1.7 Bolzano- Weierstrass Theorem

The Bolzano- Weierstrass Theorem states that every bounded sequence of real numbers has a convergent sub-sequence. There are 3 different types of proof to prove this theorem. I prefer the second proof of the Bolzano- Weierstrass. We will try to prove some implications of this theorem.

Theorem 15. *Every sequence $x : \mathbb{N} \to \mathbb{R}$ either has a sub-sequence that is increasing or a sub-sequence that is decreasing.*

Proof. This statement comes from Proof 2 of the Bolzano Theorem. In order to prove the statement, it requires knowledge of increasing or decreasing sequences from Advanced Calculus 1.

Assume there is an arbitrary sequence. Lets choose an arbitrary point, called a_m. Now, consider the successive term of the sequence, called n, where $n \geq m$. If $a_m > a_n$ and there is no terms in the sequence larger than a_m, the sequence is decreasing. We can call this a_m value the 'peak'. An increasing sequence will have no peaks since we can also find a term larger than a_m. Using the definition of a peak, a decreasing sequence will have a peak at each successive term since each term is smaller than the previous. When I think of the word peak, it reminds me of the max and min of a function. Lets consider two cases:

Case 1: The sub sequence is decreasing. Assume that $n_1 < n_2 < n_3 ...$ and $a_{n_1} \geq a_{n_2} \geq a_{n_3}$. This means that each successive term is smaller than the first, so the sub sequence must be decreasing. This also means multiple peaks exist since for an arbitrary value, such the previous value is always larger than the values after it.

Case 2: The sub sequence is increasing. Assume there exists an infinite number of peaks, such as what we saw when we investigated $sin(n)$ on Mathematica. Lets call these peak values $a_{n_1}, a_{n_2}, a_{n_3}, ..., a_{n_k}$. Now, lets call $s_1 = n_k + 1$ the next value after the previous peak a_n. We can assume a_{s_1} is not a peak since there exists only a finite amount. So, we move to s_2 where $s_1 < s_2$ and $a_{s_1} \leq a_{s_2}$. Again, s_2 cannot be a peak since each successive term is larger than the previous. We can continue this process to find that $a_n < a_{n+1}$ for each given term. So, we can form a sub sequence $n_1 < n_2 < n_k$ where $a_{n_1} < a_{n_2} < a_{n_k}$.

Remarks: This means a sub-sequence can either be increasing or decreasing. It all depends on whether the successive terms are all smaller or larger than the previous term. And, the sequence must be monotonic. □

Theorem 16. *Every increasing sequence $x : \mathbb{N} \to \mathbb{R}$ that is bounded above converges to a limit, and every decreasing sequence $x : \mathbb{N} \to \mathbb{R}$ that is bounded below converges to a limit.*

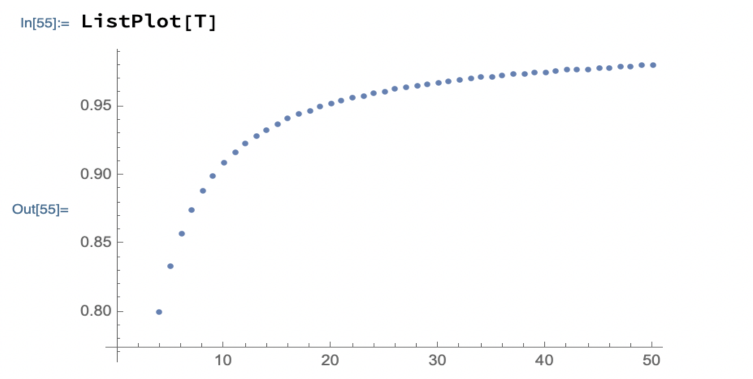
Proof. This is the second part to proof two of the Bolzano- Weierstrass Theorem. When I read this theorem, I though it probably involves the inf and sup from Riemann integrals. I also remember what it means to be bounded above and below. For example, if we look at $1 - 1/n$ for $n > 0$, we see that it is increasing and bounded above.

