The Schwarz-Christoffel Transformation and its Use in MATLAB

Chapter 11 of A MATLAB COMPANION TO COMPLEX VARIABLES BY

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The Schwarz-Christoffel transformation is an analytic transformation between domains in two complex planes. It is capable of mapping the upper half of one plane into a domain in another plane where the second domain is bounded by a polygon or by connected straight lines some of which extend to infinity. The analyticity breaks down at branch points and branch cuts. If we reverse a given transformation, we will have succeeded in mapping a domain whose boundary is a polygon into a half plane whose boundary is the real axis. If we wish to solve Laplace's equation within the polygonal domain, subject to certain boundary conditions as in electrostatics, fluid mechanics, and heat transfer problems we can instead solve the problem in an upper half space where now the boundary conditions are imposed on the real axis. Such problems are often solvable by means of the Poisson integral formula for the upper half plane .(See W sec. 47 and S p.146).

(Here W means the book Complex Variables with Applications by A.D Wunsch, 3rd ed. while S is the Schaum's Outline Series book on complex variables, 2nd ed by M.R Spiegel et. al.)

Most complex variables texts give a derivation of the formula known as the Schwarz-Christoffel transformation. We will begin with that result and hope to make it plausible. Imagine a portion of a polygon (see Figure 11.1(a)) in the complex z plane. We have shown only two vertices and three sides. The *interior* angles, which are *always non-negative*, measured positive counterclockwise, are $\alpha_1, \alpha_2, ... \alpha_n$. The exterior angles are $\pi - \alpha_1, \pi - \alpha_2, ... \pi - \alpha_n$. The path along the polygon is taken so that its interior is on the left as we move along the contour from z_1 to z_2 etc.. If the polygon is closed then you should recall that the sum of the exterior angles must be 2π. We now state without proof that a conformal mapping of the real axis in the $w = u + iv$ plane into the polygon in the z plane is possible with a transformation of this form:

$$z(w) = A\int^{w} (w-u_1)^{((\alpha_1/\pi)-1)}(w-u_2)^{((\alpha_2/\pi)-1)}.....(w-u_n)^{((\alpha_n/\pi)-1)} dw + B \qquad (11.1)$$

The upper half of the w plane is mapped one to one into the interior of the polygon. Note that our notation here is a little questionable: we are using w as both the variable of integration and as an upper limit of integration. But this will not present a difficulty—you can change w to w' under the integral sign, if you wish.

Here $u_1, u_2, ... u_n$ are real numbers *in ascending order*, lying on the real axis in the w plane, and are the images of the complex vertices of the polygon $z_1, z_2, ... z_n$ (see Figure 11.1(b)) encountered in order as we go around the polygon . For the moment we assume that no element of these 2 sets of numbers is infinite.
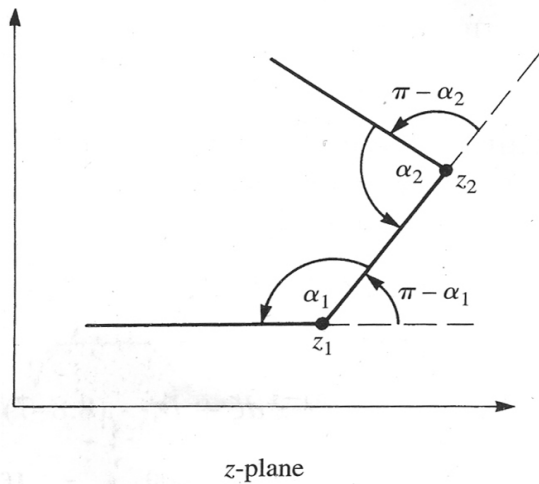
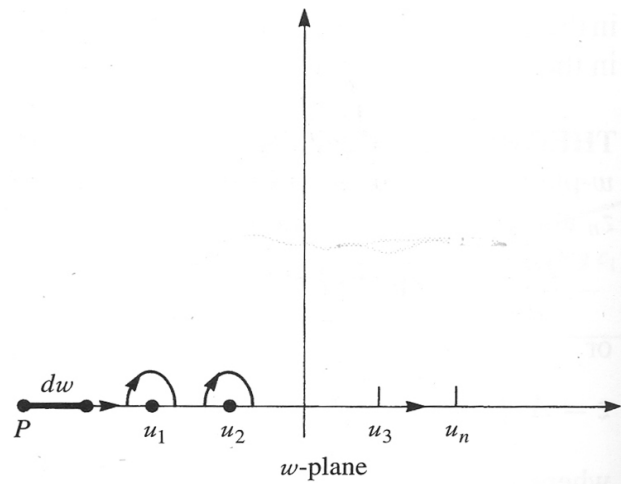Figure 11.1(a)   Figure 11.1 (b)

The lower limit has not been specified, as any constant of integration can be absorbed into the quantity B. The integration is carried out in the complex w plane and yields $z(w)$. The quantities A and B must be chosen to ensure that the desired mapping will occur. The quantity w in the upper limit of integration cannot lie in the lower half of the complex plane in Fig, 11.1(b) . The branches of all functions of the form $(w-u_k)^{((\alpha_k/\pi)-1)}$ are defined by branch cuts in the w plane starting at u_k and extending into the lower half space. This could include principal branches which are here defined by a cut just below the real axis.

Notice that if we were to take the lower limit as 0 and B=0 than the transformation would automatically map $w=0$ into $z=0$ and this can be useful. And if we were to take the lower limit as a and B=0 than the transformation would map $w=a$ into $z=0$. And if the lower limit is u_k, then the image of $z(u_k)=z_k=B$. These are all useful shortcuts.

To make Eq.(11.1) plausible, we differentiate that equation with respect to w. From the fundamental theorem of integral calculus, we have

$$\frac{dz}{dw} = A(w-u_1)^{((\alpha_1/\pi)-1)}(w-u_2)^{((\alpha_2/\pi)-1)}.....(w-u_n)^{((\alpha_n/\pi)-1)} \qquad (11.2)$$

We now assume that w moves along a locus in the w plane, and we are taking branch cuts as described above, i.e. they emanate from u_1,u_2...and can extend downward in Figure 11.1 (b) or just below the real axis, i.e. along $\mathrm{Im}(w)=0-$

.

We begin at the point P on the real axis of the w plane. The point is chosen at $w = u$ where $w < u_1$. From Eq.(11-2) we have $dz = A(u-u_1)^{((\alpha_1/\pi)-1)}(u-u_2)^{((\alpha_2/\pi)-1)}.....(u-u_n)^{((\alpha_n/\pi)-1)}du$. Notice that each term in the parenthesis, e.g., $u-u_k$, is a negative real number. Using principal values for the argument of all terms of the form $u-u_k$, they will all have an angle of π. Thus $u-u_k = |u-u_k| \angle \pi$ and therefore

$$(u-u_k)^{(\alpha_k/(\pi)-1)} = (|u-u_k| \angle \pi)^{(\alpha_k/(\pi)-1)} = |u-u_k|^{(\alpha_k/(\pi)-1)} \angle (\alpha_k - \pi)$$

Hence the argument of dz is given by

$$\arg(dz) = \arg(A) + (\alpha_1 - \pi) + (\alpha_2 - \pi) + ...(\alpha_n - \pi). \qquad (11.3)$$

This remains the angle of dz until we make u sufficiently less negative such that the point P advances past u_1 but not past u_2. We have $u-u_1 > 0$. Because all branch cuts are in the lower half plane the argument of $u-u_1$ is no longer π but zero. Thus, we have when P lies between u_1 and u_2 that $\arg(dz) = \arg(A) + (\alpha_2 - \pi) + ...(\alpha_n - \pi).$

Now the angle of dz has decreased by $\alpha_1 - \pi$ or increased by $\pi - \alpha_1$. The same kind of change in argument appears if P is allowed to move from between u_1 and u_2 to a location between u_2 and u_3. Thus our expression now becomes $\arg(dz) = \arg(A) + (\alpha_3 - \pi) + ...(\alpha_n - \pi).$ Note the α_3.

We could have chosen the point P to lie at $-\infty$ on the real axis in Fig.(11.1 b) and end at ∞. At the start of our journey $\arg(dz)$ is given by Eq.(11.3) and at the end it is $\arg(A)$. The locus taken by z is a set of connected broken lines like that shown in Figure 11.1 . If the sum of the exterior angles which are of the form $\pi - \alpha_k$ is 2π then the locus in the z plane is a polygon. We have for this condition $\pi - \alpha_1 + \pi - \alpha_2 + ... \pi - \alpha_n = 2\pi$ which easily leads to the result

$$\sum_{k=1}^{n} \alpha_k = \pi(n-2) \qquad (11.4)$$

The above does not hold if one of the vertices of the polygon is at ∞ i.e. we are dealing with an open structure.

How should the transformation Eq.(11.1) be modified if the image of one of the vertices of the polygon lies at $w = \infty$. Let us call this image u_n. We rewrite Eq.(11.2) as

$$\frac{dz}{dw} = (-u_n)^{((\alpha_n/\pi)-1)} A(w-u_1)^{((\alpha_1/\pi)-1)} (w-u_2)^{((\alpha_2/\pi)-1)} \dots \left[\frac{w-u_n}{-u_n}\right]^{((\alpha_n/\pi)-1)}$$

Letting $u_n \to \infty$ we will take the final factor on the right as 1. We allow A to grow so that the product $A(-u_n)^{((\alpha_n/\pi)-1)}$ remains finite and nonzero and is again designated as a constant, which we might as well still call A. Thus in Eq.(11.1) , if z_n has an image at infinity in the w plane, then the term involving u_n does not appear in the integration.

