Can. J. Math., Vol. XXVIII, No. 5, 1976, pp. 1076–1081

COMPATIBLE TIGHT RIESZ ORDERS ON THE GROUP OF AUTOMORPHISMS OF AN 0-2-HOMOGENEOUS SET

GARY DAVIS AND COLIN D. FOX

Introduction. Davis and Bolz (1974) considered, and to some extent classified, compatible tight Riesz order on the group of all order-preserving permutations of a totally ordered field. Glass (1976) carried out a more general study of compatible tight Riesz orders on ordered permutation groups and, in particular, showed the importance of determining compatible tight Riesz orders on 0-primitive ordered permutation groups. However, the general problems of existence and classification of compatible tight Riesz orders on 0-primitive ordered permutation groups remained open.

In this paper we consider these problems in relation to the group $A(\Omega)$ of all order-preserving permutations of a totally-ordered set Ω with $A(\Omega)$ acting 0-2-transitively on Ω. Such a group has compatible tight Riesz orders (Theorem 7), which answers an implicit question of Glass (1976) and, with a further restriction on Ω, we can describe certain maximal compatible tight Riesz orders on $A(\Omega)$ (Theorem 8). The final section deals with the maximal tangents of the compatible tight Riesz orders we have found.

We are grateful to the referee for his comments: especially a neater proof of Theorem 10. We are also grateful to Andrew Glass for his comments on the paper, and for pointing out to us that Rick Ball independently, and about the same time, proved our Theorem 7.

For a totally-ordered set Ω we denote by $A^+(\Omega)$ the positive set of $A(\Omega)$ with the usual lattice order. That is $A^+(\Omega) = \{g \in A(\Omega) : xg \geqq x \text{ for all } x \in \Omega\}$. For $x \in \Omega$ the stabilizer of x in $A(\Omega)$ is $A_x(\Omega) = \{g \in A(\Omega) : xg = x\}$, and we write $A_x^+(\Omega)$ for $A_x(\Omega) \cap A^+(\Omega)$.

In the sequel we shall call an order-preserving permutation of Ω an *automorphism* of Ω, and we shall assume always that $A(\Omega)$ is a non-trivial group.

We recall that a subgroup G of $A(\Omega)$ acts 0-2-transitively on Ω if for all $x_1 < x_2$ and $y_1 < y_2$ in Ω there is a $g \in G$ satisfying $x_i g = y_i$ $(i = 1, 2)$. We say that Ω is *homogeneous* (respectively, 0-2-*homogeneous*) if $A(\Omega)$ acts transitively (respectively, 0-2-transitively) on Ω.

For the record we provide a proof of the following piece of folklore (apparently originating with Wielandt), since it is the key to our constructions.

THEOREM 1. *For a totally-ordered set Ω the following are equivalent:*

Received January 8, 1976 and in revised form, April 30, 1976.

Downloaded from https://www.cambridge.org/core. 27 Jan 2021 at 22:44:27, subject to the Cambridge Core terms of use.

(1) Ω *is* 0-2-*homogeneous.*

(2) Ω *has neither least nor greatest element and all closed intervals of* Ω *with more than one point have the same order-type.*

Proof ((1) implies (2)). Since Ω is homogeneous it can have neither least nor greatest element. If $x_1 < x_2$ and $y_1 < y_2$ in Ω then $x_i g = y_i$ $(i = 1, 2)$ for some $g \in A(\Omega)$. Clearly the restriction of g to the closed interval $[x_1, x_2]$ is an order-isomorphism onto $[y_1, y_2]$.

