REPRESENTATION AND EXTENSION OF SEMI-PRIME RINGS

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Introduction

In many respects the theory of semi-prime rings (i.e. rings without proper nilpotents) is similar to that for lattice-ordered groups. In this paper semi-prime rings are faithfully represented as subrings of continuous global sections of sheaves of integral domains with Boolean base spaces. This representation allows a simple description of a particular extension of a semi-prime ring as the corresponding ring of all continuous global sections. The ideals in a semi-prime ring R that give rise to the stalks in the sheaf representation are then characterized when R is projectable. Finally equivalent conditions are given for a semi-prime ring R to satisfy a condition, that in the case of lattice-groups, was termed "weak projectability" by Spirason and Strzelecki [8]. Some of the results that are common to semi-prime rings and lattice-groups (and semi-prime semigroups) have been extended to certain universal algebras by Davey [3].

1. Sheaf Representation

Let R be a semi-prime ring. That is, $x^2 = 0$ is possible only for $x = 0$ in R; this is equivalent to the fact that R has no non-zero nilpotents. For $A \subseteq R$ define

$$A^0 = \{x \in R: ax = 0 \text{ for all } a \in A\},$$

and $A^{00} = (A^0)^0$. If $A = \{x\}$ is a singleton set then A^0, A^{00} are denoted by x^0, x^{00} respectively.

The class of all subsets of R of the form A^0 is denoted by $\mathscr{B}(R)$ and, ordered by inclusion, $\mathscr{B}(R)$ is a complete Boolean algebra with

(i) $\bigwedge_\alpha A_\alpha^0 = \cap_\alpha A_\alpha^0 = (\cup_\alpha A_\alpha)^0$

(ii) $\bigvee_\alpha A_\alpha^0 = \cap\{B^0 : B^0 \supseteq \cup_\alpha A_\alpha^0\}$

and

(iii) A^{00} as the complement of A^0.

The Stone space of $\mathscr{B}(R)$ is denoted by $\mathscr{Q}$: thus $\mathscr{Q}$ is the set of prime ideals of $\mathscr{B}(R)$ and is furnished with the hull-kernel topology, for which the closed-open sets

$$\mathscr{Q}_{A^0} = \{t \in \mathscr{Q} : A^0 \in t\}$$

form a base for the open sets. For each $t \in \mathscr{Q}$ a subset R_t of R is defined by

$$R_t = \{x \in R : x^{00} \in t\}.$$

It is readily seen that each R_t is a two-sided ideal of R. Furthermore $\cap_{t \in \mathscr{Q}} R_t = (0)$ for if $x^{00} \in t$ for all $t \in \mathscr{Q}$ then $x^{00} = 0$ so $x = 0$.

A sheaf of rings $(\mathscr{R}, p, \mathscr{Q})$ is now defined as follows: $\mathscr{R}$ is the disjoint union of the rings R/R_t, $t \in \mathscr{Q}$; p is the map from $\mathscr{R}$ into $\mathscr{Q}$ defined by $p(r) = t$ if $r \in R/R_t$; a topology is placed on $\mathscr{R}$ by taking the sets

$$\{x + R_t : t \in \mathscr{Q}_{A^0}\},$$

with $x \in R$, $A^0 \in \mathscr{B}(R)$, as basic open sets. It follows, as for instance in Dauns and Hofmann [2], that $(\mathscr{R}, p\,\mathscr{Q})$ is a sheaf of rings. The ring of continuous global sections of this sheaf is denoted by $\Gamma(\mathscr{R})$. If $x \in R$ and $A^0 \in \mathscr{B}(R)$ then the pair (x, A^0) defines an element $I(Q_{A^0}; x)$ of $\Gamma(R)$ by

$$I(\mathscr{Q}_{A^0}; x)(t) = \begin{cases} x + R_t & \text{if } t \in \mathscr{Q}_{A^0} \\ 0 + R_t & \text{if } t \notin \mathscr{Q}_{A^0} \end{cases}$$

When $\mathscr{Q}_{A^0} = \mathscr{Q}$, $I(\mathscr{Q}_{A^0}; x)$ is denoted by $\hat{x}$, and if R has an identity 1 then $I(\mathscr{Q}_{A^0};\ 1)$ is denoted by $I(\mathscr{Q}_{A^0})$.

PROPOSITION 1.1. *Let R be a semi-prime ring. Then,*

(1) *$(\mathscr{R}, p, \mathscr{Q})$ is a sheaf of integral domains*

(2) *the map $x \mapsto \hat{x}$ from R into $\Gamma(\mathscr{R})$ is a ring isomorphism*

(3) *if $\sigma \in \Gamma(\mathscr{R})$ then there is a finite closed-open partition $\{\mathscr{Q}_{A_1^0}, \cdots, \mathscr{Q}_{A_r^0}\}$ of $\mathscr{Q}$ and $x_1, \cdots, x_r \in R$ such that $\sigma = \sum I(\mathscr{Q}_{A_i^0}; x_i)$*

(4) *if R has an identity 1 then for every non-empty subset $A \subseteq \Gamma(\mathscr{R})$ there is a central idempotent $e \in \Gamma(\mathscr{R})$ such that $A^0 = e\Gamma(\mathscr{R})$.*