There is a convenient trick involved in using Eq.(11.1) that we will use in several problems below.

Suppose we want a point z_k $k=1,\dots n$ to have image w_k. We take the lower limit of integration in that equation as w_k and $B = w_k$. Note that $z(w_k) = z_k$ and that Eq.(11.2) is still satisfied.

As mentioned, the Schwarz-Christoffel transformation is most useful if, having done the integration in Eq.(11.1) and choosing appropriate constants so that the real axis is mapped into a polygon, with the interior of the polygon the image of the upper half plane , we can solve the resulting formula for w as a function of z . Then we will have succeeded in mapping a certain polygon, or a connected broken line, onto the real axis in the w plane and are well positioned to solve problems involving fluid mechanics, heat transfer and electrostatics with polygonal or broken line boundaries. Here is a simple example.

EXAMPLE 1

Let us map the region *inside* the semi-infinite strip, shown in Figure 11.2a below, into the upper half of the w plane. The boundary of the strip is to be mapped into the real axis in the w plane. The point $z_1 = 0$ is to be mapped into $w_1 = -1$. The point $z_2 = i$ is to be mapped into $w_2 = 1$. And z_3 which is at ∞ is to be mapped into $w_3 = \infty$. The result will be sufficiently simple that having obtained $z(w)$ we can then solve for $w(z)$
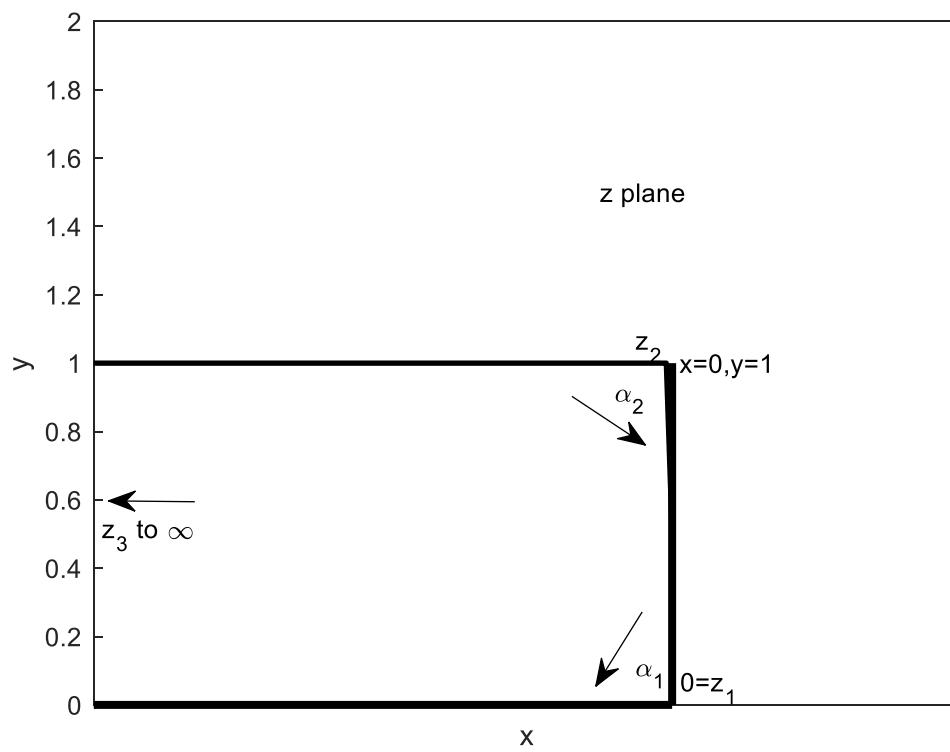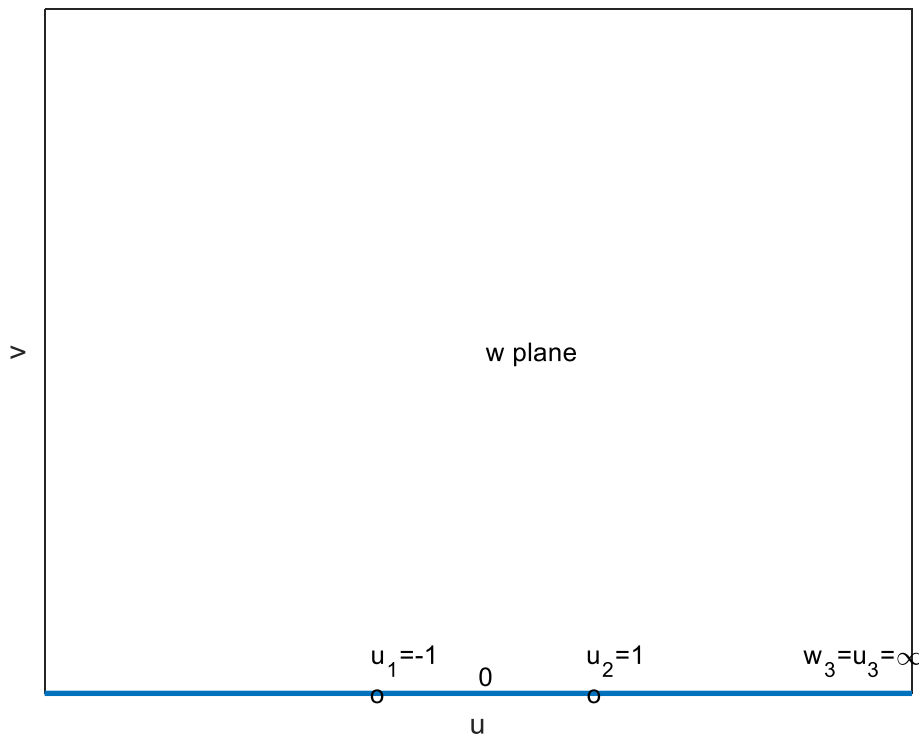
Figure 11.2 (a)

(a)Solution :

Fig. 11.2 (b)

In the z plane the two semi-infinite horizontal boundaries of the strip can be assumed to meet at infinity. The points z_1, z_2, z_3 are at $0, i,$ and ∞ respectively. Thus the value of u_3 will not appear in the integral in Eq.(11.1) where we take $u_1 = -1$ and $u_2 = 1$ as shown above. Note, see Fig. 11.2(a), that $\frac{\alpha_1}{\pi} - 1 = -1/2 = \frac{\alpha_2}{\pi} - 1$ We have to do this integration

$$z(w) = A \int^{w} (w+1)^{-1/2} (w-1)^{-1/2} \, dw + B \, .$$

We take the lower limit of integration as -1 and put B=0. In this way the point $w = u_1 = -1$ is mapped into $z_1 = 0$

We can readily find the integral in a table but MATLAB will do the integration

>> syms t

>> int(1/(sqrt(t^2-1)))

ans =

 log(t + (t^2 - 1)^(1/2))

Evaluating this at the upper limit *w* and lower limit we have from Eq.(11.1) that

$z(w) = A\log(w + (w^2 - 1)^{1/2}) - A\log(-1) = A\log(w + (w^2 - 1)^{1/2}) - Ai\pi$. We must use the principal branch of the log which we designate by Log.

To find A we require that if *w*=1 we need $z = i$. Thus

$i = A\mathrm{Log}(1) - Ai\pi$

Thus, $A = \dfrac{-1}{\pi}$. To summarize we have

$$z(w) = \frac{-1}{\pi}\log(w + (w^2 - 1)^{1/2}) + i \qquad (11.5)$$

Notice that if $|w| \to \infty$ then $|z| \to \infty$, and recall that we wanted to have infinity in the z plane to have an image of infinity in the *w* plane. Another check is to set $w = 0$. The image of this point should lie on the portion of the imaginary z axis between 0 and 1. We get $z = -\dfrac{1}{\pi}\mathrm{Log}(i) + i = \dfrac{1}{2}i$

-which is a good result.

Suppose that the vertical portion of the strip $x = 0, 0 < y < 1$ is maintained at 1 volt, while the two semi-infinite horizontal portions are kept at zero volts. We seek the electrostatic potential inside the strip, which may be regarded as an infinite channel whose cross section is shown in

Figure 11.2 a .

The corresponding problem in the $w = u + iv$ plane is to obtain the potential in the space $v \geq 0$ while the line segments (thought of as the cross-section of metallic conductors) $v = 0, |u| > 0$ are

maintained at a potential of zero volts while the line segment $v = 0, 0 < u < 1$ is kept at 1 volt.

The function $\phi(u, v)$ must satisfy Laplace's equation in the upper half of the w plane and must satisfy the required boundary conditions on the *u* axis.

This problem is readily solved with the Poisson integral formula for the upper half plane .

The formula, which can be found in W p.220 and in S Chap5, gives $\phi(u, v) = \dfrac{v}{\pi}\displaystyle\int_{-\infty}^{\infty}\dfrac{\phi(u', 0)du'}{(u' - u)^2 + v^2}$

which in the present case becomes $\phi(u, v) = \dfrac{v}{\pi}\displaystyle\int_{-1}^{1}\dfrac{du'}{(u' - u)^2 + v^2}$.

This is readily evaluated with a table but for fun we can do it in MATLAB.