((2) implies (1)). By a result of Holland (1965, Theorem 4) we need only show that for all $x < y < z$ in Ω there is a $g \in A_x^+(\Omega)$ satisfying $yg = z$. (Since Ω is without a least element this also shows immediately that Ω is homogeneous). For each *integer* n take $a_n \in \Omega$ satisfying $x < a_n < a_{n+1}$, $y = a_0$ and $z = a_1$. This is possible since Ω is dense in itself and has no greatest element. Now for each n let ϕ_n be an order-isomorphism from $[a_n, a_{n+1}]$ onto $[a_{n+1}, a_{n+2}]$. The map $g : \Omega \to \Omega$ defined by

$$wg = \begin{cases} w\phi_n & \text{if } w \in [a_n, a_{n+1}) \text{ for some } n \\ w & \text{otherwise} \end{cases}$$

is an element of $A_x^+(\Omega)$ and $yg = z$ (both facts being easy to verify).

By Lemma 9 of Holland (1963) we have the following result, which is important for us:

COROLLARY 2. *If* Ω *is* 0-2-*homogeneous then* $A(\Omega)$ *is divisible.*

Compatible tight Riesz orders. A *compatible tight Riesz order* on $A(\Omega)$ is a subset T of $A(\Omega)$ satisfying the following:

(1) T is a proper dual ideal of $A^+(\Omega)$

(2) T is normal in $A(\Omega)$

(3) $T = TT$

(4) $\inf T = 1$

Our objective in this section is to show that $A(\Omega)$ *has* a compatible tight Riesz order when Ω is 0-2-homogeneous and then, in some cases, to determine maximal compatible tight Riesz orders.

We equip Ω with the order topology. The collection of all open dense subsets of Ω is denoted by $D(\Omega)$. Clearly $D(\Omega)$ is a filter of the lattice of open subsets of Ω. The *support* of $g \in A(\Omega)$ is the set $\text{supp}(g) = \{x \in \Omega : xg \neq x\}$. Each support set is open for the order topology.

The collection $\Sigma(\Omega) = \{\text{supp}(g) : g \in A(\Omega)\}$, ordered by inclusion, is a sublattice of the lattice of open sets of Ω and is called the *support lattice* of Ω. Thus $\Sigma(\Omega)$ is a distributive lattice with least element $\Box = \text{supp}(1)$, but in general without a greatest element. We denote the annihilator of Δ in $\Sigma(\Omega)$ by Δ^*. Thus $\Delta^* = \{\Delta' \in \Sigma(\Omega) : \Delta \cap \Delta' = \Box\}$, and we denote $\Sigma(\Omega) \cap D(\Omega)$ by $\delta(\Omega)$.

We say that a closed interval $[a, b]$ in Ω supports a non-identity automorphism if $A([a, b]) \neq \langle 1 \rangle$.

Downloaded from https://www.cambridge.org/core. 27 Jan 2021 at 22:44:27, subject to the Cambridge Core terms of use.

LEMMA 3. *If each closed interval of Ω with more than one point supports a non-identity automorphism then* $\delta(\Omega) = \{\Delta \in \Sigma(\Omega) : \Delta^* = \{\Box\}\}$.

Proof. Let Ω satisfy the hypothesis of the lemma and take any Δ in $\delta(\Omega)$. If $\Delta' \cap \Delta = \Box$, with Δ' in $\Sigma(\Omega)$, then $\Delta' = \Box$ (otherwise Δ', being open, meets the open dense set Δ). Thus $\Delta^* = \{\Box\}$. Conversely, suppose that $\Delta \in \Sigma(\Omega)$ and that the closure $\bar{\Delta}$ of Δ is not Ω. Then $[y, z] \subseteq \Omega \backslash \bar{\Delta}$ for some $y < z$ in Ω, so if we let h be a non-identity automorphism of $[y, z]$ and define $g : \Omega \rightarrow \Omega$ by

$$xg = \begin{cases} xh & \text{if } x \in [y, z] \\ x & \text{otherwise} \end{cases}$$

then $g \in A(\Omega)$. Since $\text{supp}(g) \neq \Box$ and $\Delta \cap \text{supp}(g) = \Box$ we have $\Delta^* \neq \{\Box\}$. Thus for Δ in $\Sigma(\Omega)$, $\Delta^* = \{\Box\}$ implies $\Delta \in D(\Omega)$.