PROOF. The homomorphism $x \leftrightarrow \hat{x}$ is an isomorphism since $\cap_{t \in \mathscr{Q}} R_t = (0)$. If $\sigma \in \Gamma(\mathscr{R})$ then for each $t \in \mathscr{Q}$ there is an $x_t \in R$ such that $\sigma(t) = \hat{x}_t(t)$. Since $(\mathscr{R}, p, \mathscr{Q})$ is a sheaf there is a basic closed-open neighbourhood $\mathscr{Q}_{A_t^0}$ of t such that $\sigma = \hat{x}_t$ on $\mathscr{Q}_{A_t^0}$. Then $\{\mathscr{Q}_{A_t^0} : t \in \mathscr{Q}\}$ is an open cover for $\mathscr{Q}$ and since $\mathscr{Q}$ is compact there is a finite subcover $\{\mathscr{Q}_{A_{t_1}^0}, \cdots, \mathscr{Q}_{A_{t_r}^0}\}$. Put

$$\mathscr{Q}_{A_1^0} = \mathscr{Q}_{A_{t_1}^0},\ \mathscr{Q}_{A_i^0} = \mathscr{Q}_{A^0_{t_i}} = \mathscr{Q}_{A_{t_i}^0} \Big/ \bigcup_{1 \leqq j \leqq i} \mathscr{Q}_{A_{t_j}^0}$$

for $i > 1$. Then $\{\mathscr{Q}_{A^0}, \cdots, \mathscr{Q}_{A^0}\}$ is a closed-open partition of $\mathscr{Q}$, and if $x_i = x_{i_i}$ then $\sum_i I(\mathscr{Q}_{A_0}; x_i)$ is just σ, for if $t \in \mathscr{Q}_{A^0}$ then

$$\sum_i I(\mathscr{Q}_{A_i^0}; x_i)(t) = \hat{x}_j(t) = \sigma(t).$$

The sheaf $(\mathscr{R}, p, \mathscr{Q})$ is a sheaf of integral domains since the ideals R_t are prime (i.e. $xy \in R_t$ is possible only if $x \in R_t$ or $y \in R_t$). This follows from the fact that $(xy)^{00} = x^{00} \cap y^{00}$ in a semi-prime ring.

For $\sigma = \sum_i I(\mathscr{Q}_{A_i^0}; x_i) \in \Gamma(\mathscr{R})$, with $\{\mathscr{Q}_{A_1^0}, \cdots, \mathscr{Q}_{A_r^0}\}$ a partition of $\mathscr{Q}$, the set $S(\sigma) = \{t \in \mathscr{Q}: \sigma(t) \neq 0\}$ is just $\cup_i \mathscr{Q}_{A_i^0} \cap \mathscr{Q}_{xt}^{00}$ which is closed-open, so that, assuming R has an identity 1, $I(S(\sigma)) \in \Gamma(\mathscr{R})$. For an arbitrary subset $\{\sigma_\alpha\} \subseteq \Gamma(\mathscr{R})$ the closure S of $\cup_\alpha S(\sigma_\alpha)$ is closed-open since $\mathscr{Q}$ is extremally-disconnected so that $I(S) \in \Gamma(\mathscr{R})$. Since the R/R_t are integral domains, $\sigma_\alpha \cdot \sigma = 0$ for all α is equivalent to $I(S)I(S(\sigma)) = 0$ so that $\{\sigma_\alpha\}^0 = [\hat{1} - I(S)](\mathscr{R})$ and $\hat{1} - I(S)$ is a central idempotent.

The above argument is essentially that given by Kist [5]. Notice also that an entirely similar argument gives the following:

PROPOSITION 1.2. *Let Γ be the ring of all continuous global sections of a sheaf of integral domains with identities over a Boolean base space X. Then for every $x \in R$ there is a unique central idempotent e such that $x^0 = \{y \in \Gamma: xy = 0\} = e\Gamma$. If X is extremely-disconnected then for every subset $A \subset R$ there is a unique central idempotent e such that $A^0 = \{y \in \Gamma : xy = 0$ for all $x \in A\} = e\Gamma$.*

Koh [6] has extended Grothendieck and Dieudonné's sheaf representation of a commutative ring with identity to semi-prime rings. In his representation a semi-prime ring is isomorphic to the ring of *all* continuous global sections of a sheaf of semi-prime rings over a compact base-space: however the semi-prime rings that comprise the stalks are not necessarily integral domains and the base space of the sheaf is not necessarily Boolean.

2. Extensions

DEFINITIONS 2.1. A ring S with identity 1 is said to be *completely-projectable* if for every non-empty subset $A \subseteq S$ there is a central idempotent e such that $A^0 = eS$. Let R be a semi-prime ring: a completely-projectable cover for R is a triple $(S, \Psi, \overline{\Psi})$ where

(1) S is a completely-projectable ring
(2) $\Psi: R \to S$ is a ring isomorphism into S
(3) $\overline{\Psi}: \mathscr{B}(R) \to \mathscr{B}(S)$ is a Boolean bijection
(4) $\overline{\Psi}(x^0) = \Psi(x)^0$, for $x \in R$.