We have

```
syms up u v
assume(u,'real')

assume(v,'real')

f=1./((up-u)^2+v.^2);

result=int(f,up,[-1  1]);

result=v/pi*result
```

 The output is

result =

-(atan((u - 1)/v) - atan((u + 1)/v))/pi

This is more familiarly written as

$$\phi(u,v) = \frac{\arctan((u+1)/(v)) - \arctan((u-1)/(v))}{\pi} \qquad (11.6)$$

Since $\arctan(s) + \arctan(1/s) = \pi/2$ we have $\phi(u,v) = \dfrac{\arctan(v/(u-1)) - \arctan(v/(u+1))}{\pi}$

Notice that the above is $\phi(u,v) = \text{Re}(\Phi(w))$ where

$$\Phi(w) = \frac{-i}{\pi}\left[\text{Log}(w-1) - \text{Log}(w+1)\right] \quad w = u+iv. \quad v \geq 0 \ (11.7)$$

If we had not recognized this, we could have obtained the real part of $\Phi(w)$ from its imaginary part through the use of the Cauchy-Riemann equations (see W. p 67 or S. Chap. 3).

We have in the above the complex electrostatic potential in the upper half of the complex *w* plane. (See W sec. 2.6).

We are fortunate in being able to solve Eq.(11.5) for $w(z)$. Entering that function in Eq.(11.6) we can obtain the complex potential in the z plane.

Let $p = e^{-\pi z}$. We have from Eq.(11.5) $-p = (w + (w^2 - 1)^{1/2})$. We can solve for *p* using MATLAB as follows
```
syms p w
eqn=p+w+sqrt(w^2-1)==0
solw=solve(eqn,w)
```

The output is

solw =

-(p^2 + 1)/(2*p)

Or equivalently

$$w = -\frac{1}{2}\left(p + 1/p\right) \text{ so}$$

$$w = -\cosh(\pi z) \tag{11.8}$$

Placing this result with $p = e^{-\pi z}$ in Eq.(11.6) we obtain:

$$\Phi(z) = \frac{-i}{\pi}\left[\mathrm{Log}(-1-\cosh(\pi z)) - \mathrm{Log}(1-\cosh(\pi z))\right] = \frac{-i}{\pi}\left[\mathrm{Log}\frac{\cosh(\pi z)+1}{\cosh(\pi z)-1}\right] \quad w = u + iv \tag{11.9}$$

Let us plot the real part of this expression in the interior of the "polygon", the strip $\mathrm{Re}(z) \leq 0,\ 0 \leq \mathrm{Im}(z) \leq 1$.Notice that we must avoid the points $z = 0$ and $z = i$ where the previous expression has singularities, which is to be expected as the potential is discontinuous at these points. We have made contour plots.

The real part is shown in Figure 11.3 a below. It shows the cross-sections of equipotential surfaces describing the voltage inside the strip. We chose our values for the potentials after some experimentation. We have also plotted, in Figure 11.3 b , the contours on which the imaginary part is constant. This is the stream function and its contours of constant value are tangent to the electric field in the strip (see W. Chapter 4 and S Chapter 9). Again, we experimented in order to choose values for the streamlines.

Had we chosen to work directly with Eq.(11.6) we could obtain $\phi(x, y)$ and we would obtain from Eq.(11.8) that $u = -\cosh(\pi x)\cos(\pi y)$ and $v = -\sinh(\pi x)\sin(\pi y)$ and could use these expressions in Eq.(11.6). The stream function $\psi(x, y)$ can be obtained from Eq.(11.6) and the use of the Cauchy-Riemann equations
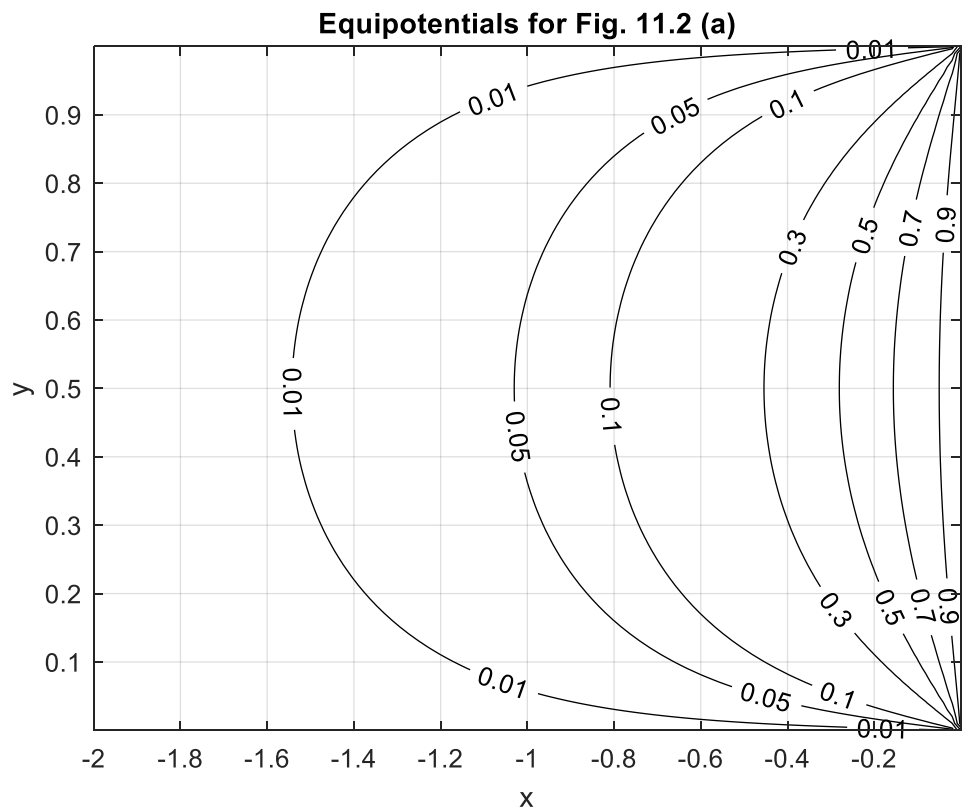
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Figure 11.3 a

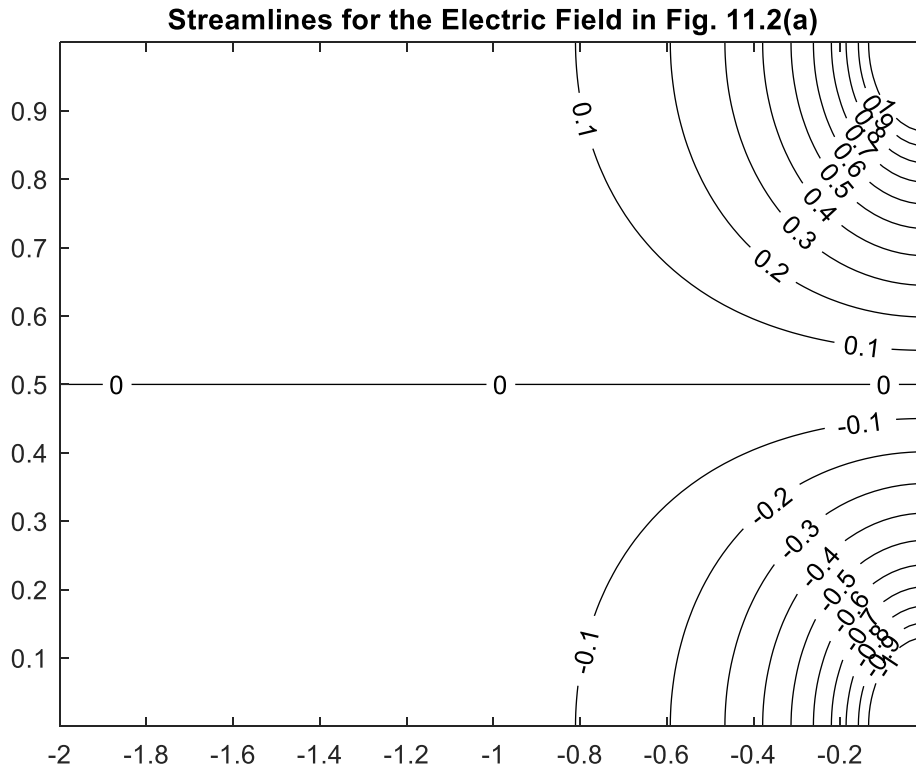**Streamlines for the Electric Field in Fig. 11.2(a)**

Figure 11.3b

The code used in generating these two plots is given below

```
x=linspace(-2,-10*eps,100);
y=linspace (10*eps,1-10*eps,100);
[X Y]=meshgrid(x,y);
z=X+i*Y;
Phi=-i/pi*log((cosh(pi*z)+1)./(cosh(pi*z)-1))
phi=real(Phi);
figure(1)
[c,h]=contour (X,Y,phi,[.01 .05 .1 .3 .5 .7 .9])
clabel(c,h);colormap([0 0 0])
grid
psi=imag(Phi);
figure(2)
n=0:10;
n=n*.1;
n=[-n n];
[p,q]=contour (X,Y,psi,n);colormap([0 0 0])
clabel(p,q)
```

Note that these two plots could have been obtained without our using the complex potential. We had obtained at first the electrostatic potential $\phi(u,v)$ and from this could have plotted the

equipotential surfaces. The streamlines can be obtained from a **quiver** plot of the gradient of $\phi(u,v)$ (see W. section 4.3). We might also have obtained $\psi(u,v)$, which is the harmonic conjugate of $\phi(u,v)$ through the use of the Cauchy Riemann Equations (see W sec 4.4).