COROLLARY 4. *If Ω is* 0-2-*homogeneous then* $\delta(\Omega) = \{\Delta \in \Sigma(\Omega) : \Delta^* = \{\Box\}\}$.

Proof. Let $[x, y]$, with $x < y$, be a proper closed interval of Ω. A non-identity automorphism of $[x, y]$ can be constructed as in Theorem 1.

Now we define a candidate for a compatible tight Riesz order on $A(\Omega)$:

$$T_\delta = \{g \in A^+(\Omega) : \text{supp}(g) \text{ is dense in } \Omega\}.$$

LEMMA 5. *T_δ is either empty or a proper normal dual ideal of $A^+(\Omega)$.*

Proof. Suppose $T_\delta \neq \Box$. Take $f, g \in T_\delta$ and any $h \in A(\Omega)$. Recall that $D(\Omega)$ is a filter of the lattice of open subsets of Ω. Since $f \leqq h$ implies $\text{supp}(f) \subseteq \text{supp}(h)$, and since $\text{supp}(f \wedge g) = \text{supp}(f) \cap \text{supp}(g)$, it follows that T_δ is a dual ideal of $A^+(\Omega)$. Also $\text{supp}(h^{-1}fh) = \text{supp}(f)h$, and h is a homeomorphism of Ω, so that T_δ is normal in $A(\Omega)$. Clearly $1 \notin T_\delta$ so T_δ is either empty or a proper normal dual ideal of $A(\Omega)$.

In fact, when Ω is 0-2-homogeneous T_δ is not empty. The next lemma describes the elements of $\delta(\Omega)$ in this case.

We shall say that a pairwise disjoint collection $\{K_i : i \in I\}$ of subsets of Ω is a *topological partition* of Ω if $\bigcup\{K_i : i \in I\}$ is dense in Ω (for the order topology). If $K \subseteq \Omega$ we say that $S \subseteq K$ is *terminal* in K if for all $x \in K$ there are $a, b \in S$ such that $a \leqq x \leqq b$.

LEMMA 6. *If Ω is dense in itself then there is a topological partition $\{K_i : i \in I\}$ of Ω for which each K_i is a convex set with a countable terminal subset.*

Proof. Let X denote the set of all collections $\{K_i : i \in I\}$ where each K_i is a convex subset of Ω with a countable terminal subset, and $K_i \cap K_j = \Box$ if $i \neq j$. Then X, ordered by inclusion, is an inductive set, so let $\{K_i : i \in I\}$ be a maximal element of X. If $\Delta = \bigcup\{K_i : i \in I\}$ is not dense in Ω then there is a non-empty open interval (x, y) contained in $\Omega \backslash \bar{\Delta}$. Since Ω is order-dense the interval (x, y) contains a convex set K with a countable terminal subset, and

Downloaded from https://www.cambridge.org/core. 27 Jan 2021 at 22:44:27, subject to the Cambridge Core terms of use.

for this K we have $K \cap K_i = \square$ for all $i \in I$. However this contradicts the maximality of $\{K_i : i \in I\}$ in X.

THEOREM 7. *If Ω is 0-2-homogeneous then T_δ is a compatible tight Riesz order on $A(\Omega)$.*

Proof. Suppose that Ω is 0-2-homogeneous. To show that both $T_\delta \neq \square$ and $\inf T = 1$, it is sufficient to take any $w \in \Omega$ and then find $g \in T_\delta \cap A_w(\Omega)$. So take $w \in \Omega$ and let $\Omega_1 = \{x \in \Omega : x < w\}$ and $\Omega_2 = \{x \in \Omega : x > w\}$. By Lemma 6 we can write $\Omega_1 = \overline{\dot{\bigcup}\{K_i : i \in I\}}$, where $K_i = \bigcup\{[x_{i(n)}, x_{i(n+1)}] : n \in \mathbf{Z}\}$ with $x_{i(n)} < x_{i(n+1)}$ for all $n \in \mathbf{Z}$. For each $i \in I$ and $n \in \mathbf{Z}$ let $\phi_{i(n)}$ be an order-isomorphism from $[x_{i(n)}, x_{i(n+1)}]$ onto $[x_{i(n+1)}, x_{i(n+2)}]$. Then $g_1 : \Omega_1 \rightarrow \Omega_1$ defined by

