What is ... the Kakeya Needle Problem?

by Daniel Glasscock.

In 1917, Japanese mathematician Sōichi Kakeya (1886–1947) posed the following problem:

"What is the oval of the smallest area, within which a given oval is revolvable?"

Here, "oval" is a closed, convex curve, and "revolvable" means able to be moved continuously so that some point on the oval completes a 360° revolution.

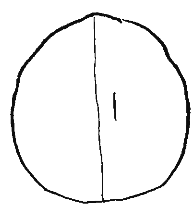
The line segment is one of the simplest ovals, and Kakeya considered this special case of the problem. This led to the now famous, Kakeya Needle Problem:

"What is the planar figure of least area in which a unit line segment may be turned through 360° by a continuous movement?"

Call such a figure a <u>Kakeya needle set</u>.

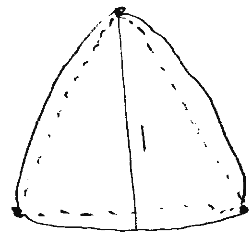
(Needle refers to a unit line segment.)

What are some examples of Kakeya needle sets?



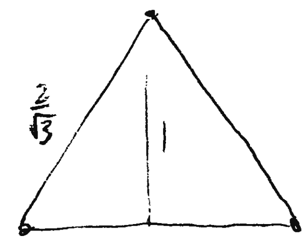
Circle, radius $\frac{1}{2}$

$A = \frac{\pi}{4} \approx 0.785$



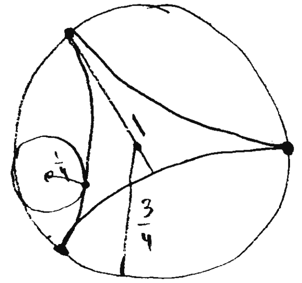
Reuleaux triangle*

$A = \frac{\pi - \sqrt{3}}{2} \approx 0.705$



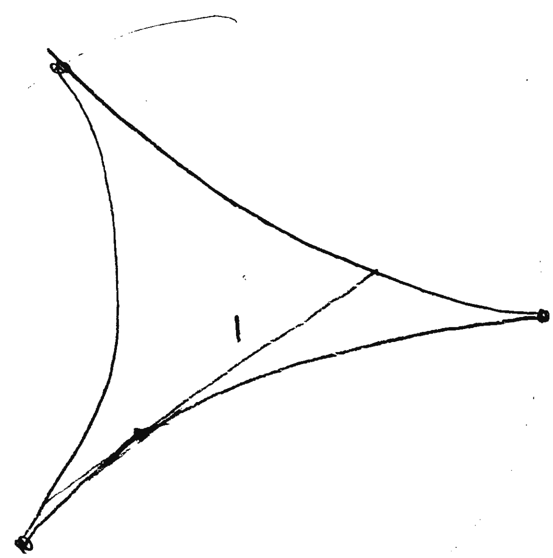
Equilateral triangle

$A = \frac{1}{\sqrt{3}} \approx 0.577$



Hypocycloid in a circle of radius of $\frac{3}{4}$

$A = \frac{\pi}{8} \approx 0.393$



Any tangent of the hypocycloid intersects the hypocycloid at two points a distance of exactly 1 apart.

It was conjectured at the time that the hypocycloid may be the solution to Kakeya's problem. G.D. Birkhoff(‡) mentioned this problem in his book The Origin, Nature and Influence of Relativity in 1925.

(‡) George David Birkhoff, 1884–1944, American Mathem.

* Intersection of 3 disks; smallest area of any curve of constant width (Blaschke-Lebesgue theorem

Russian mathematician Abram S. Besicovitch disproved the conjecture when he solved the Kakeya needle problem in 1928:

<u>Theorem</u> (Besicovitch) There exist Kakeya needle sets of arbitrarily small area; that is, given $\varepsilon > 0$, there exists a Kakeya needle set of area less than ε.