By an abuse of language, S is sometimes said to be a completely-projectable cover for R if $(S, \Psi, \overline{\Psi})$ has this property. When R has an identity $\Gamma(\mathscr{R})$ is a completely-projectable ring. The Boolean algebra $\mathscr{B}(R)$ is isomorphic, $A^0 \mapsto \mathscr{Q}_{A^0}$, to the Boolean algebra of closed-open subsets of $\mathscr{Q}$, and this latter algebra is isomorphic to $\mathscr{B}(\Gamma(\mathscr{R}))$, for if $\mathscr{Q}' \subseteq \mathscr{Q}$ is closed-open then

$$\{\sigma \in \Gamma(\mathscr{R}) : S(\sigma) \subseteq \mathscr{Q}'\} = \{\sigma \in \Gamma(\mathscr{R}) : S(\sigma) \subseteq \mathscr{Q} \setminus \mathscr{Q}'\}^0,$$

and conversely if $\{\sigma_\alpha\} \subseteq \Gamma(\mathscr{R})$ then

$$\{\sigma_\alpha\}^0 = \{\sigma \in \Gamma(\mathscr{R}) : S(\sigma) \subseteq \mathscr{Q} \setminus \text{closure} \bigcup_\alpha S(\sigma_\alpha)\}.$$

Denote this isomorphism between $\mathscr{B}(R)$ and $\mathscr{B}(\Gamma(\mathscr{R}))$ by Ψ. Then for $x \in R$,

$$\Psi(x^0) = \{\sigma \in \Gamma(\mathscr{R}) : S(\sigma) \subseteq \mathscr{Q}_{x^{00}}\}^0$$

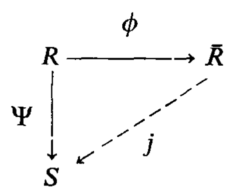
whilst $\hat{x}^0 = \{\sigma \in \Gamma(\mathscr{R}) : S(\sigma) \subseteq \mathscr{Q} \setminus S(x)\}$ so $\Psi(x^c) = x^0$. Thus,

PROPOSITION 2.2. *If R is semi-prime with identity then $\Gamma(\mathscr{R})$ is a completely-projectable cover for R.*

DEFINITION 2.3. *A completely-projectable extension for a semi-prime ring R* is a triple $(\bar{R}, \phi, \bar{\phi})$ where

(1) $(\bar{R}, \phi, \bar{\phi})$ is a completely-projectable cover for R

(2) If $(S, \Psi, \overline{\Psi})$ is a completely-projectable cover for R there is an isomorphism $j : \bar{R} \to S$ such that the diagram



is commutative.

LEMMA 2.4. *If R is a completely-projectable ring then R is semi-prime and $x \mapsto \hat{x}$ is an isomorphism onto $\Gamma(\mathscr{R})$.*

PROOF. It is well-known that completely-projectable rings (otherwise known as Baer rings) are semi-prime.

For $A^0 \in \mathscr{B}(\mathscr{R})$, $I(\mathscr{Q}_{A^0})$ agrees on $\mathscr{Q}$ with the map $\hat{x}$ where x is the unique element of $A^0 \subseteq R$ for which $1 - x \in A^{00}$, so that all continuous global sections are of the form $\hat{x}$ for some $x \in R$.

THEOREM 2.5. *If R is a semi-prime ring with identity then $\Gamma(\mathscr{R})$ is a completely-projectable extension of R.*

PROOF. For a semi-prime ring S the sheaf of integral domains obtained from S, as in 1.1 will be denoted by $(\mathscr{R}_s, p_s, \mathscr{Q}^s)$.

If $(S, \Psi, \bar{\Psi})$ is a completely-projectable cover for R then by the previous lemma S can be replaced, without restriction, by $\Gamma(\mathscr{R}_s)$. Since $\bar{\Psi}: \mathscr{B}(R) \to \mathscr{B}(S)$ is an isomorphism satisfying $\bar{\Psi}(x^0) = \Psi(x)^0$ for $x \in R$ then a map from $\Gamma(\mathscr{R}_R)$ into $\Gamma(\mathscr{R}_s)$ can be defined by

$$\sum_i I(\mathscr{Q}^{R_0}_{A_i})\hat{x}_i \mapsto \sum_i I(\mathscr{Q}^{s}_{\Psi(A_i^0)})\Psi(x_i).$$

This map is an isomorphism for which the appropriate diagram commutes, with S replaced by $\Gamma(\mathscr{R}_s)$.

PROPOSITION 2.6. *If $M \subseteq R$ is a minimal prime ring ideal then $M = R_t$ for some prime ideal $t \subseteq \mathscr{B}_0(R)$.*

PROOF. Take $x_1, \cdots, x_m \in M$ and suppose that for some $y \in x_1^{00} \vee \cdots \vee x_m^{00}$, $y^{00} \nsubseteq M$. Then, $(0) = y^{00} \cap y^0$ so $y^0 \subseteq M$. Then

$$x_1^0 \cap \cdots \cap x_m^0 \subseteq y^0 \subseteq M$$

so $x_i^0 \subseteq M$ for some i. Since M is minimal prime, R is commutative and semi-prime, and $x_i \in M$, there is an $a \notin M$ such that $ax_i = 0$. Thus $a \in x_i \subseteq M$ which is a contradiction. Hence

$$x_1^{00} \vee \cdots \vee x_m^{00} \subseteq M.$$

Now let t_0 be the ideal in $\mathscr{B}_0(R)$ generated by the set $\{x^{00} \vee y^0 : x \in M,\ y \notin M\}$. that is,

$$t_0 = \left\{ \begin{array}{l} A^0 \in \mathscr{B}_0(R) : A^0 \subseteq x_1^{00} \vee \cdots \vee x_m^{00} \vee y_1^{00} \vee \cdots \vee y_n^{00}, \\ \text{for some } x_i \in M,\ y_j \notin M. \end{array} \right\}$$