●

It is natural for the reader to wonder how many points from the complex z plane can be mapped into predetermined points in the *w* plane.[1] The answer, *not proved here, is 3*, where a point at infinity, if required in the mapping, will count as one of the three.
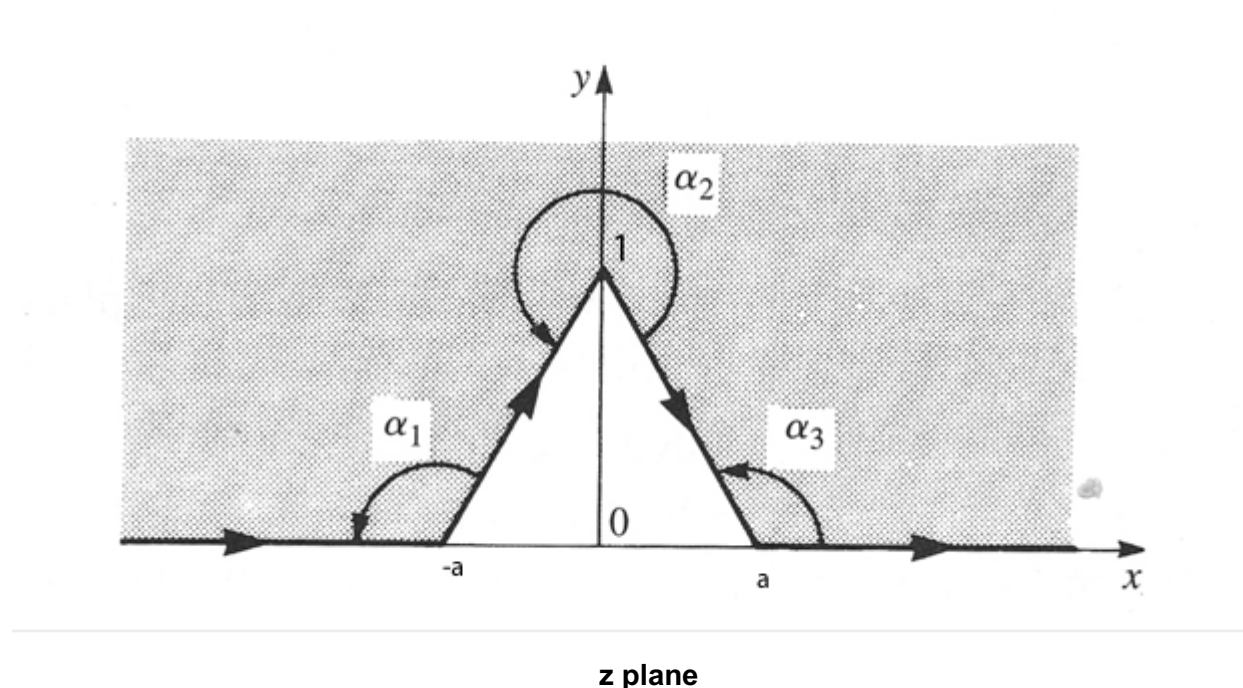
Example 2

z plane

Figure 11.4 a

We wish to map the shaded region shown in Figure 11.4(a) into the upper half of the *w* plane in such a way that the points at $z = \pm a$ are mapped into are mapped into $w = \pm 1$. The point at $z = i$ is to be mapped into $w = 0$. We will then let $a \to 0+$ so that $-a \to 0-$. Thus, in the limit, the region shown in Figure 11.4a becomes the region shown in Figure 11.4b and the shaded area in Fig. 11.4(a) is mapped into the upper half of the *w* plane.

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[1] This question is answered in *Complex Variables and Applications* , 6th edition, By RV Churchill, and J.W. Brown, section 94.
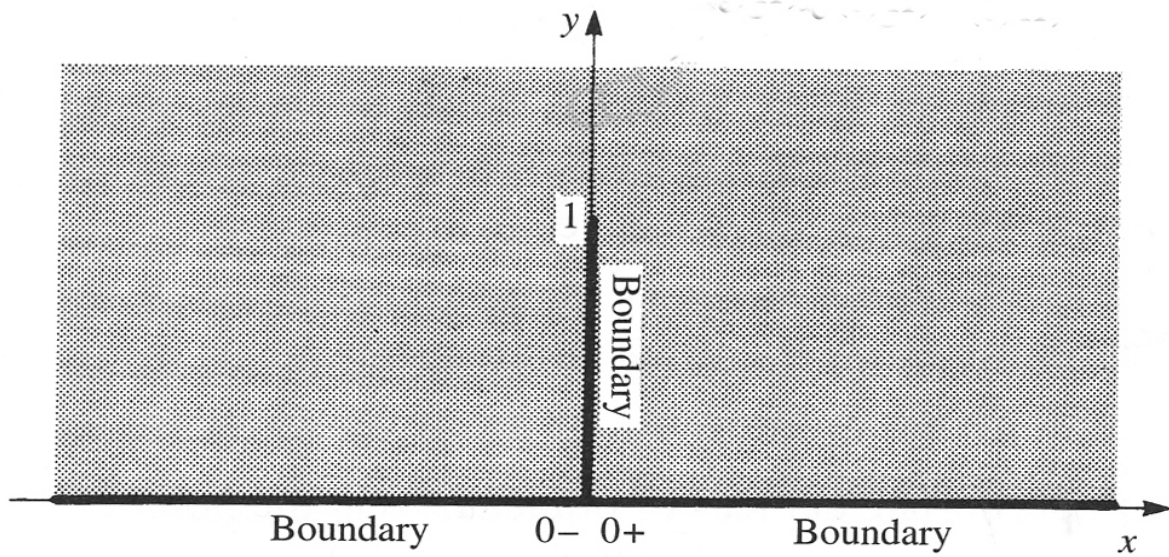
Figure 11.4 (b)

We should anticipate that the line $x = 0, 0 \le y \le 1$ becomes a branch cut of the resulting transformation because of the discontinuity in $z(w)$ that is required,

Suppose we wish to map the region above and exterior to the line shown in Fig. 11.4 b into the upper half plane. We move along the path shown in Figure 11.4(a) keeping the interior on the left.

Solution

We take $\alpha_1 = \alpha_3 = \alpha$. We will map the points $z_1 = -a$ into $w_1 = -1$ and $z_3 = a$ into $w_3 = 1$ and $z_2 = i$ gets mapped into w=0. Observe that since the sum of the interior angles of a triangle is π we requires $2\pi - 2\alpha + 2\pi - \alpha_2 = \pi$ or $\alpha_2 = 3\pi - 2\alpha$. Our integral Eq.(11.1) becomes

$$z(w) = A \int^w (w+1)^{\alpha/\pi - 1} (w-1)^{\alpha/\pi - 1} (w)^{\frac{1}{\pi}(3\pi - 2\alpha) - 1} dw + B.$$

Assuming we can swap the order of integration and the limiting process we will take $\alpha = \pi/2$ and $a \to 0+$. Our integral becomes $z(w) = A \int^w (w^2 - 1)^{-1/2} w\, dw + B = A \int^w \dfrac{w}{(w^2 - 1)^{1/2}} dw + B$

This integral is easily performed and we get $z = A(w^2 - 1)^{1/2} + B$.

Since we want $z = 0+$ to have as its image $w = 1+$ we see that $B = 0$. We want $z = i$ to have as its image $w = 0$ and this shows $A = 1$. We were careful to evaluate the branch of $(w^2 - 1)^{1/2}$ at zero as i because we are using the principal branch of this function, with branch cuts extending into the lower half plane. Thus

$$z = \left(w^2 - 1\right)^{1/2} \tag{11.10}$$

Notice that the requirement that $z = 0-$ has its image at $w = -1$ is also fulfilled.

And from the preceding equation we find

$$w = \left(z^2 + 1\right)^{1/2} \tag{11.11}$$

Observe that $z = \infty$ has the image $w = \infty$. The above function is defined by a branch cut : $x = 0, -1 \le y \le 1$ which leads to the required discontinuous behavior at $z = 0$.
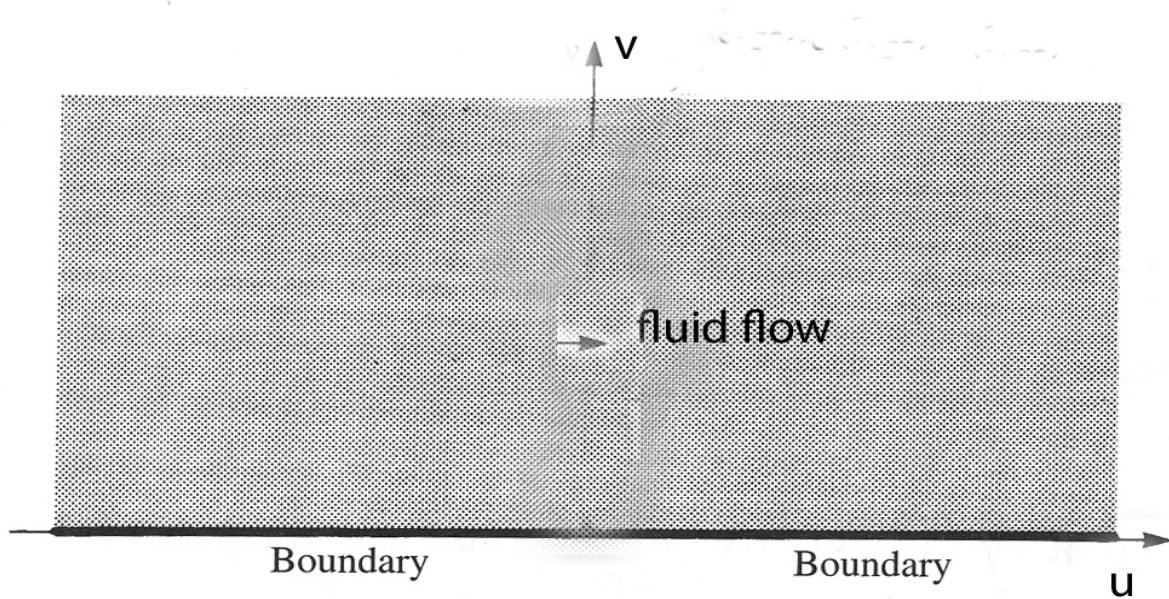