Let's make this a bit more formal: A function

$$f : [0,1] \to \mathbb{R}^2 \times S^1$$

describes, at time t, the location $f_1(t)$ of the center of a line segment in the direction $f_2(t)$. Then

i) "Continuous movement" means f is continuous

ii) "rotated through 360°" is implied by $\pi_2 \circ f : [0,1] \to S^1$ onto

iii) "in the set K" means $K \supseteq \bigcup_{t \in [0,1]} \left\{ \begin{array}{c} \text{unit line segment with} \\ \text{center at } f_1(t) \text{ in direction} \\ \text{of } f_2(t) \end{array} \right\}$

So, $K \subseteq \mathbb{R}^2$ is a Kakeya needle set if and only if there exists a function $f : [0,1] \to \mathbb{R}^2 \times S^1$ satisfying i), ii), and iii).

By "area," we mean planar Lebesgue measure. Think of this as a notion of area which generalizes the usual area you know and is mathematically consistent. For this talk, it will be enough to be able to distinguish between "zero area" and "positive / non zero area".

Def: A set $E \subseteq \mathbb{R}^2$ is of measure 0 (that is, has zero area) if for all $\varepsilon > 0$, there exists balls $B_1, \ldots, B_N$ $(N = N(\varepsilon))$ with radii $r_1, \ldots, r_N$, respectively, so that $E \subseteq B_1 \cup \cdots \cup B_N$ and $\sum_{i=1}^{N} r_i < \varepsilon$.

The theorem of Besicovitch is only a solution to the Kakeya Needle Problem once you have the following exercise:

Exercise*: Prove that a Kakeya Needle set necessarily has positive measure (area).

* For those of you with some analysis background :).

Before outlining Besicovitch's solution, let's take a historical step back to see why Besicovitch was primed to solve the problem. Prior to 1920, while still in Russia, Besicovitch considered the following problem:

"Given $X \subseteq \mathbb{R}^2$ and $f: X \to \mathbb{R}$ Riemann integrable, does there exist a pair of mutually orthogonal directions for which the double integral of f along these two directions exists and is equal to the integral of f?"

Besicovitch published the solution in a Russian periodical in 1919; he constructed a planar set B of zero measure which contains a unit line segment in every direction, then showed that such a set may be used to answer "No" to the question above.

Def: Call a set $B \subset \mathbb{R}^2$ a Besicovitch set (sometimes, Kakeya set) if it contains a unit line segment in every direction.

Kakeya posed his problem 3 years earlier in 1917, but because of the state of Russia at the time, Kakeya's 1917 paper never made it into and Besicovitch's 1920 paper never made it out of Russia. Besicovitch had not heard of Kakeya's Needle Problem, nor did anyone outside of Russia hear about Besicovitch's result.

In 1924, Besicovitch emigrated to Europe, spending time in Copenhagen, Oxford, Liverpool, and Cambridge where he remained until his death in 1970.

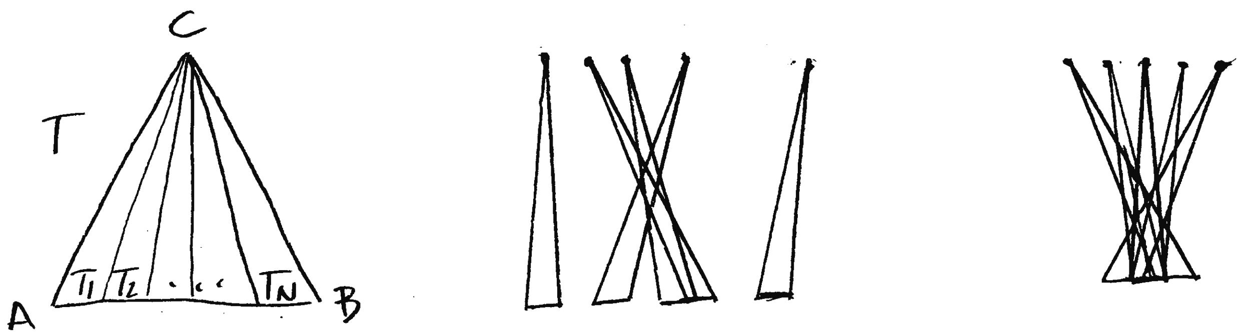
While in Copenhagen, he met a Hungarian mathematician named Gyula Pál. Pál knew well about the Kakeya Needle problem, having solved the convex variant 3 years prior to Besicovitch's arrival (see K19-20). Pál introduced Besicovitch to the problem, and had the key idea of how to turn a Besicovitch set into a Kakeya needle set.