If $t_0 = \mathscr{B}_0(R)$ then

$$R = (0)^0 = x_1^{00} \vee \cdots \vee x_m^{00} \vee y_1 \vee \cdots \vee y_n^0$$

for some $x_i \in M$, $y_j \notin M$. Then

$$\begin{aligned} y_1^{00} \cap \cdots \cap y_n^{00} &= y_1^{00} \cap \cdots \cap y_n^{00} \cap R \\ &= (y_1^{00} \cap \cdots \cap y_n^{00}) \cap (x_1^{00} \vee \cdots \vee x_m^{00}) \end{aligned}$$

so that

$$y_1^{00} \cap \cdots \cap y_n^{00} \subseteq x_1^{00} \vee \cdots \vee x_m^{00} \subseteq M$$

and therefore $y_i \in M$ for some i, contrary to the choice of the y_i. Thus t_0 is contained in a prime ideal $t \subseteq \mathscr{B}_0(R)$ and it is readily seen that $M = R_t$. (c.f. Spirason and Strzelecki [7]).

Keimel [4] has considered the problem of Stone and Baer extensions for commutative semi-prime semigroups and rings respectively. It is to be noted that in [4] a *Baer envelope* of a commutative semi-prime ring R with identity is a commutative Baer ring Γ (i.e. a ring in which for every $A \subseteq \Gamma$, $A^0 = e\Gamma$ for some idempotent $e \in \Gamma$; since Γ is commutative it is also completely projectable) minimally containing an isomorphic copy of R. In the following section it is seen that Keimel's Γ is the $\Gamma(R)$ of this section and hence a more functorial statement can be made about the ring Γ. In the case of semigroups, however, no such statement is apparent. Keimel has also remarked that every commutative semi-prime ring R with identity has a weak Baer envelope Γ (i.e. a commutative ring Γ minimally containing an isomorphic copy of R and in which for every $x \in \Gamma$ there is an idempotent e satisfying $x_0 = e\Gamma$). In fact any such Γ is a Baer extension in the sense of Kist [5] and also has functorial properties similar to those of the completely-projectable extension of R. In the remainder of the section this point is considered in some detail: let R be a commutative semi-prime ring with identity 1, and denote by $\mathscr{B}_0$ (R) the Boolean subalgebra of $\mathscr{B}$ (R) generated by polar sets of the form x^0, $x \in R$. Thus, $A^0 \in \mathscr{B}_0(R)$ if and only if $A = \wedge_i \vee_j Aij$ where $\{Aij\}$ is a finite set of polars with, for each i, j either $A\overset{0}{ij} = x\overset{0}{ij}$ or $A\overset{0}{ij} = x\overset{00}{ij}$ for elements $xij \in R$.

LEMMA 2.7. *If $\mathscr{Q}_0$ is the Stone space of $\mathscr{B}_0(R)$ and $\mathscr{Q}_0^1 \subseteq \mathscr{Q}_0$ is closed-open then there is an $x \in R$ such $\mathscr{Q}_0^1 = \mathscr{Q}_0\,(x) = \{t \in \mathscr{Q} : x^{00} \notin t\}$.*

PROOF. If $\mathscr{Q}_0^1 \subseteq \mathscr{Q}_0$ is closed-open then

$$\mathscr{Q}_0^1 = \mathscr{Q}_{A^0} = \{t \in \mathscr{Q}_0;\, A^0 \notin t\}$$

for some $A^0 \in \mathscr{B}_0(R)$. Suppose that $A^0 = \wedge_i \vee_j A\overset{0}{ij}$, where for each i, j, $A\overset{0}{ij} = x\overset{0}{ij}$ or $A\overset{0}{ij} = x\overset{0,0}{ij}$, for some $xij \in R$. Then $\vee_j A\overset{0}{ij} \notin t$ for each i, so that for each i there is a $j(i)$ such that $A\overset{0}{ij}(i) \notin t$. Conversely, if for each i there is a $j(i)$ such that $A\overset{0}{ij}(i) \notin t$ then $\wedge_i \vee_j A\overset{0}{ij} \notin t$.

Thus, there is a finite set $x_1, \cdots, x_m,\ y, \cdots, y_n \subseteq R$ such that

$$\mathscr{Q}_0^1 = \mathscr{Q}_{A^0} = \{t \in \mathscr{Q} : x_i^{00} \notin t,\ y_j^0 \notin t, \text{ for all } i,j\}.$$

Now if $y^0 \notin t$ then $(1-y)^0 \in t$ for if $a \in y^0 \cap (1-y)^0$ then $ya = a - (1-y)a$ so $a = 0$. Hence there is a finite set $\{x_1, \cdots, x_p\} \subseteq R$ such that

$$\mathscr{Q}_0^1 = \mathscr{Q}_{A^0} = \{t \in \mathscr{Q}_0 : (x_1 \cdot \cdots \cdot x_p)^{00} \in t\}.$$

Then $(x_1 \cdot \cdots \cdot x_p)^{00} = x_1^{00} \cap \cdots \cap x_p^{00}$ so that

$$\bigcap_i \{t \in \mathscr{Q}_0 : x_i^{00} \notin t\} = \mathscr{Q}_0^1$$

and thus $x = x_1 \cdot \cdots \cdot x_p$ is the required element of R.