Figure 11.4 (c)

Imagine that fluid is flowing from left to right in Figure 11.4 (c). The line $v = 0+$ is a streamline because we assume that $v = 0$ is a rigid impenetrable barrier. The flow is uniform and we will say that the complex velocity potential describing the flow is given by

$$\Phi(w) = V_o w = V_o(u + iv) = \phi(u,v) + i\psi(u,v) \text{ where } V_0 > 0 \tag{11.12}$$

See W sec. 2.6 and S (chap 9). From W, (p.93) we have that the complex velocity is

$V(w) = \overline{\left(\dfrac{d\Phi}{dw}\right)} = V_0$. At any point above the plane in Fig. 11.4(b) the velocity is a positive

constant, which means that flow is uniform and to the right. Since $\psi(u,v) = V_o v$ the plane whose edge is v=0 must be a streamline on which $\psi = 0$.

From Eq(11.11) and Eq.(11.12) we have that the complex potential is the z plane is

$$\Phi(z) = V_0(z^2 + 1)^{1/2} = \phi(x, y) + \psi(x, y) \qquad (11.13)$$

The complex velocity potential in the z plane is $\Phi(w(z))$ and the complex velocity in that plane is

$$V(z) = \overline{\left(\frac{d\Phi}{dz}\right)} = \overline{\left(\frac{d\Phi}{dw}\frac{dw}{dz}\right)} = V_0\overline{\left(\frac{z}{(z^2 + 1)^{1/2}}\right)} \qquad (11.14)$$

By studying the selected branch of $(z^2 + 1)^{1/2}$ we see that when $z \to \infty$ the expression in the brackets tends to 1. Thus, in the z plane, the fluid flow is constant (uniform) as infinity is approached and, as a vector, points from left to right and indicates a velocity V_0.

From Eq.(11.13) we have $\phi(x, y) = V_0 \operatorname{Re}(z^2 + 1)^{1/2} = \operatorname{Re}(z^2 + 1)^{1/2}$ and
$\psi(x, y) = V_0 \operatorname{Im}(z^2 + 1)^{1/2} = \operatorname{Im}(z^2 + 1)^{1/2}$ where we take $V_0 = 1$. There is *a subtlety here* that requires our attention. The expression $z^2 + 1$ is negative real on the lines $x = 0, y > 1$ and $x = 0, y < -1$. Thus MATLAB, which uses principal branches, would place the branch cuts of this function along these two semi-infinite lines. We want MATLAB to place the branch cut along the line segment $x = 0, -1 \le y \le 1$. This is accomplished by putting

$(z^2 + 1)^{1/2} = z(\dfrac{z^2 + 1}{z^2})^{1/2}$. Notice that the expression in the parenthesis on the right is negative and

real on the line $x = 0, -1 \le y \le 1$ and nowhere else. Thus if we use the right hand side of the preceding equation in lieu of the left side, in MATLAB, we will have the branch cut where we want it .

Example 3

Plot the streamlines for the configuration shown in Figure 11.4 b, i.e. plot the contour lines for $\psi(x, y)$ given in Eq.(11.13). Take $V_0 = 1$

Solution: it should be apparent that $\psi(x, y) = V_0 \operatorname{Im}(z^2 + 1)^{1/2} = \operatorname{Im}(z^2 + 1)^{1/2}$ is zero on the lines $y = 0, -\infty < x < \infty$ and $x = 0, 0 \le y \le 1$. Thus we will not need to plot $\psi = 0$. Choosing some positive values for ψ we obtain the following plot from this code where we use

$(z^2 + 1)^{1/2} = z(\dfrac{z^2 + 1}{z^2})^{1/2}$.

Solution

```
x=linspace (-5,5,1000);
y=linspace (0,3,10000);
[X Y]=meshgrid(x,y);
z=X+i*Y;
psi=imag(z.*sqrt(1+1./z.^2));
[c h]=contour(X,Y,psi,[.1 .5 1 2]);
colormap([0 0 0])
```

```
xlabel('x');ylabel('y')
clabel(c,h)
hold on
y=linspace(0,1,100);
x=0*y;
%the following plots the barrier interrupting the flow.
plot(x,y,'linewidth',2)
```
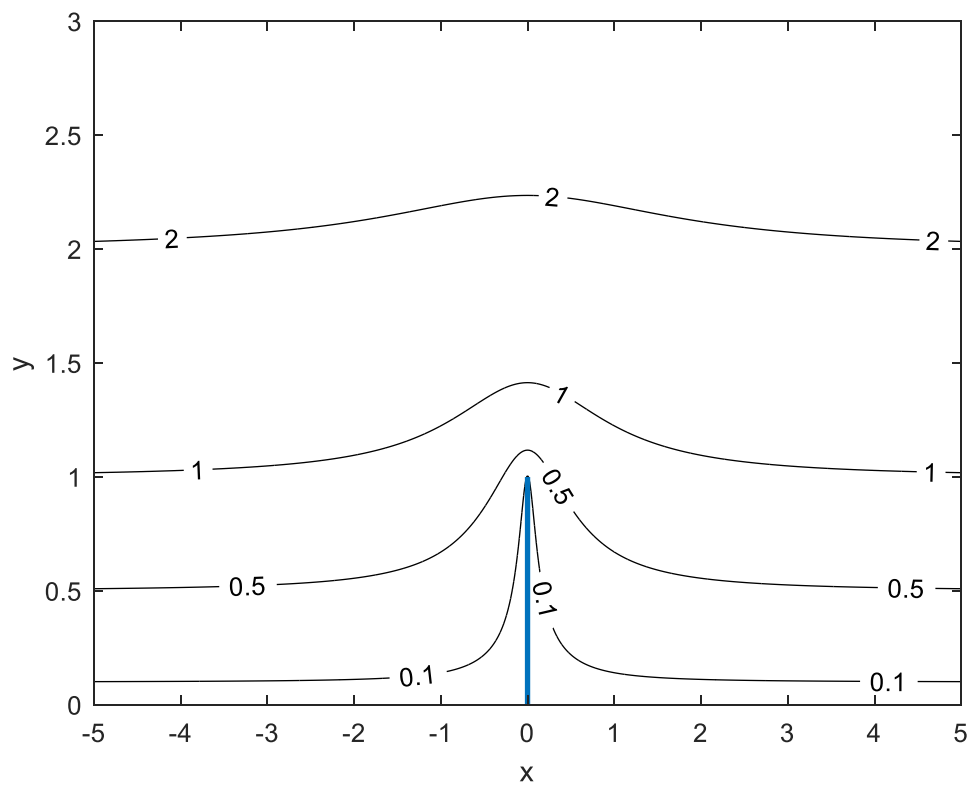
Here is the output

Figure 11.4 d

•

In the previous worked examples we were able to find $\Phi(z)$ for the given configuration by first finding $\Phi(w)$ in the transformed, simpler configuration. Because we were able to go from the

expression $z(w)$ to $w(z)$ we could then find $\Phi(w(z))$. But sometimes we cannot easily solve the equation $z = w(z)$ for *w* as an explicit function of *z*. However, we can still find useful results, as in this example.
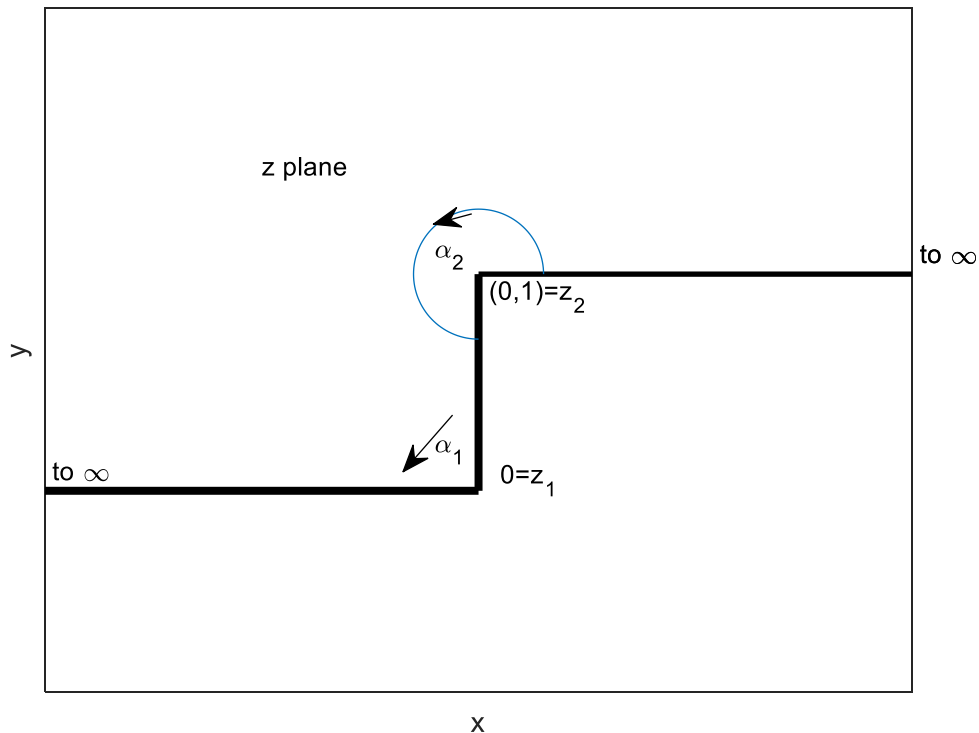
Example 4

Figure 11.5

Plot the streamlines when fluid, moving parallel to the x axis, and is uniform at infinity, encounters the barrier shown above.