Lets look at the increasing sequence. Assume that there exists a least upper bound L of the sequence x_n and that there is an arbitrary $\epsilon > 0$. I predict that the least upper bound of the sequence will be equal to $sup(x_n)$. Since L is denoted as the least upper bound of the sequence, there must be some x_{n0} where $x_{n0} \geq L - \epsilon$. We also know that the sequence is increasing for all $x_n \geq x_{n0}$ when $n \geq n0$. As a direct consequence,

$$L - \epsilon \leq x_n \leq L + \epsilon$$

By the definition of limits, we can see that the increasing sequence x_n converges to L, or the least upper bound. Using a similar argument, we could also prove that a decreasing sequence converges to the lower bound (inf).

I thought of a sequence that is increasing and bounded above $s_n = n/(n+1)$. I have attached a graph of the sequence below. The limit of the sequence is 1, which is the least upper bound (sup). Since the sequence is bounded over a closed interval, this means that it is Riemann Integrable.



Remarks: The last two statements can be summarized. Every sequence has a sub-sequence that is either increasing or decreasing. An increasing sequence that is bounded above converges to the upper sum and every decreasing sequence that is bounded below converges to the lower sum. This helps to show that every bounded sequence of real numbers has a convergent sub-sequence. □

Theorem 17. *Every sequence has a lim inf and lim sup.*

Proof. Will researching about sequential compactness, I found that a sequence has two different limits: the limit of the upper sum and the limit of the lower sum. From the previous example, we know that a monotonic sequence is either increasing or decreasing, and is convergent. As a result, these quantities will exist for all bounded real number sequences. This results in two important implications. If the lim inf = lim sup, then the sequence is convergent. If at least one of the lim inf or lim sup equals ∞, then the sequence is divergent and unbounded. Lets investigate a simple sequence to see the limits of the lower and upper sums.

Lets look at the increasing, bounded sequence $1 - 1/n$. Then there exist two quantities: $\sup(1 - 1/x, x \geq N)$ and $\inf(1 - 1/x, x \geq N)$. The limit of the sup and inf both equal 1, so the sequence converges to 1 and is bounded. In this case, none of the limits headed to infinity. □

Chapter 2

Dirichlet and Riemann's functions

Theorem 18. *Argue that the Dirichlet Function $D : \mathbb{R} \to \mathbb{R}$ is not continuous at any $x_0 \in \mathbb{R}$.*

Proof. The Dirichlet Function is defined as the following: 1 if $x_0 \in \mathbb{Q}$ and $0 \notin \mathbb{Q}$. It is hard to visualize this function because there is an infinite amount of rational numbers between irrational numbers. The Dirichlet Functions turns out to be continuous and not Riemann Integrable (the fundamental theorem of calculus fails).

First off, between every two irrational numbers there exists rational numbers. Therefore, the graph of the function would be filled with many discontinuous holes. For a function to be continuous, the right and left hand limits must be equal to each other. However, we cannot even find a one-sided limit of the Dirichlet Function. For an arbitrary number n, there exists both rational and irrational numbers nearby. Therefore, we cannot find a limit and the function D cannot be continuous.

Another result of the Dirichlet Function being discontinuous at any $x_0 \in \mathbb{Q}$ is that it does not have derivatives and is not Riemann Integrable. □

Theorem 19. *Argue that the restriction of Dirichlet's function to any closed bounded interval, $D : [a, b] \to \mathbb{R}$ with $a < b$, is not Riemann integrable*

Proof. Recall that we need to find the upper and lower sums in order to find the Riemann Integral:

$L(f, P) := \sum_{i=0}^{n-1} m_i(x_{i+1} - x_i)$

$U(f, P) := \sum_{i=0}^{n-1} M_i(x_{i+1} - x_i)$

The only two values that the Dirichlet function can take on are $0, 1$. Define the Max of the function as $x = 1$ and the Min of the function as $x = 0$. Therefore, $L(f, p) = 0$, $U(f, p) = 1$ and the difference will always be $1 - 0 = 1$. Since the difference will always be constant, there is no way to make ϵ small enough so that $|f(x) - f(y)| \leq \epsilon$. We could always let $\epsilon > 1$, and then the definition of uniform continuity and the integral would fail.

In the next example, we will see that some discontinuous functions are Integrable, such as the Riemann-Zeta Function. □