Besicovitch's solution to the Kakeya Needle Problem goes like this:

1) Construct a Besicovitch set with small area

2) Union this set with another set of small area which allows for a continuous rotation of the needle.

We'll begin with 1). The construction of such a set relies on cutting and overlaying triangles to achieve a small area.



Cut the triangle $T = \Delta ABC$ into smaller triangles $T_1, \ldots, T_N$ along lines originating from C and terminating along AB, as shown.

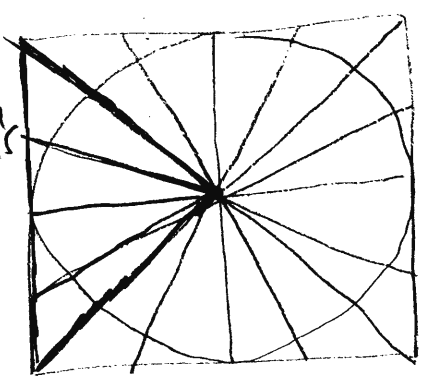
Translate each T_i horizontally so as to minimize the area of $\bigcup_{i=1}^{N} T_i$.

<u>Lemma</u> (Besicovitch) Given $\varepsilon > 0$, there exists an N such that T can be cut into triangles $T_1, \ldots, T_N$ as above and those triangles translated to $\tilde{T}_1, \ldots, \tilde{T}_N$ so that $\text{area}(\bigcup \tilde{T}_i) < \varepsilon$.

Besicovitch proved this lemma in 1920, and simpler proofs have since been developed by Perron (1929) and Schoenberg (1962).

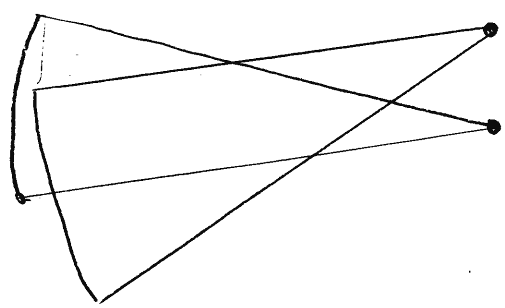
How does this lemma help? Since $T' = \bigcup_{i=1}^{N} T_i$ was obtained by cutting and translating, the set of directions in which T contains a unit line segment is the same as the set of directions in which T' contains a unit line segment.

So, lets apply this lemma to a disk of radius 1. It is an easy corollary of Besicovitch's lemma that for $\varepsilon > 0$, the disk may be cut into N sectors and those N sectors translated about so that the total area of the resulting figure is less than ε. From the remark above, the resulting figure contains a unit line segment in every direction!

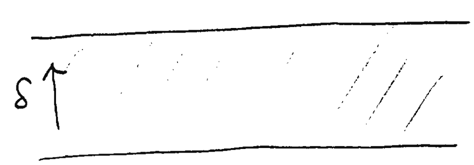


This completes 1): we have a Besicovitch set of small area.

The next step is to make this a Kakeya Needle set. So far, we are only guaranteed to be able to rotate a segment continuously through an angle of $\frac{2\pi}{N}$. How can we "jump" to the next sector?