Solution:

We wish to map the above configuration into the w plane so that $z_1 = 0$ has image $w_1 = -1$ and $z_2 = i$ is mapped into $w_2 = 1$ and we want infinity in the *z* plane to have an image of infinity in the *w* plane. The region above the broken line in the figure is to be mapped into the upper half plane. Taking $\alpha_1 = \pi/2$ $\alpha_2 = 3\pi/2$, $u_1 = -1$, $u_2 = 1$ we have that Eq.(11.1) becomes

$$z(w) = A\int^w (w+1)^{-1/2}(w-1)^{1/2}\,dw + B = A\int^w \frac{(w-1)}{(w^2-1)^{1/2}}\,dw + B \quad ,$$ Let us take the lower limit in the

integration as $w = -1$. With B=0 we see that $w = -1$ has image $z_1 = 0$ as required .

With the substitution $w = \cosh t$ this becomes

$$z(w) = A\left(\int_{\cosh^{-1}(-1)=i\pi}^{\cosh^{-1}(w)} \cosh t - 1 \right) dt + B = A(\sinh t - t)\Big]_{i\pi}^{\cosh^{-1}(w)} + B = A\left(\left(\cosh^2 t - 1\right)^{1/2} - t\right)\Big]_{i\pi}^{\cosh^{-1}(w)} =$$

$$A(w^2 - 1)^{1/2} - A\cosh^{-1}(w) - A(1-1)^{1/2} + Ai\pi =$$

$$z(w) = A(w^2 - 1)^{1/2} - A\operatorname{Log}(w + (w^2 - 1)^{1/2}) + Ai\pi$$

Or we can apply the MATLAB code

```
syms w A
f(w)=(w-1)/sqrt(w^2-1)
result=int(f);
A*(result(w)-result(-1))
```

which yields

A*((w^2 - 1)^(1/2) - log(w + (w^2 - 1)^(1/2)) + pi*1i)

We now apply the boundary conditions: when *z=i*, *w* =1 and obtain $A = \dfrac{1}{\pi}$

Thus finally

$$z(w) = \frac{1}{\pi}\left[(w^2 - 1)^{1/2} - \operatorname{Log}(w + (w^2 - 1)^{1/2})\right] + i \qquad\qquad (11.15)$$

Unlike the previous problems we have treated, we cannot here solve for *w* as a function of *z*.

There is also a subtlety here involving the branch of $(w^2 - 1)^{1/2}$. We know that the branches of $(w-1)^{1/2}$ and $(w+1)^{1/2}$ appearing in Eq.(11.1) involve branch cuts extending into the lower half plane. MATLAB will evaluate these functions properly. But it *will not* evaluate their product properly if we take it as . $(w^2 - 1)^{1/2}$. Suppose we were to evaluate the principal values of $(w-1)^{1/2}$ and $(w+1)^{1/2}$ at a point on the negative real axis to the left of $w = -1$. The arguments (angles) of both $(w-1)$ and $(w+1)$ would each be taken by MATLAB as π, and the angles of $(w-1)^{1/2}$ and $(w+1)^{1/2}$ would be each $\pi/2$. Upon multiplying $(w-1)^{1/2}$ and $(w+1)^{1/2}$ the resulting expression would have angle π and would result in a product that is a negative real. However, MATLAB would evaluate $(w^2 - 1)^{1/2}$ as the square route of a positive real and would yield a positive real answer. Thus, in our MATLAB calculation, we should take $(w^2 - 1)^{1/2}$ *as the product* $(w-1)^{1/2} (w+1)^{1/2}$.

Imagine that fluid flows from left to right in Figure 11.5 and encounters the barrier that is the vertical line shown in that figure. The fluid flow is interrupted and then continues above the

barrier. We will imagine that the flow is uniform at infinity. We wish to plot the streamlines showing the fluid flow.

The barrier described by the broken line in Figure 11.5 becomes the real axis in the *w* plane. The points *w*=-1 and *w*=1 are the images of z=0 and z=*i* respectively. Because infinity in the *w* plane has an image of infinity in the z plane we may take a fluid flow that is uniform (a constant) at infinity in the *w* plane and assert that it will be uniform in the z plane at infinity. Moreover, the streamlines of the fluid flow in the *w* plane will have images that are the streamlines in the z plane. Let us take the velocity potential in the *w* plane as $\Phi(w) = kw$ where K>0 is real. Thus,

the vector velocity is $\overline{\left(\dfrac{d\Phi}{dw} \right)} = K$. The velocity potential in the z plane will be $\Phi(z(w))$ and the

velocity vector V is $V = \overline{\left(\dfrac{d\Phi}{dz} \right)} = \overline{\left(\dfrac{d\Phi}{dw} \dfrac{dw}{dz} \right)} = \overline{\left(\dfrac{d\Phi}{dw} \right)} \overline{\left(\dfrac{dz}{dw} \right)}^{-1}$. The integral used in deriving

Eq.(11.15), which comes from Eq.(11.2) yields $\dfrac{dz}{dw} = \dfrac{1}{\pi} \dfrac{(w-1)^{1/2}}{(w+1)^{1/2}}$.

Thus

$$V = \pi K \overline{\left[\dfrac{(w+1)^{1/2}}{(w-1)^{1/2}} \right]} \qquad\qquad (11.16)$$

This yields the complex velocity vector in the z plane at any point whose image is *w* . The value of z can be found from Eq. (11.15). Notice that at infinity in the *z* plane the velocity vector is πK while it is K at infinity in the *w* plane. This confirms that the flow will be uniform in the z plane at ∞ if it is uniform at ∞ in the *w* plane. *As a general principle we cannot say that this will always be true.* This is because in the expression

$$V = \overline{\left(\dfrac{d\Phi}{dz} \right)} = \overline{\left(\dfrac{d\Phi}{dw} \right)} \overline{\left(\dfrac{dz}{dw} \right)}^{-1}$$

if $\dfrac{d\Phi}{dw}$ tends to a constant as $w \rightarrow \infty$ it does not necessarily mean that dz / dw will tend to a constant in the same limit. Thus the velocity need not be constant as $z \rightarrow \infty$.

The infinity in this vector at $w = 1$ corresponds to the discontinuous behavior of the fluid velocity as the fluid rounds the corner at $z = i$ in Figure (11.5) .

Let us map some streamlines of the flow in the w plane into the z plane where they will still be streamlines. Let us take $K = 1/\pi$ so that the velocity is 1 at ∞. The streamlines are found with the following code where the values of ψ used for each streamline were found after some experimentation with the code.

```
u=linspace(-10,20,1000);
v=[ .1 .3 .5 1 1.5 2]';

%above are values of psi on streamlines
% above is a column vector
clf
w=u+i*v;
wbr1=sqrt(w+1);
wbr2=sqrt(w-1);
wbr=wbr1.*wbr2;
z=1./pi*(wbr-log(w+wbr))+i;
z=z.'; %transposes the above
%psi=imag(z);

   plot(real(z),imag(z),'k')
   %above plots columns of imag(z)vs corresp column of real(z)
   %each value in a given column of imag(z)is a fixed value
   %of psi since it corresponds to a constant value of     imag(w)
   hold on
   psival=num2str(v);
   text(real(z(1,:)),imag(z(1,:)),psival)
   text(-6,.9,'numbers are values of{\psi}')
   text(-5,.8,1.35,'\downarrow')
   x=linspace(-3.5,6);
   %the barrier is below
   step=heaviside(x);
   plot(x,step,'linewidth',2)
  The output is
```
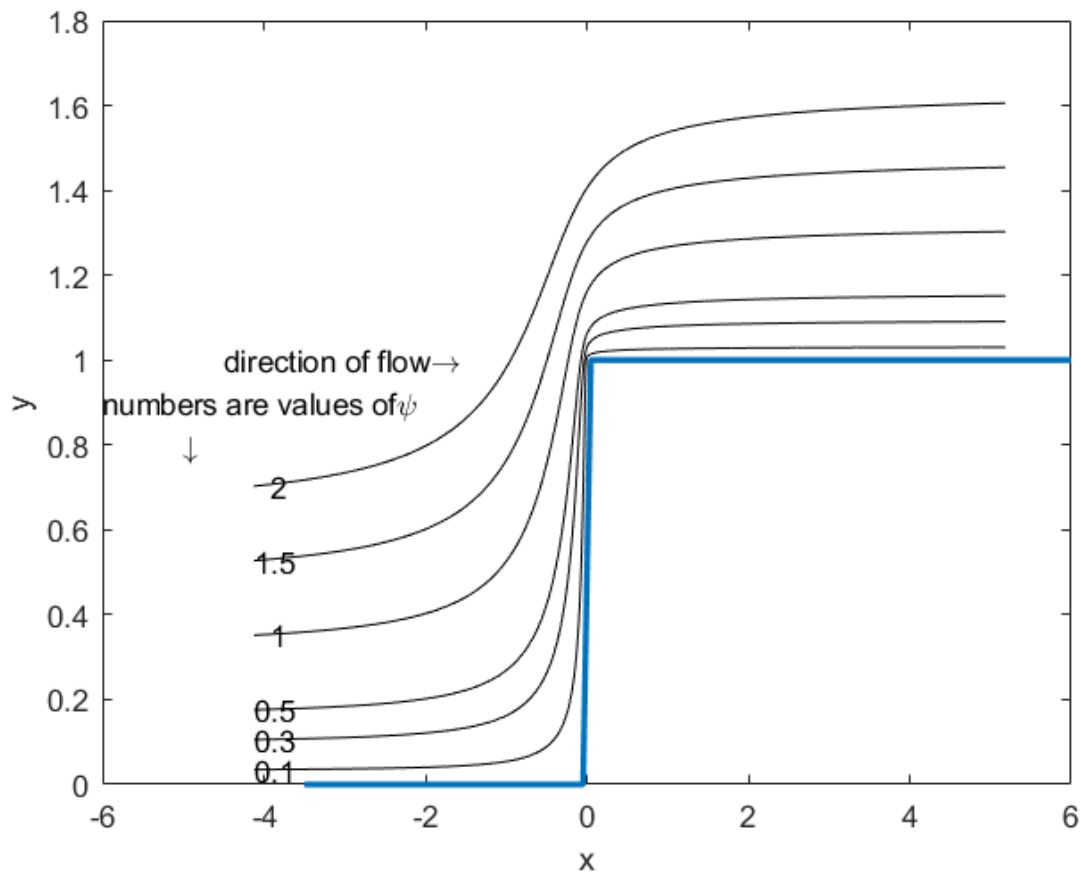
Figure 11.6