Kist [5] calls a commutative ring B a *Baer ring* if for each $x \in B$ there is an idempotent $e \in B$ satisfying $x^0 = eB$. Kist's definition of a *Baer extension* of a commutative ring R is as follows:

a Baer ring B is a Baer extension of a commutative ring R if

(1) *R is isomorphic to a subring of B containing the identity of B,*

(2) *the subring of B generated by the image of R and the idempotents of B is B,*

(3) *the semilattice* $\mu_R = \{\mathcal{M}(x): x \in R\}$, *where* $\mathcal{M}(x)$ *is the class of minimal prime ideals of R not containing* $x \in R$ *is isomorphic to a dense subsemilattice of* $\mu_B = \{\mathcal{M}(x): x \in B\}$ *and the Boolean subalgebra of* μ_B *generated by* μ_R *is* μ_B.

As before a sheaf of integral domains $(\mathcal{R}_0, p, \mathcal{Q}_0)$ is constructed with the stalks being the integral domains R/R_t, $t \in \mathcal{Q}_0$.

PROPOSITION 2.8. $\Gamma(\mathcal{R}_0)$ *is a Baer ring.*

This is just the commutative case of 1.2. As in the remarks before 1.2 it can also be seen that if R is a Baer ring then $R \simeq \Gamma(\mathcal{R}_0)$.

PROPOSITION 2.9. μ_R *is* $\wedge$*-isomorphic to the* $\wedge$*-subsemilattice of the Boolean algebra* $B(\Gamma\mathcal{R}_0))$ *of idempotents of* $\Gamma(\mathcal{R}_0)$.

PROOF. As in Kist [5] the idempotents of $\Gamma(\mathcal{R}_0)$ are seen to be the sections $I(\mathcal{Q}_0(x))$, $x \in R$. Consider the assignment $\mathcal{M}(x) \to I(\mathcal{Q}_0(x))$: this is a map from μ_R into the idempotents of $\Gamma(\mathcal{R}_0)$ for if $\mathcal{M}(x) = \mathcal{M}(y)$ then the continuous sections $I(\mathcal{Q}_0(x))$, $I(\mathcal{Q}_0(y))$ are equal on the dense subset of those $t \in \mathcal{Q}_0$ for which R_t is minimal prime. Since $(xy)^{00} = x^{00} \cap y^{00}$ for $x, y \in R$ it follows that

$$\mathcal{M}(xy) = \mathcal{M}(x) \cap \mathcal{M}(y) \text{ and } I(\mathcal{Q}_0(xy)) = I(\mathcal{Q}_0(x)) \cdot I(\mathcal{Q}_0(y)).$$

Finally, if $I(\mathcal{Q}_0(x)) = I(\mathcal{Q}_0(y))$ then $x \in R_t$ iff $y \in R_t$ so $\mathcal{M}(x) = \mathcal{M}(y)$.

COROLLARY 2.10. $\Gamma(R_0)$ *is a Baer extension of R.*

A *projectable extension* of a not necessarily commutative semi-prime ring R with identity can be defined as follows: a ring R is projectable if for every $x \in R$ there is a central idempotent $e^2 = e$ satisfying $x_0 = eR$. A projectable extension of R is then defined as in 2.1 and 2.3 with "completely-projectable" replaced by "projectable", and "$\mathcal{B}(R)$" replaced by "$\mathcal{B}_0(R)$". The following theorem then holds:

THEOREM 2.11. $\Gamma(\mathcal{R}_0)$ *is a projectable extension of R.*

NOTE. In all cases the ring R has been assumed to have an identity. If R is semi-prime but without an identity then R can be embedded in the ring $\bar{R}$ of all generalized left translations on R: a group endomorphism $\Phi: R \to R$ is a generalized left translation if $\Phi(xy) = \Phi(x)y$. The ring $\bar{R}$ is minimal with respect to the properties

(1) $\bar{R}$ is semi-prime with an identity

(2) $\bar{R}$ contains an isomorphic copy of R

(3) $\bar{R}$ is isomorphic to R if R has an identity.

3. The ideals R_t

The following result gives an internal description of the ideals R_t for a class of semi-prime rings R.

DEFINITIONS 3.1. A ring R with identity is said to be *projectable* if for each $x \in R$ there is a central idempotent e such that $x_0 = eR$. Since the above idempotent e is central it is uniquely determined by $x_0 = eR$, and e is denoted by $\mathrm{id}(x)$. A ring ideal $I \subseteq R$ is a *projection ideal* if $x \in I$ is equivalent to $1 - \mathrm{id}(x) \in I$.

THEOREM 3.2. *Let R be a projectable ring. If $B(R)$ is the Boolen algebra of central idempotents of R then the ideals $R_t = \{x \in R\colon x^{00} \in t\}$, for $t \in \mathscr{Q}$, are the projection ideals $I \subseteq R$ such that $I \cap B(R)$ is a prime ideal in $B(R)$.*

PROOF. Firstly see that the ideals R_t are characterized as those ideals $I \subseteq R$ satisfying

(1) $x \in I$ implies $x^0 \neq (0)$
(2) $x \in I$ implies $x^{00} \subseteq I$
(3) $xy = 0$ implies $x \in I$ or $y \in I$;

let $I \subseteq R$ be an ideal satisfying (1)-(3).