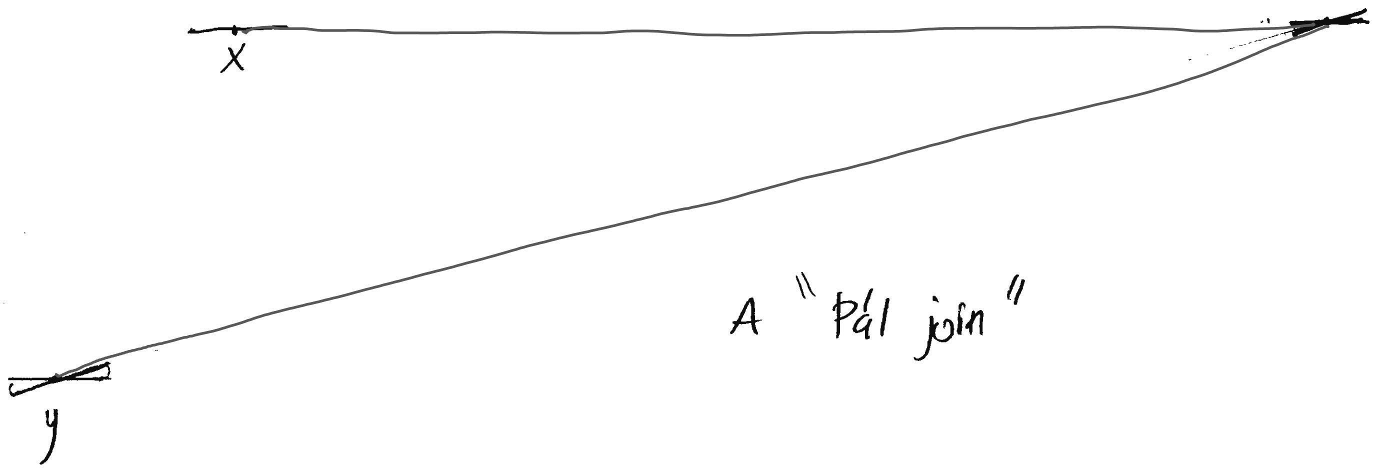


We need to be able to translate the needle (ie. move it with rotating it) to a new position. Naïvely translating may result in leaving the set - in fact, it likely will since translating a unit segment a distance δ can take up as much area as δ.



The question becomes, can we possibly translate a needle using a very small area?

Besicovitch credits Gyula Pál with the solution!



A "Pál join"

To translate from x to y, slide the needle in the direction it points very far away, turn it so that it points toward y, slide it to y, then turn it back. As you send the needle further and further out, this move requires less and less area! This shape is called a "Pál join".

So, suppose our Besicovitch set from 1) is comprised of N sectors and has area less than $\varepsilon/2$. We need only to union this set with N "joins" as above. If we make each join take less than $\frac{\varepsilon}{2N}$ area, then the area of the resulting Kakeya Needle set will be less than ε.

As mentioned before, Besicovitch was primed to solve the Kakeya Needle Problem due to his work on the "twin problem" (as he called it):

<u>Theorem</u> (Besicovitch, 1920) There exists a Besicovitch set of zero measure.

To prove this, Besicovitch took a suitable limit of Besicovitch sets with smaller and smaller area. He showed that the limiting set remains a Besicovitch set and has zero measure.

This begs the question: <u>just how large must Besicovitch sets be</u>? Area is just one notion of the "size" of a set; there are many other natural notions of size which allow us to distinguish between sets of zero area. Indeed, both a point and a line have zero planar area, but surely we can say more than that!

Interestingly, the question of how large Besicovitch sets must be is currently driving very modern research in several diverse fields!

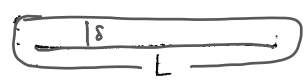
One common way to measure the size of sets in the plane (or in $\mathbb{R}^n$ in general) is called the (upper) Minkowski dimension.

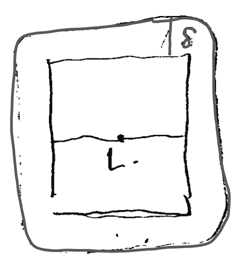
For a set $E \subseteq \mathbb{R}^2$ and a positive number $\delta > 0$, the δ-neighborhood of E, denoted $N_\delta(E)$ is the set of points a distance δ away from E:

$$N_\delta(E) = \{x \in \mathbb{R}^2 \mid \inf_{e \in E} |x - e| < \delta\}.$$

The set $N_\delta(E)$ will have positive area (measure) if E is nonempty, so we can ask: for what number c does the area of $N_\delta(E)$ tend to zero like δ^c?

Here are some examples:





$\text{Area}(N_\delta(\text{point})) \approx \delta^2$. So, $c = 2$.

$\text{Area}(N_\delta(\text{line})) \approx \delta L$ So, $c = 1$

$\text{Area}(N_\delta(\text{square})) \approx L^2 + 4\delta$ So, $c = 0$.