Streamlines for uniform flow encountering the barrier in Figure 11.5

EXAMPLE 5.

For the configuration shown above, write a program such that if you enter a value of z as a complex number that the program will yield the velocity as a complex number (a vector) at that point. Assume as in the above that the vector fluid flow at ∞ is 1.

Solution

The vector velocity in the complex w plane is given by Eq.(11.16). Given a value of z we must find the corresponding value of w to employ this formula. This is accomplished with the following code which prompts you to enter z and then finds the velocity.

```
z=input('z=')
while z~=0
syms w wr
digits(6)
```

```
wr=(w-1).^(1/2).*(w+1).^(1/2);
warning('off')
syms w
w=solve(1./pi*wr-1/pi*log(w+wr) +i==z);
velocity=(sqrt(w+1)/(sqrt(w-1)));
velocity=conj(velocity)
z=input('z=')
end
```

 We will test the code by finding the velocity at $z = -.1+.1i$ and $z = 0+1.1i$. The results are as follows
0.0791768 + 0.0778853i and 1.61529 + 0.697552i .
These are consistent with the directions of the streamlines in Figure 11.6.

EXAMPLE 6

In Figure 11.5 assume that what is shown is the cross-section of a perfect metallic conductor. The vertical portion of the conductor $x = 0, 0 < y < 1$ is maintained at 1 volt while the horizontal portions $y = 0, -\infty < x < 0$ and $y = 1, 0 < x < \infty$ are maintained at 0 volts (ground potential). Find the voltage everywhere in the complex z plain above the boundary and map it with contour lines (equipotentials).

Solution: We have succeeded above in mapping the boundary of Figure 11.5 onto the real axis in the w plane. The image is shown if Fig. 11.2(b). The transformation is from Eq.(11.15). As stated, we cannot solve this equation for w as a function of z. We note that the portion of the boundary at 1 volt in the z plane is mapped into the segment $v = 0, \; -1 < u < 1$ in the w=(u,v) plane. The remainder of the boundary in the z plane is mapped into $v = 0, \; |u| > 1$ We solved for the electrostatic voltage on and above the real axis in the w plane in Example 1 and found it to be

$$\phi(u,v) = \frac{\arctan((u+1)/(v)) - \arctan((u-1)/(v))}{\pi}.$$

This is the real part of the complex potential given in Eq.(11.7) For each value of z of interest, we will solve for the corresponding value of w, using Eq.(11.15) to obtain u and v, and from this find the potential at each value of z, and plot the contour lines. We have plotted contour lines for the equipotential surfaces using the code shown below. Notice that because of the boundary we have divided the z space of interest into two parts: One part is where y>1, $-\infty < x < \infty$ and other for y<1, x<0. The values chosen for the voltage on the equipotentials were decided after some experimentation. We ran the program taking n =100 which means that 20,000 data points were used, each requiring the use of **solve**. The program took just under 3 hours to run on a laptop using Windows 10 and the resulting figure is below. However, an adequate but less smooth plot of the contour lines is obtained by taking n=10 which involves 200 data points, and this took about 15 seconds. Notice the command **digits(6)** which reduces the computation time.
Here is the code
```
warning off all
n=input('n=')
digits(6)
x=linspace (-1,1.,n);
```

```matlab
y=linspace(1.01,2,n);
[X Y]=meshgrid(x,y);
tic
syms w
z=X+i*Y;
W=ones(n,n);
%the following gives you z  for each value of w
for j=1:n
  for k= 1:n
    W(j,k)= solve (1/pi*sqrt(w-1).*sqrt(w+1)-1/pi*log(w+sqrt(w+1).*sqrt(w-1))+i==z(j,k));
    end
end
% the following gives you the potential at the value of z %of interest
Phi=i/pi*log(W+1)-i/pi*log(W-1);
phi=real(Phi);
clf
figure(1)
levels=[.01 .05 .2 .3 .4 .5 .7 .9];
[c,h]=contour (X,Y,phi,levels);colormap([0 0 0]);
clabel(c,h)
hold on
x=linspace (-1,-.01,n);
y=linspace(0,.99,n);
[X Y]=meshgrid(x,y);
syms w
z=X+i*Y;
W=ones(n,n);
for j=1:n
  for k= 1:n
    W(j,k)= solve (1/pi*sqrt(w-1).*sqrt(w+1)-1/pi*log(w+sqrt(w+1).*sqrt(w-1))+i==z(j,k));
    end
end
 Phi=i/pi*log(W+1)-i/pi*log(W-1);
phi=real(Phi);
figure(1)
 [c,h]=contour (X,Y,phi,levels);colormap ([0 0 0])
xlim([-1.1 1.1]);
clabel(c,h)
toc
hold on
```
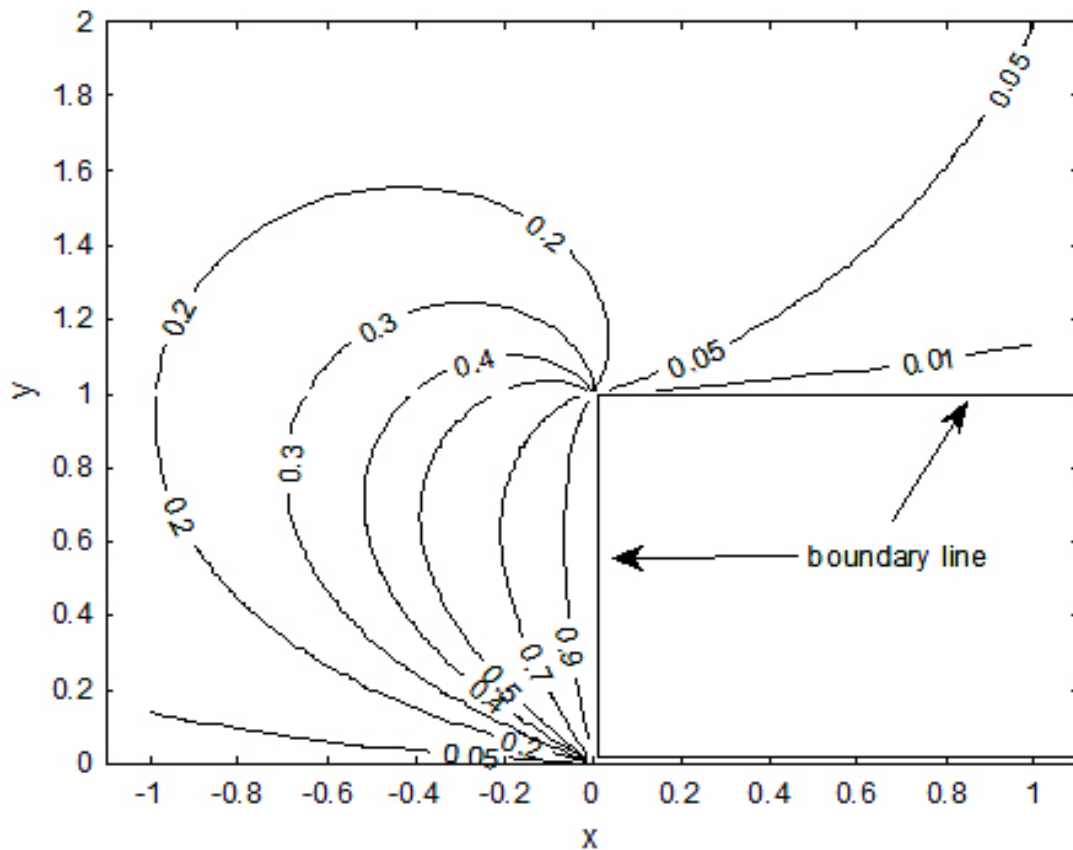And here is the output.