$$xg_1 = \begin{cases} x\phi_{i(n)} & \text{if } x \in [x_{i(n)}, x_{i(n+1)}) \text{ for some } i(n) \\ x & \text{otherwise} \end{cases}$$

is an element of $A^+(\Omega_1)$, and $\operatorname{supp}(g_1) = \bigcup\{K_i : i \in I\}$ is dense in Ω_1. Similarly we can find $g_2 \in A^+(\Omega_2)$ with $\operatorname{supp}(g_2)$ dense in Ω_2. Then $g : \Omega \rightarrow \Omega$ defined by

$$xg = \begin{cases} xg_1 & \text{if } x \in \Omega_1 \\ x & \text{if } x = w \\ xg_2 & \text{if } x \in \Omega_2 \end{cases}$$

is an element of $T_\delta \cap A_w(\Omega)$. By Lemma 5 it remains to show that $T_\delta = T_\delta T_\delta$. Since T_δ is a dual ideal of $A^+(\Omega)$ it is also a subsemigroup, and since $A(\Omega)$ is divisible (Corollary 2) and $\operatorname{supp}(g) = \operatorname{supp}(g^2)$ for all $g \in A(\Omega)$ we have $T_\delta \subseteq T_\delta T_\delta$.

There are two obvious compatible tight Riesz orders larger than T_δ. Namely $T_\rho = \{g \in A^+(\Omega) : \operatorname{supp}(g) \cap [x, \infty) \text{ is dense in } [x, \infty) \text{ for some } x \in \Omega\}$, and its dual T_λ. (Here $[x, \infty) = \{y \in \Omega : y \geqq x\}$). When are these compatible tight Riesz orders maximal? Not always, we suspect. The following theorem gives a partial answer.

THEOREM 8. *If Ω is 0-2-homogeneous and has a countable cofinal (coinitial) subset then $T_\rho(T_\lambda)$ is a maximal compatible tight Riesz order on $A(\Omega)$.*

Proof. Suppose Ω is 0-2-homogeneous with countable cofinal subset $z_1 < z_2 < \ldots$ (that is, for each $x \in \Omega$ there is an n for which $x \leqq z_n$). Assume that T is a compatible tight Riesz order properly containing T_ρ. Then there is a $g \in T$ with fixed intervals $[x_n, y_n]$ such that $z_n \leqq x_n < y_n \leqq x_{n+1}$ for all natural numbers n. We choose arbitrary elements x_n, y_n $(n = 0, -1, -2, \ldots)$ in Ω satisfying $y_{n-1} < x_n < y_n < z_1$. Then (as in our previous constructions) there is an $h \in A(\Omega)$ satisfying $x_n h = y_n$ and $y_n h = x_{n+1}$ for all integers n. We see that the support of $g \wedge h^{-1}gh$ is bounded above. If $x \geqq x_1$, then $x \in [x_n, y_n]$ for some integer n, in which case $xg = x$, or $x \in [y_n, x_{n+1}]$ for some integer n, in which case $x(h^{-1}gh) = x$. Since these are the only possibilities for $x \geqq x_1$ it follows that $\operatorname{supp}(g \wedge h^{-1}gh)$ is bounded above by x_1. We can then find $k \in T_\rho$ with $xk = x$ for $x \leqq x_1$, and therefore $1 = k \wedge g \wedge h^{-1}gh \in T$—a contradiction.

Downloaded from https://www.cambridge.org/core. 27 Jan 2021 at 22:44:27, subject to the Cambridge Core terms of use.