In fact,

$$\frac{\text{Area}(N_\delta(\text{point}))}{\delta^{2-d}} \xrightarrow[\delta \to 0]{} 0$$

for all $d > 0$

$$\frac{\text{Area}(N_\delta(\text{line}))}{\delta^{2-d}} \xrightarrow[\delta \to 0]{} 0$$

for all $d > 1$

$$\frac{\text{Area}(N_\delta(\text{disk}))}{\delta^{2-d}} \xrightarrow[\delta \to 0]{} 0$$

for all $d > 2$.

This leads us to define the (upper) Minkowski dimension as

$$\overline{\dim}_M E = \inf \{ d \geq 0 \mid \frac{\text{Area}(N_\delta(E))}{\delta^{2-d}} \xrightarrow[\delta \to 0]{} 0 \}$$

If the precise definition eludes you, think of it this way: every set has an (upper) Minkowski dimension; it's some number between 0 and 2 which measures how large the set is based on the area of the sets' neighborhoods. Points have dimension 0, lines dimension 1, and sets with positive area have "full" dimension, that is dimension 2.

Exercise: Show that any countable set which is dense in $[0,1]^2$ has zero measure but full (upper) Minkowski dimension.

So, Besicovitch sets may have zero measure, but now it's natural to ask whether they can have dimension less than 2. In fact, the answer is No! (in more ways than 1).

<u>Theorem</u> (Davies, 1971) A Besicovitch set necessarily has <u>Hausdorff</u> dimension 2. In particular, it has Minkowski dim 2.

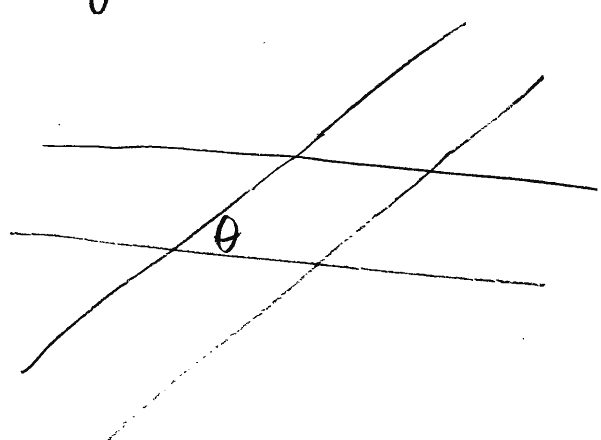
I won't define Hausdorff dimension; what is important to know is that the Hausdorff dim of E is always $\leq \dim_M E$. Therefore, it is an immediate corollary of Davies' theorem that the Minkowski dimension of a Besicovitch set is 2. In other words, having a unit line segment in every direction <u>does</u> force the set to be quite large, in the sense of having full dimension.

Let $B \subseteq \mathbb{R}^2$ be a Besicovitch set. Since B contains a line segment, clearly $\dim_m B \geq 1$. Let's show via an elementary argument that $\dim_m B \geq \frac{3}{2}$.

Let $\delta > 0$; we're interested in bounding $\mathrm{vol}(N_\delta(B))$ from below. Here's the idea: B contains a line segment in every direction, so $N_\delta(B)$ contains a δ-tube (pill shaped) in every direction. These tubes overlap a lot, but two tubes which are sufficiently transverse don't overlap too much. So, if we can pick out many tubes which have pairwise small intersections, then we'll know that $\mathrm{vol}(N_\delta(B))$ must be large.

Lemma: Two δ-tubes in $\mathbb{R}^2$ forming an angle of $0 < \theta \leq \frac{\pi}{2}$ intersect in an area of less than $\frac{\delta^2}{\theta}$.

(In fact, the area is exactly $\frac{\delta^2}{2\sin\theta}$)



The more tubes we look at, the (potentially) larger their union but the smaller the angle we'll be able to guarantee between them.

