts inappropriate interpretation. We avoided emphasizing formal fraction notation initially, preferring to refer to the operators verbally. Although Copycat displayed the fraction of interest (if it was not hidden by a screen), no explicit attention was drawn to it. The computer setting allowed the children to consider relationships between inputs and outputs from the perspective of their whole-number experience. Another cause of whole-number interference is the introduction of an unfamiliar task. Children often return to naive strategies in the face of a new level of difficulty. Even Shannon (11 May) was observed to lapse in spite of our efforts to avoid the problem. In that case it was possible to explicitly invoke the Equal Inputs scheme to set up a contradictory situation.

This study underscores the importance of children having a firm foundation in whole-number strategies and relationships established in the early years of schooling. Other children in this teaching experiment were given similar tasks and problems initially. Their ability to succeed with those tasks and problems was constrained

This content downloaded from
71.232.212.173 on Sun, 18 Oct 2020 18:51:25 UTC
All use subject to https://about.jstor.org/terms

by their facility with, and knowledge of, whole numbers and their relationships. The kinds of rational number tasks and problems that were given in this study had a positive effect of stimulating growth and exercising existing whole-number knowledge. Although it is true to say that to succeed in the given tasks a foundation of whole-number knowledge was needed, it is equally true that the tasks stimulated and encouraged further facility with whole numbers and their relationships. Thus, for all children who undertook the teaching experiment, their knowledge of whole numbers was strengthened and improved as a result of the experiences they had. This in turn allowed them to experience greater success with rational-number tasks.

In conclusion, we learned from this study that elementary students' whole-number knowledge could be tapped in a positive way to teach basic fraction concepts. Further, development of rational-number knowledge and whole-number knowledge was interdependent. Rational-number tasks in operator settings such as that used with Copycat helped stimulate and extend children's whole-number knowledge. Fraction settings provided by the Copycat program allowed students to investigate problems of order and equivalence meaningfully by drawing on and expanding their prior knowledge of whole-number relationships. Novel cognitive schemes, which reorganized the students' whole-number knowledge in new ways, were used to solve fraction comparison tasks. These schemes were an Equal Outputs scheme, limited in effectiveness to comparisons of unit fractions; an Equal Inputs scheme that activated strategies for determining a common multiple; and a Scaling scheme in which fractions were scaled up or down using ratios. The operator interpretation of rational numbers would seem to be a viable approach to teaching basic fraction concepts and might be used to complement other interpretations such as part-whole and measures.

REFERENCES

Behr, M. J., Wachsmuth, I., Post, T., & Lesh, R. (1984). Order and equivalence of rational numbers: A clinical teaching experiment. *Journal for Research in Mathematics Education, 15,* 323–341.

Behr, M. J., Harel, G., Post, T., & Lesh, R. (1992). Rational number, ratio, and proportion. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 296–333). New York: Macmillan.

Behr, M. J., Harel, G., Post, T., & Lesh, R. (1993). Rational numbers: Toward a semantic analysis—emphasis on the operator construct. In T. P. Carpenter, E. Fennema, & T. A. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 13–47). Hillsdale, NJ: Erlbaum.

Braunfeld, P., & Wolfe, M. (1966). Fractions for low achievers. *Arithmetic Teacher, 13,* 647–655.

Carpenter, T. P. (1985). Learning to add and subtract: An exercise in problem solving. In E. A. Silver (Ed.), *Teaching and learning mathematical problem solving: Multiple research perspectives* (pp. 17–40). Hillsdale, NJ: Erlbaum.

Carpenter, T. P., Fennema, E., & Romberg, T. A. (1993). Toward a unified discipline of scientific inquiry. In T. P. Carpenter, E. Fennema, & T. A. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 1–11). Hillsdale, NJ: Erlbaum.

Carpenter, T. P., & Moser, J. M. (1982). The development of addition and subtraction problem solving skills. In T. P. Carpenter, J. M. Moser, & T. A. Romberg (Eds.), *Addition and subtraction: A cognitive perspective* (pp. 9–24). New York: Academic Press.

Carraher, D. W. (1993, January). *Building rational number concepts upon students' prior understanding.* Paper presented at the International Conference on Rational Number Learning, University of Georgia, Athens, GA.

Charles, R., & Silver, E. (1989). *The teaching and assessing of mathematical problem solving.* Reston, VA: National Council of Teachers of Mathematics.

This content downloaded from
71.232.212.173 on Sun, 18 Oct 2020 18:51:25 UTC
All use subject to https://about.jstor.org/terms

Cobb, P., Wood, T., & Yackel, E. (1990). Classrooms as learning environments for teachers and researchers. In R. B. Davis, C. A. Maher, & N. Noddings (Eds.), *Constructivist views on the teaching and learning of mathematics. Journal for Research in Mathematics Education* Monograph Series, Number 6 (pp. 125–146). Reston, VA: National Council of Teachers of Mathematics.