The ideal $t_0 \subseteq \mathscr{B}(R)$ generated by the set $\{x^{00} \vee y^0\colon x \in I,\ y \notin I\}$ is then a proper ideal: if $x_1, x_2 \in I$ then

$$x_1^{00} \vee x_2^{00} = (1 - \mathrm{id}(x_1)^{00} \vee (1 - \mathrm{id}(x_2))^{00} = [(1 - \mathrm{id}(x_1)) \vee (1 - \mathrm{id}(x_2))]^{00}$$

which is contained in I, since

$$1 - \mathrm{id}(x_i) \in (1 - \mathrm{id}(x_i))^{00} = x_i^{00} \subseteq I \ (i = 1, 2)$$

gives $(1 - \mathrm{id}(x_1)) \vee (1 - \mathrm{id}(x_2)) \in I$.

Induction shows that if $x_1, \cdots, x_m \in I$ then $x^{00} \vee \cdots \vee x_m^{00} \subseteq I$. If t_0 is not a proper ideal then there exist $x_1, \cdots, x_m \notin I$ and $y_1, \cdots, y_n \in I$ such that

$$R = \bigvee_{i=1}^{m} x_i^{00} \vee \bigvee_{i=1}^{n} y_i^0$$

Then

$$\bigcap_{i=1}^{n} y_i^{00} = \bigcap_{i=1}^{n} y_i^{00} \cap R = \bigvee_{i=1}^{m} x_i^{00} \cap \bigcap_{i=1}^{n} y_i^{00}$$

so that

$$\bigcap_{i=1}^{n} y_i^{00} \subseteq \bigvee_{i=1}^{m} x_i^{00} \subseteq I$$

and thus $\bigwedge_{i=1}^{n} 1 - \mathrm{id}(y_i) \in I$.

If e, f are central idempotents and $ef \in I$ then

$$(ef-e)(ef-f)=0$$

shows that $ef-e\in I$ or $ef-f\in I$ so $e\in I$ or $f\in I$.

Induction gives that if $\wedge_{i=1}^{n} e_i\in I$ then $e_i\in I$ for some i. Consequently, $1-\mathrm{id}(y_i)\in I$ for some i, so that $y_i\in I$, contrary to the choice of y_i. Hence t_0 is proper ideal in $\mathscr{B}(R)$ and is therefore contained in a maximal ideal t. It is readily seen that $R_t = I$. Conversely it is easily seen that every $R_t\subseteq R$ satisfies (1)$-$(3). Now let I be a projection ideal such that $I\cap B(R)$ is a prime ideal in $B(R)$. Then $1\notin I$, and if $x\in I$, $x^0=(0)$ then $(1-\mathrm{id}(x))=(0)$ so $\mathrm{id}(x)=.0$ and thus $1=1-\mathrm{id}(x)\in I$. That is, I satisfies (1). Suppose $x\in I$, $y\in x^{00}$. Then $\mathrm{id}(x)\leqq\mathrm{id}(y)$ so

$$1-\mathrm{id}(y)\leqq 1-\mathrm{id}(x)\in I,$$

which gives $1-\mathrm{id}(y)\in I$ and hence $y\in I$. Finally, suppose that $xy=0$. Then

$$[1-\mathrm{id}(x)]\wedge[1-id(y)]=0$$

so $1-\mathrm{id}(x\in I$ or $1-\mathrm{id}(y)\in I$, and therefore $x\in I$ or $y\in I$. Hence I satisfies (1)-(3).

Conversely, any ideal satisfying (1)-(3) is a projection ideal whose intersection with $B(R)$ is a prime ideal.

For a semi-prime ring R the class of proper ideals $R_t\subseteq R$ can be given a topology that is compact if R has an identity, and a Boolean space if R is projectable. Denote by $\mathscr{V}(R)$ the class of ideals $R_t\neq R$, $t\in\mathscr{Q}$. For $x\in R$, put $\mathscr{V}(x) = \{R_t: x\notin R_t\}$.

Proposition 3.3. *The class $v_R=\{\mathscr{V}(x): x\in R\}$ is an intersection semi-lattice and so forms a base for the open sets for a topology on $\mathscr{V}(R)$. If R has an identity then $\mathscr{V}(R)$ is compact. If R is projectable then $v_R=\{\mathscr{V}(x):\ e\in B(R)\}$ is a lattice for union and intersection and $\mathscr{V}(R)$ is a Boolean space.*

Proof. $\mathscr{V}(x)\cap\mathscr{V}(y)$

$$\begin{aligned}&=\{R_t\neq R: x^{00}\notin t,\ y^{00}\notin t\}\\&=\{R_t\neq R: x^{00}\cap y^{00}\notin t\}\\&=\{R_t\neq R: (xy)^{00}\notin t\}\\&=\mathscr{V}(xy).\end{aligned}$$