Maximal tangents. If F is a filter of the distributive lattice $A^+(\Omega)$ of positive elements of $A(\Omega)$ then any subset of $A^+(\Omega)$ maximal with respect to being a lattice ideal not meeting F is a prime ideal (this is a specialization of a well-known theorem of M. H. Stone). When T is a compatible tight Riesz order on $A(\Omega)$ the subsets of $A(\Omega)$ that are maximal with respect to being convex sub-lattice subgroups not meeting T, are called the *maximal tangents* of T. Since convex sublattice subgroups of $A(\Omega)$ are generated by their intersection with $A^+(\Omega)$ as lattice ideals it follows that the maximal tangents of a compatible tight Riesz order are *prime subgroups* of $A(\Omega)$ (i.e. convex sublattice subgroups M of $A(\Omega)$ for which $A^+(\Omega)\backslash M$ is a dual ideal).

We shall denote the set of maximal tangents for a compatible tight Riesz order T by $\text{Max}(T)$. A fundamental theorem due to Norman Reilly (1973) asserts that, always, $T = A^+(\Omega)\backslash \bigcup \text{Max}(T)$.

Our objective in this section is to determine the maximal tangents of T_δ, and this turns out to be a piece of lattice theory.

We recall that a distributive lattice $\mathscr{L}$ with least element 0 is *quasi-pseudo-complemented* (or a distributive *-lattice) if for each $x \in \mathscr{L}$ there is a $y \in \mathscr{L}$ such $x \wedge y = 0$ and $(x \vee y)^* = (0)$ where, for $z \in \mathscr{L}$, $z^* = \{z' \in \mathscr{L} : z \wedge z' = 0\}$.

If we denote by R the congruence on $\mathscr{L}$ defined by xRy if $x^* = y^*$, and by D the set $\{z \in \mathscr{L} : z^* = (0)\}$ of dense elements of $\mathscr{L}$, then the following conditions, amongst others, are known to be equivalent (see, for instance, T. P. Speed (1969)):

(1) $\mathscr{L}$ is quasi-pseudo-complemented

(2) $\mathscr{L}/R$ is Boolean

(3) for any $x \in \mathscr{L}$ there is a $y \in \mathscr{L}$ satisfying $x^{**} = y^*$

(4) for any ideal I of $\mathscr{L}$ with $I \cap D = \square$ there is a minimal prime ideal $\supseteq I$.

Since a quasi-pseudo-complemented lattice $\mathscr{L}$ *has* dense elements the set D of dense elements of $\mathscr{L}$ is a filter and the prime ideals of $\mathscr{L}$ not meeting D are precisely the minimal prime ideals (Grätzer (1971), p. 169).

THEOREM 9. *If Ω is 0-2-homogeneous then $\Sigma(\Omega)$ is quasi-pseudo-complemented.*

Proof. Take any $g \in A^+(\Omega)$. Then $\Omega\backslash\text{supp}(g)$ is closed for the order topology and can be written as a disjoint union of maximal closed intervals (whose end-points may be in $\bar{\Omega}$, the Dedekind completion of Ω). For each such interval $[x, y]$ we can find an automorphism of Ω whose support set is contained in and dense in $[x, y]$ by Theorem 7. If g' is the join of these automorphisms of Ω then $g \wedge g' = 1$ so that $\text{supp}(g) \cap \text{supp}(g') = \square$ and $\text{supp}(g) \cup \text{supp}(g') = \text{supp}(g \vee g')$ is dense in Ω. By Corollary 5 we then have $(\text{supp}(g) \cup \text{supp}(g'))^* = \{\square\}$.

We recall that a prime subgroup M of a lattice-ordered group G is minimal prime if and only if for all $m \in M \cap G^+$ there is a $g \in G^+\backslash M$ such that $m \wedge g = 1$.

THEOREM 10. *If Ω is 0-2-homogeneous then the maximal tangents of T_δ are precisely the minimal prime subgroups of $A(\Omega)$.*

Downloaded from https://www.cambridge.org/core. 27 Jan 2021 at 22:44:27, subject to the Cambridge Core terms of use.