Cobb, P., Perlwitz, M., & Underwood, D. (1994). Individual construction, mathematical acculturation, and the classroom community. *Revue des Sciences de l'Education, 28*, 41–62.

Cobb, P., & Steffe, L. P. (1983). The constructivist researcher as teacher and model builder. *Journal for Research in Mathematics Education, 14*, 83–94.

Confrey, J. (1988). Multiplication and splitting: Their role in understanding exponential functions. *Proceedings of the Tenth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Northern Illinois University: DeKalb, IL.

Davis, G. (1991). Fractions as operators and as cloning machines. In R. P. Hunting & G. Davis (Eds.), *Early fraction learning* (pp. 91–102). New York: Springer-Verlag.

Davis, G., & Hunting, R. P. (1991). On clinical methods for studying young children's mathematics. In R. P. Hunting & G. Davis (Eds.), *Early fraction learning* (pp. 209–224). New York: Springer-Verlag.

Davis, G., Hunting, R. P., & Pearn, C. (1993). What might a fraction mean to a child and how would a teacher know? *Journal of Mathematical Behavior, 12*, 63–76.

Dienes, Z. P. (1967). *Fractions: An operational approach.* Harlow, Essex, Great Britain: Educational Supply Association.

Freudenthal, H. (1983). *Didactical phenomenology of mathematical structures.* Dordrecht, Netherlands: Reidel.

Fuson, K. (1988). *Children's counting and concepts of number.* New York: Springer-Verlag.

Gray, E. M. (1993, January). *The transition from whole number to fraction: The need for flexible thinking.* Paper presented at the International Conference on Rational Number Learning, University of Georgia, Athens, GA.

Hart, K. (1981). *Children's understanding of mathematics* 11–16. London: Murray.

Hilton, P. (1983). Do we still need to teach fractions? In M. Zweng (Ed.), *Proceedings of the Fourth International Congress on Mathematical Education.* Boston: Birkhauser.

Hunting, R. P. (1983). Emerging methodologies for understanding internal processes governing children's mathematical behaviour. *Australian Journal of Education, 27*(1), 45–61.

Hunting, R. P. (1986). Rachel's schemes for constructing fraction knowledge. *Educational Studies in Mathematics, 17*, 49–66.

Hunting, R. P., & Davis, G. (Eds.). (1991). *Early fraction learning.* New York: Springer-Verlag.

Hunting, R. P., Davis, G., & Bigelow, J. C. (1991). Higher order thinking in young children's engagements with a fraction machine. In R. P. Hunting, & G. Davis (Eds.), *Early fraction learning* (pp. 73–90). New York: Springer-Verlag.

Hunting, R. P., & Korbosky, R. K. (1990). Context and process in fraction learning. *International Journal of Mathematical Education in Science and Technology, 21*, 929–948.

Kaput, J. (1985). *Multiplicative word problems and intensive quantities: An integrated software response* (Technical Report). Harvard Graduate School of Education, Educational Technology Center.

Kieren, T. (1976). On the mathematical, cognitive, and instructional foundations of rational numbers. In R. Lesh (Ed.), *Number and measurement: Papers from a research workshop* (pp. 101–144). Columbus, OH: ERIC/SMEAC.

Kieren, T. E. (1988). Personal knowledge of rational numbers—Its intuitive and formal development. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 162–181). Hillsdale, NJ: Erlbaum.

Kieren, T. E. (1993). Rational and fractional numbers: From quotient fields to recursive understanding. In T. P. Carpenter, E. Fennema, & T. A. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 49–84). Hillsdale, NJ: Erlbaum.

Kieren, T., & Nelson, D. (1978). The operator construct of rational numbers in childhood and adolescence—An exploratory study. *Alberta Journal of Educational Research, 24*, 22–30.

Kieren, T., & Southwell, B. (1979). The development in children and adolescents of the construct of rational numbers as operators. *Alberta Journal of Educational Research, 25*, 234–247.

Lamon, S. J. (1993a). Ratio and proportion: Children's cognitive and metacognitive processes. In T. P. Carpenter, E. Fennema, & T. A. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 131–156). Hillsdale, NJ: Erlbaum.

This content downloaded from
71.232.212.173 on Sun, 18 Oct 2020 18:51:25 UTC
All use subject to https://about.jstor.org/terms

Lamon, S. J. (1993b). Ratio and proportion: Connecting content and children's thinking. *Journal for Research in Mathematics Education, 24*, 41–61.

Mack, N. K. (1990). Learning fractions with understanding: Building on informal knowledge. *Journal for Research in Mathematics Education, 21*, 16–32.

Mack, N. K., (1993). Learning rational numbers with understanding: The case of informal knowledge. In T. P. Carpenter, E. Fennema, & T. A. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 85–106). Hillsdale, NJ: Erlbaum.