If R has an identity then the map $\phi: t\mapsto R_t$ maps $\mathscr{Q}$ onto $\mathscr{V}(R)$. For $x\in R$,

$$\phi^{-1}(\mathscr{V}(x))=\{t\in\mathscr{Q}: Rt\in\mathscr{V}(x)\}=\mathscr{Q}_{x^{00}}$$

so ϕ is continuous. Since $\mathscr{Q}$ is compact so is $\mathscr{V}(R)$. Now assume R is projectable. Then every $\mathscr{V}(x)$ is of the form $\mathscr{V}(e)$ for some central idempotent e: in fact $\mathscr{V}(x)=\mathscr{V}(1-\mathrm{id}(x))$. Also, $\mathscr{V}(R)$ is a union semi-lattice since

$$\begin{aligned}\mathscr{V}(x) \cup \mathscr{V}(y) &= \{R_t \neq R : x^{00} \notin t \text{ or } y^{00} \notin t\}\\ &= \{R_t \neq R : x^{00} \vee y^{00} \notin t\}\\ &= \{R_t \neq R : [(1-\mathrm{id}(x)) \vee (1-\mathrm{id}(y))]^{00} \notin t\}\\ &= \mathscr{V}(1-\mathrm{id}(x) \wedge \mathrm{id}(y)).\end{aligned}$$

If $R_{t_1} \neq R_{t_2}$ then there is an $x \in R_{t_1}$, $x \notin R_{t_2}$. Then $1 = x_1 + x_2$ with $x_1 \in x^{00}$, $x_2 \in x^0$, so that $x_1 \in R_{t_1}$ but $x_2 \notin R_{t_1}$, for otherwise $1 \in R_{t_1}$. Thus $R_{t_1} \in \mathscr{V}(x_2)$, $R_{t_2} \in \mathscr{V}(x)$ and $\mathscr{V}(x_2) \cap \mathscr{V}(x)$ is void. That is, $\mathscr{V}(R)$ is a Hausdorff space. Finally, let e be a central idempotent. Then for $t \in \mathscr{Q}$ either $e \in R_t$ or $1-e \in R_t$ but not both since $1 \notin R_t$, and therefore

$$\begin{aligned}\mathscr{V}(R)(e) &= \{R_t : e \in R_t\}\\ &= \{R_t : 1-e \notin R_t\}\\ &= \mathscr{V}(1-e)\end{aligned}$$

so the basic open sets $\mathscr{V}(e)$ are closed-open.

Note that the ideals R_t in a semi-prime ring R are just those used by Keimel [4] and Adams [1]. These ideals were also used by Veksler [8] in a more general setting. In a commutative semi-prime ring R every minimal prime ideal is an R_t, and in the next section the converse of this is considered. The minimal prime ideals in a non-commutative semi-prime ring R are characterized as those prime ideals P satisfying

$$P = 0_p = \{x \in R : xa = 0 \text{ for some } a \notin P\},$$

Koh [6], and it then follows as in 2.6 that every minimal prime ideal of R is an R_t.

4. Commutative semi-prime rings

For a commutative semi-prime ring R there are several conditions that imply that $\mathscr{V}(R)$ is a Hausdorff topological space, and if the annihilators x^{00}, $x \in R$, form a sublattice of $\mathscr{B}(R)$ then these conditions are equivalent to the Hausdorff property of $\mathscr{V}(R)$. Throughout this section R will be assumed commutative and semi-prime.

The class of minimal prime ideals of R is denoted by $\mathscr{M}(R)$, and the sets $\mathscr{M}(x) = \{\mathscr{M} \in \mathscr{M}(R) : x \notin \mathscr{M}\}$, for $x \in R$, form a closed-open base for the open sets for a Hausdorff topology on $\mathscr{M}(R)$.

THEOREM 4.2. *Consider the following statements*:

(1) *For every* $x \in R$ *there is an* $x' \in R$ *such that* $x^{00} = (x')^0$
(2) *For all* $x, y \in R$ *there is an* $a \in x^{00} \oplus x^0$ *such that* $y^0 = a^0$
(3) $\mathscr{V}(R) = \mathscr{M}(R)$
(4) *Each* $\mathscr{V}(x)$ *is closed in* $\mathscr{V}(R)$
(5) $\mathscr{V}(R)$ *is Hausdorff*
(6) v_R *is relatively complemented.*

Then

[*a*] (1) *implies* (2)
[*b*] *If R has an identity then* (2) *implies* (1)
[*c*] (2) *implies* (3)
[*d*] (3), (4), (5) *are equivalent*
[*e*] (2) *implies* (6)

If the intersection semi-lattice $\{x^{00}: x \in R\}$ *is a sub-lattice of* $\mathscr{B}(R)$ *then* (2)-(6) *are equivalent.*

PROOF. (1) *implies* (2): for $x, y \in R$ suppose that $(x')^0 = x^{00}$. Then $xy \in x^{00}$, $x'y \in x^0$ and

$$(xy + x'y)^{00} = ((x + x')y)^{00} = (x + x')^{00} \cap y^{00} = [x^{00} \vee (x')^{00}] \cap y^{00} = R \cap y^{00} = y^{00}.$$

If R has an identity 1 then (2) implies (1): for $x \in R$, $1 = a + b$ with $a \in x^{00}$, $b \in x^0$ and

$$(0) = 1^0 = (a + b)^0 = a^0 \cap b^0,$$

so that $a^{00} = b^0$. Then $x^{00} \subseteq b^0$ and $b^0 = a^{00} \subseteq x^{00}$, so $b^0 = x^{00}$.