So, consider k tubes $T_1, \ldots, T_k \subseteq N_\delta(B)$. We can guarantee an angle of $\frac{2\pi}{k}$ between each of them. Then, using the previous lemma and inclusion-exclusion, we write

$$\begin{aligned}
\mathrm{vol}(N_\delta(B)) &\geq \mathrm{vol}\left(\bigcup_{i=1}^{k} T_i\right) \\
&\geq \sum_{i=1}^{k} \mathrm{vol}(T_i) - \sum_{i \neq j} \mathrm{vol}(T_i \cap T_j) \\
&\geq k \cdot 2\delta - \binom{k}{2} \frac{\delta^2}{2\pi/k} \\
&\geq 2\delta k - k^3 \delta^2 / 2\pi \\
&\geq 2\delta k - k^3 \delta^2 .
\end{aligned}$$

We want to maximize this in k; basic calculus tells us that $k = \left(\frac{2}{3\delta}\right)^{\frac{1}{2}}$, in which case $\mathrm{vol}(N_\delta(B)) \geq c\,\delta^{\frac{1}{2}}$ for some positive constant c. This tells us that $\frac{\mathrm{vol}(N_\delta(B))}{\delta^{2-d}} \xrightarrow[\delta \to 0]{} 0$ only if $d > \frac{3}{2}$, whereby $\dim_M B \geq \frac{3}{2}$.

This argument goes through in exactly the same way in higher dimensions to give that the (upper) Minkowski dimension of a Besicovitch set in $\mathbb{R}^d$ must be $\geq \frac{d+1}{2}$.

In the nearly 100 years since its inception, the Kakeya Needle Problem and its variants have generated a tremendous amount of exceptional mathematics. It continues to be a focal point of modern research in many diverse fields:

- Incidence geometry
- Geometric measure theory
- Harmonic analysis
- Additive combinatorics
- Algebraic geometry
- Partial differential equations

$\vdots$

The biggest open problem here is:

Kakeya Needle Conjecture: A Besicovitch set $B \subseteq \mathbb{R}^d$ must have Hausdorff dimension d.

At the time of writing, the best known bounds for $B \subseteq \mathbb{R}^d$ are

$$\text{Hausdorff dim}(B) \geq \begin{cases} \frac{d+2}{2} & \text{for } d \leq 4 \\ (2-\sqrt{2})(d-4)+3 & \text{for } d \geq 5 \end{cases}$$

$$\overline{\text{Minkowski}} \text{ dim}(B) \geq \begin{cases} \frac{d+2}{2} & \text{for } d \leq 4 \\ (2-\sqrt{2})(d-4)+3 & \text{for } 5 \leq d \leq 22 \\ 0.597d + 0.403 & \text{for } d \geq 23. \end{cases}$$

Instead of listing 100 references, the reader is pointed to a recent beautiful comprehensive survey of Łaba:

Łaba, Izabella, From Harmonic Analysis to Arithmetic Combinatorics, Bull. Amer. Math. Soc. 2008 no. 1, 77–115.

(K19)

A random assortment of Kakeya problem variants:

"What is the planar figure in class $\mathcal{C}$ of minimal area in which a unit segment may be continuously rotated through 360°?"

$\mathcal{C}$ = all planar figures (original): Besicovitch, 1928

$\mathcal{C}$ = Convex figures: Gyula Pál (later, Julius Pál) proved in 1921 that the equilateral triangle of height 1 is the solution. ($A = \frac{1}{\sqrt{3}} \approx 0.57$)

$\mathcal{C}$ = regions of diameter $\leq 4+\varepsilon$: H. J. van Alphen, 1942.

$\mathcal{C}$ = Simply connected regions of diameter ≤ 2: F. Cunningham 1971

In both cases, sets of arbitrarily small area may be found.

Kakeya in $\mathbb{R}^3$ • Exercise: rotate a Kakeya needle set about a line to get a set in $\mathbb{R}^3$ with volume $< \varepsilon$.

• Marstrand 1979: If $E \subseteq \mathbb{R}^3$ is such that for all $x \in S^2$, E contains a plane normal to x, then E must have positive volume

(Generalizations of this to higher dimensions are due to Falconer 1980)

Q: What about "continuous rotation version" with unit squares instead of planes?

Varying the needle

- Exercise (?) : Kakeya with lines (in place of line segments) may still be accomplished in zero measure.
- Besicovitch & Rado 1968 : There exists a set of plane measure 0 containing circumferences of every radius.
- Davies 1971 : There exists a set in the plane of measure 0 containing a translate of every <u>polygonal arc</u>. Also, a figure consisting of a finite # of line segments can be rotated inside a planar set of arbitrarily small measure if and only if the segments are parallel.