Proof. Let M be a maximal tangent. Then M is a prime subgroup. If $m \in M^+$ then, since $m \notin T_\delta$ and $\Sigma(\Omega)$ is quasi-pseudo-complemented, there exists an $m^* \in A^+(\Omega)$ with $m \wedge m^* = 1$ and $m \vee m^* \in T_\delta$. Thus $m^* \notin M$ and M is a minimal prime.

Conversely, let M be a minimal prime and $m \in M^+$. Then $m \wedge m^* = 1$, for some $m^* \in A^+(\Omega)\backslash M$. Therefore $m \notin T_\delta$ and $M \cap T_\delta = \emptyset$. Hence M is contained in a maximal tangent which, by the first part of the proof, is a minimal prime and therefore equal to M.

COROLLARY 11. *If $\{M_\lambda : \lambda \in \Lambda\}$ is a non-empty collection of minimal prime subgroups of $A(\Omega)$ left invariant by conjugation then $T = A^+(\Omega)\backslash \bigcup\{M_\lambda : \lambda \in \Lambda\}$ is a compatible tight Riesz order on $A(\Omega)$.*

We denote by A the normal convex sublattice subgroup of $A(\Omega)$ consisting of all $g \in A(\Omega)$ for which $\text{supp}(g) \subseteq \Omega\backslash[x, \infty)$ for some $x \in \Omega$, and by B the dual normal convex sublattice subgroup of $A(\Omega)$.

COROLLARY 12. *The maximal tangent of the compatible tight Riesz order $T_\rho(T_\lambda)$ are precisely the minimal prime subgroups of $A(\Omega)$ lying above $A(B)$.*

Proof. Let M be a maximal tangent of T_λ. Since $T_\lambda \supseteq T_\delta$, M is a prime subgroup of $A(\Omega)$ not meeting T_δ and therefore M is contained in a maximal tangent of T_δ. That is, M is a minimal prime subgroup. Suppose that there is a $g \geqq 1$ in $A\backslash M$, so that, for some $x \in \Omega$, $[x, \infty) \subseteq \text{fix}(g) = \Omega\backslash\text{supp}(g)$. We can then find $h \in T_\rho$ satisfying $g \wedge h = 1$, so that either $g \in M$ or $h \in M$—both contradictory.

Suppose on the other hand that M is a minimal prime subgroup lying above A and that $M \cap T_\rho \neq \square$. Then there is a $g > 1$, $g \in M$ such that $[x, \infty) \cap \text{supp}(g)$ is dense in $[x, \infty)$ for some $x \in \Omega$. Since M is a minimal prime subgroup there is an $h \geqq 1$ satisfying $g \wedge h = 1$ and $h \notin M$. Then we have $zh = z$ on $[x, \infty) \cap \text{supp}(g)$—a dense subset of $[x, \infty)$—so $[x, \infty) \subseteq \text{fix}(h) = \Omega\backslash\text{supp}(h)$. That is, $h \in A \subseteq M$—a contradiction.

REFERENCES

1. G. Davis and E. Bolz, *Compatible tight Riesz orders on ordered permutation groups*, Journal Aust. Math. Soc. (to appear).
2. A. M. W. Glass, *Compatible tight Riesz orders*, Can. J. Math. *28* (1976) 186–200.
3. G. Grätzer, *Lattice theory* (W. H. Freeman, San Francisco, 1971).
4. C. Holland, *The lattice-ordered group of automorphisms of an ordered set*, Michigan Math. J. *10* (1963), 399–408.
5. ——— *Transitive lattice-ordered permutation groups*, Math. Zeitschr. *87* (1965), 420–433.
6. N. R. Reilly, *Compatible tight Riesz orders and prime subgroups*, Glasgow Math. Journal *14* (1973), 145–160.
7. T. P. Speed, *Some remarks on a class of distributive lattices*, Journal Aust. Math. Soc. *9* (1969), 289–296.

La Trobe University,
Melbourne, Australia

Downloaded from https://www.cambridge.org/core. 27 Jan 2021 at 22:44:27, subject to the Cambridge Core terms of use.