(2) *implies* (3): suppose $x \in R_t \in \mathscr{V}(R)$. Then there is a $y \notin R_t$ and $y_1 \in x^{00}$, $y_2 \in x^0$ such that

$$y^0 = (y_1 + y_2)^0 = y_1^0 \cap y_2^0.$$

If $y_2 \in R_t$ then $y^0 = y_1^0 \cap y_2^0 \notin t$ so $y \in R_t$. Thus $y_2 \notin R_t$ and $y_2 x = 0$ so R_t is minimal prime. The preceding lemma says that every minimal prime is an R_t, so that $\mathscr{V}(R) = \mathscr{M}(R)$.

(3) *implies* (4): if each $R_t \in \mathscr{V}(R)$ is minimal prime then for $x \in R$, $\mathscr{V}(x) = \mathscr{M}(x)$ is closed in $\mathscr{V}(R) = \mathscr{M}(R)$.

(4) *implies* (3): suppose $x \in R_t \in \mathscr{V}(R)$. Then $t \notin \mathscr{V}(x)$ so there is a basic open set $\mathscr{V}(y)$ such that

$$t \in \mathscr{V}(y) \subseteq \mathscr{V}(R) \setminus \mathscr{V}(x)$$

and $\mathscr{V}(y) \cap \mathscr{V}(x)$ is void. That is, $y \notin R_t$ and $xy = 0$ so R_t is minimal prime.

(3) *is equivalent to* (5): if $\mathscr{V}(R) = \mathscr{M}(R)$ then $\mathscr{V}(R)$ is Hausdorff. Conversely, if $\mathscr{V}(R) \neq \mathscr{M}(R)$ then there is a proper R_t that properly contains a minimal prime ideal M. Then R_t and M cannot be Hausdorff separated.

(2) *implies* (6): suppose $\mathscr{V}(x) \in v_R$ and $\mathscr{V}(y) \subseteq \mathscr{V}(x)$. Then $y^{00} \subseteq x^{00}$, and there exist $a \in y^0$, $b \in y^{00}$ such that

$$x_{00} = (a + b)^{00} = a^{00} \vee b^{00},$$

so that $xa \in y_0 \cap x^{00}$ and

$$\begin{aligned}(xa)^{00} \vee y^{00} &= (x^{00} \cap a^{00}) \vee y^{00} \\ &= x^{00} \cap (a^{00} \vee y^{00}) \\ &= a^{00} \vee y^{00} = x^{00},\end{aligned}$$

since $x^{00} \supseteq a^{00} \vee y^{00} \supseteq a^{00} \vee b^{00} = x^{00}$. Thus, $\mathscr{V}(xa) \cap \mathscr{V}(y)$ is void and $\mathscr{V}(xa) \cup \mathscr{V}(y) = \mathscr{V}(x)$, so that v_R is relatively complemented.

Now suppose that $\{x^{00} : x \in R\}$ is a sub-lattice of $\mathscr{B}(R)$. That is, for all $x, y \in R$ there is an $a \in R$ such that $x^{00} \vee y^{00} = a^{00}$.

(3) *implies* (2): suppose that R does not have property (2). Then there exist $x, y \in R$ such that for all $a \in x^0$, $y \notin (x + a)^{00}$. The subset

$$t_0 = \{A^0 \in \mathscr{B}(R) : A^0 \subseteq (x + a)^{00} \vee y^0, \text{ for some } a \in x^0\}$$

is then an ideal of $\mathscr{B}(R)$, since for $a, b \in x^0$,

$$(x + a)^{00} \vee y^0 \vee (x + b)^{00} \vee y^0 = x^{00} \vee a^{00} \vee b^{00} \vee y^0 = x^{00} \vee c^{00} \vee y^0,$$

for some $c \in R$ and $c \in a^{00} \vee b^{00} \subseteq x^0$. If t_0 is not a proper ideal then

$$R = (x + a)^{00} \vee y^0 = x^{00} \vee a^{00} \vee y^0$$

for some $a \in R$, so $y^0 = y^{00} \cap R = (x^{00} \vee a^{00}) \cap y^{00}$ and therefore $y \in y^{00} \subseteq x^{00} \vee a^{00}$, contrary to assumption. Then t_0 is contained in a prime ideal t and $R_t \neq R$ since $y^{00} \notin t$. If $a \in x^0$ then

$$a^{00} \subseteq x^{00} \vee a^{00} \vee y^0 = (x + a)^{00} \vee y^0 \in t$$

so $a \in R_t$. That is, $x \in R_t$ and $x^0 \subseteq R_t$ so R_t is not minimal prime.

(6) *implies* (2): if v_R is relatively complemented and $x, y \in R$ then $\mathscr{V}(x) \subseteq \mathscr{V}(a)$, where $a^{00} = x^{00} \vee y^{00}$, so there is an $x' \in R$ such that $x'x = 0$ and $(x')^{00} \vee x^{00} = x^{00} \vee y^{00}$. Then

$$(x'y + xy)^{00} = (x' + x)^{00} \cap y^{00} = (x^{00} \vee y^{00}) \cap y^{00}$$

and $x'y \in x^0$, $xy \in x^{00}$.

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