Opper, S. (1977). Piaget's clinical method. *Journal of Children's Mathematical Behavior, 1*, 90–107.

Piaget, J. (1970). *Genetic epistemology.* (E. Duckworth, Trans.). New York: Columbia University Press.

Piaget, J. (1972). *The principles of genetic epistemology.* London: Routledge & Kegan Paul.

Pickert, G. (1968). Die Bruchrechnung als Operieren mit Abbildungen. [Treatment of elementary fractions by means of mappings.] *Mathematisch-Physikalische Semesterberichte, Band XV*, 32–47.

Post, T. R., Cramer, K. A., Behr, M., Lesh, R., & Harel, G. (1993). Curriculum implications of research on the learning, teaching and assessing of rational number concepts. In T. P. Carpenter, E. Fennema, & T. A. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 327–362). Hillsdale, NJ: Erlbaum.

Resnick, L. B., & Singer, J. A. (1993). Protoquantitative origins of ratio reasoning. In T. P. Carpenter, E. Fennema, & T. A. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 107–130). Hillsdale, NJ: Erlbaum.

Sáenz-Ludlow, A. (1992) Ann's strategies to add fractions. In W. Geeslin and K. Graham (Eds.), *Proceedings of the Sixteenth Psychology of Mathematics Education Conference*, Vol. 2, (pp. 266–273). Durham, NH: University of New Hampshire.

Sáenz-Ludlow, A. (1994). Michael's fraction schemes. *Journal for Research in Mathematics Education, 25*, 50–85.

Schwartz, J. L. (1988). Intensive quantity and referent transforming arithmetic operations. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 41–52). Hillsdale, NJ: Erlbaum.

Southwell, B. (1984). An analysis of the development of several subconstructs of rational numbers. In B. Southwell (Ed.), *Proceedings of the Eighth International Conference for the Psychology of Mathematics Education* (pp. 387–396). Sydney: Mathematical Association of New South Wales.

Steffe, L. P. (1984). The teaching experiment methodology in a constructivist research program. In M. Zweng (Ed.), *Proceedings of the Fourth International Congress on Mathematical Education* (pp. 469–471). Boston: Birkhauser.

Steffe, L. P. (1988). Children's construction of number sequences and multiplying schemes. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 119–140). Hillsdale, NJ: Erlbaum.

Steffe, L. P. (1993, January). *Learning an iterative fraction scheme.* Paper presented at the International Conference on Rational Number Learning, University of Georgia, Athens, GA.

Steffe, L. P., Cobb, P., and von Glasersfeld, E. (1988). *Construction of arithmetical meanings and strategies.* New York: Springer-Verlag.

Steffe, L. P., & Olive, J. (1993, January). *Children's mathematical learning in computer microworlds.* Paper presented at the International Conference on Rational Number Learning, University of Georgia, Athens, GA.

Steffe, L. P., & von Glasersfeld, E. (1985). Helping children to conceive of number. *Recherches en Didactique des Mathématiques, 6*(2-3), 269–303.

Steffe, L. P., von Glasersfeld, E., Richards, J., & Cobb, P. (1983). *Children's counting types: Philosophy, theory, and application.* New York: Praegar Scientific.

Streefland, L. (1984). Unmasking N-Distracters as a source of failures in learning fractions. In B. Southwell (Ed.), *Proceedings of the Eighth International Conference for the Psychology of Mathematics Education.* Sydney: Mathematical Association of New South Wales.

Streefland, L. (1991). *Fractions in realistic mathematics education.* Dordrecht, Netherlands: Kluwer.

Tirosh, D., Wilson, J., Graeber, A., & Fischbein, E. (1993, January). *Conceptual adjustments in progressing from whole to rational numbers.* Paper presented at the International Conference on Rational Number Learning, University of Georgia, Athens, GA.

This content downloaded from
71.232.212.173 on Sun, 18 Oct 2020 18:51:25 UTC
All use subject to https://about.jstor.org/terms

Usiskin, Z. (1979). The future of fractions. *Arithmetic Teacher*, *27*, 18–20.

Wagner, S., & Kieren, C. (Eds.). (1989). *Research issues in the learning and teaching of algebra.* Reston, VA: National Council of Teachers of Mathematics.

AUTHORS

ROBERT P. HUNTING, Senior Lecturer, Graduate School of Education, La Trobe University, Bundoora Victoria 3083, Australia; e-mail: hunting@luga.latrobe.edu.au

GARY DAVIS, Professor, School of Education, University of Southampton, Highfield SO17 1BJ, United Kingdom; e-mail: gary@soton.ac.uk

CATHERINE A. PEARN, Research Assistant, Graduate School of Education, La Trobe University, Bundoora Victoria 3083, Australia; e-mail: cpearn@luga.latrobe.edu.au

This content downloaded from
71.232.212.173 on Sun, 18 Oct 2020 18:51:25 UTC
All use subject to https://about.jstor.org/terms