Figure 11.7 Equipotential Surfaces for Fig11.5,vertical portion is at 1 volt, remainder at 0

Example 7
It is possible to obtain the preceding contour lines *without using* solve. The method is to *determine the equations* of the equipotentials in the w plane and then map them into the z plane using the formula for *z(w)* given by Equation (11.1) as applied to the present case.

In Example 1 we solved the problem of finding the complex potential $\Phi(w)$ in the space $\text{Im}(w) \geq 0$ when the boundary conditions are $\phi(w) = \text{Re}(\Phi) = 1$ for $v = 0, -1 < u < 1$, and $\phi(w) = 0$ for $v = 0, |u| > 1$. The result is $\Phi(w) = \dfrac{-i}{\pi} \text{Log} \dfrac{w-1}{w+1}$. We saw above that the real part of the above can be written as

$$\phi(u,v) = \frac{\arctan((u+1)/(v)) - \arctan((u-1)/(v))}{\pi} \qquad =.$$

Now recall the formula (see W. p. 48, problem 43)

$$\arctan(a) - \arctan(b) = \arctan \frac{a-b}{1+ab}.$$

Thus, we have

$$\phi(w) = \frac{1}{\pi}(\arctan(\frac{u+1}{v}) - \frac{1}{\pi}\arctan(\frac{u-1}{v})) = \frac{1}{\pi}\arctan\frac{2v}{u^2+v^2-1}$$

Recalling that $\arctan(a) = \pi/2 - \arctan(1/a)$ we rewrite the above as

$$\phi(u,v) = \frac{1}{\pi}\left[\frac{\pi}{2} - \arctan\frac{u^2+v^2-1}{2v}\right] \qquad (11.17)$$

Our reason for putting the expression in this form is that MATLAB always yields the arctangent of any real number as satisfying $-\pi/2 \le \arctan \le \pi/2$. With this constraint the potential must always, lie in the space $v \ge 0$, at a value between 0 and 1, as it must because of physical constraints.

The equipotentials satisfy $\frac{u^2+v^2-1}{2v} = \beta$ where β is any real constant. We can rewrite this condition as

$$u^2 + (v-\beta)^2 = 1 + \beta^2 \qquad (11.18)$$

which means that the equipotentials must lie on circles of radius $\sqrt{1+\beta^2}$ with center at $u=0$, $v=\beta$. We can find β for any voltage (potential) of interest satisfying $0 \le V_0 \le 1$ since (see Eq.(11.17).

$$V_0 = \frac{1}{\pi}\left[\frac{\pi}{2} - \arctan\beta\right] \qquad (11.19)$$

so that

$$\beta = \tan(\pi/2 - V_0\pi).$$

Since $0 \le V_0 \le 1$ we see that $-\infty < \beta < \infty$. The following code will generate the equipotentials in the w plane. We will supply the values of V_0 of interest, the program will determine the values of β and then plot the circles describing each equipotential for the space $v \ge 0$. It is easy to prove that each circle intersects the v axis at $v = \beta + \sqrt{\beta^2+1}$ in the upper half plane. Just set u=0 in Eq.(11.18).

Here is the code
```
%example 7 chap 11
clf
%voltages of interest below
Vo=[.1 .2 .5 .9];
beta=tan(pi/2-Vo*pi);
beta=beta.';
for(j=1:length(Vo))
beta2=beta.^2;
alpha=atan(beta);
psi=linspace(-alpha(j),pi+alpha(j),99);
mybeta=beta(j);
w=i*beta(j)+sqrt((1+beta2(j))).*exp(i*psi);
plot(real(w),imag(w),'k')
hold on
```

```
grid
volts=num2str(Vo(j))
intersect=beta(j)+sqrt(1+mybeta^2);
text(0,intersect,volts)
end
axis square
x=linspace(-4,-1.1,10);
x1=-x;
y=0*x;
y1=0*x1;
plot(x,y,'k','linewidth',2);
plot(x1,y1,'k','linewidth',2)
text(1,0,'1')
x=linspace(-.9,.9, 20);
y=0*x;
plot(x,y,'k','linewidth',2);
text(-.2,4,'numbers are voltages on surface')
xlabel('u'); ylabel('v');
```
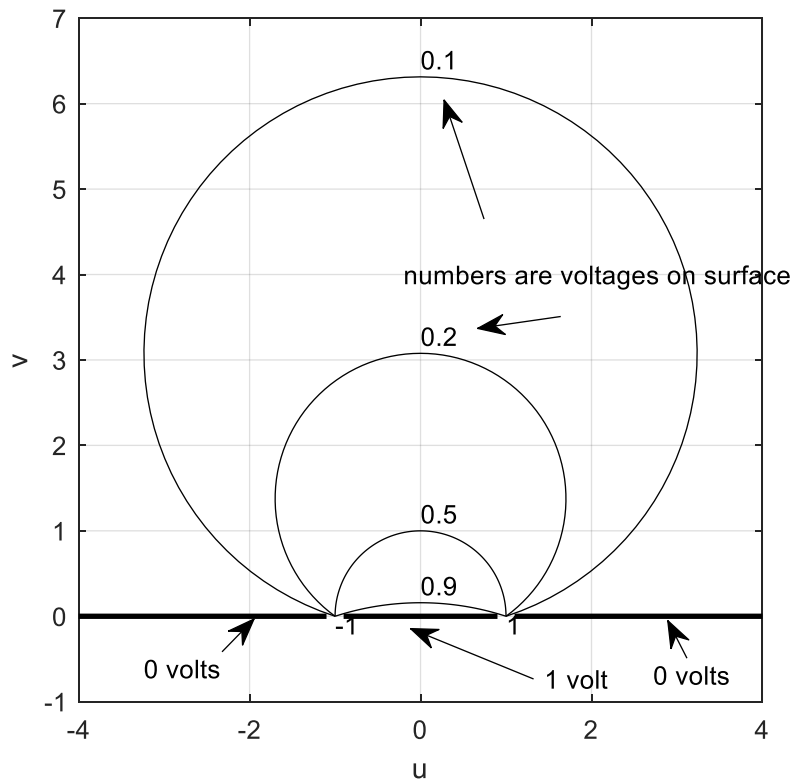
And here is the output

Figure 11.8 a

Our goal now is to map the above equipotentials, from the above *w* plane, into the *z* plane by means of Eq.(11.15). In this way we will not have to use the time consuming **solve** command. The following code required only .045 seconds to execute and resulted in the figure shown below the code.

```
%example 7 part 2 chap 11
clf
%voltages of interest below
tic
Vo=[.1 .2 .5 .9];
beta=tan(pi/2-Vo*pi);
beta=beta.';
for(j=1:length(Vo))
beta2=beta.^2;
alpha=atan(beta);
psi=linspace(-alpha(j),pi+alpha(j),99);
mybeta=beta(j);
w=i*beta(j)+sqrt((1+beta2(j))).*exp(i*psi);
z=1./pi*(sqrt(w-1).*sqrt(w+1)-log(w+sqrt(w-1).*sqrt(w+1)))+i;
plot(real(z),imag(z),'k');plots the equipotentials
hold on
grid
axis square
volts=num2str(Vo(j))
w=beta(j)+sqrt(1+mybeta^2);
inter=i*(beta(j)+sqrt(1+mybeta^2));
%plots voltage on each locus
%we map the points at which each circle intersects the vertical axis in figure
%11.9 into its image in the z plane
z=1./pi*(sqrt(inter-1).*sqrt(inter+1)-log(inter+sqrt(inter-1).*sqrt(inter+1)))+i
text(-.2+real(z),imag(z), volts)
end
x=linspace(-2,.5,1000);
%boundary is plotted below
plot(x,heaviside(x),'linewidth',2)
toc
```
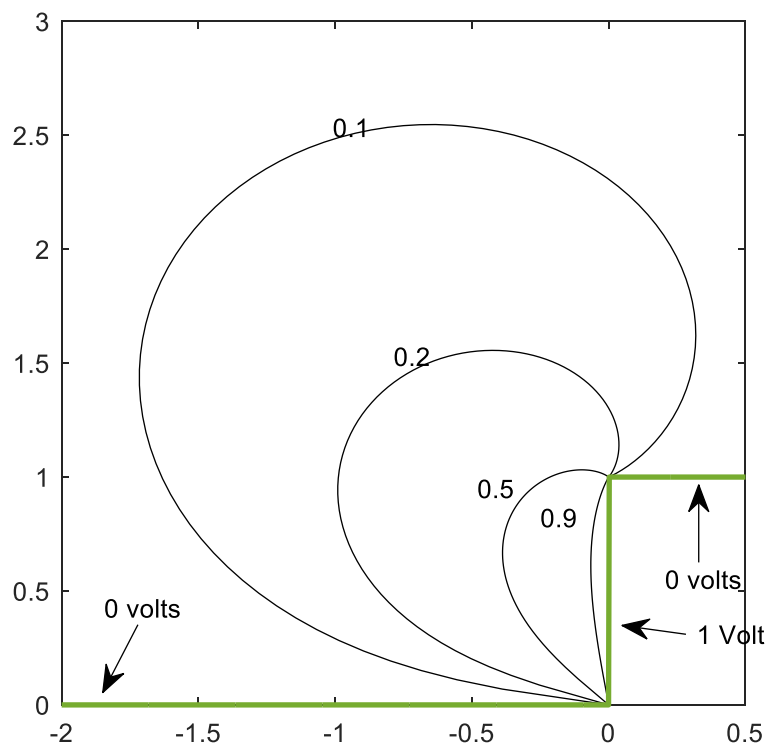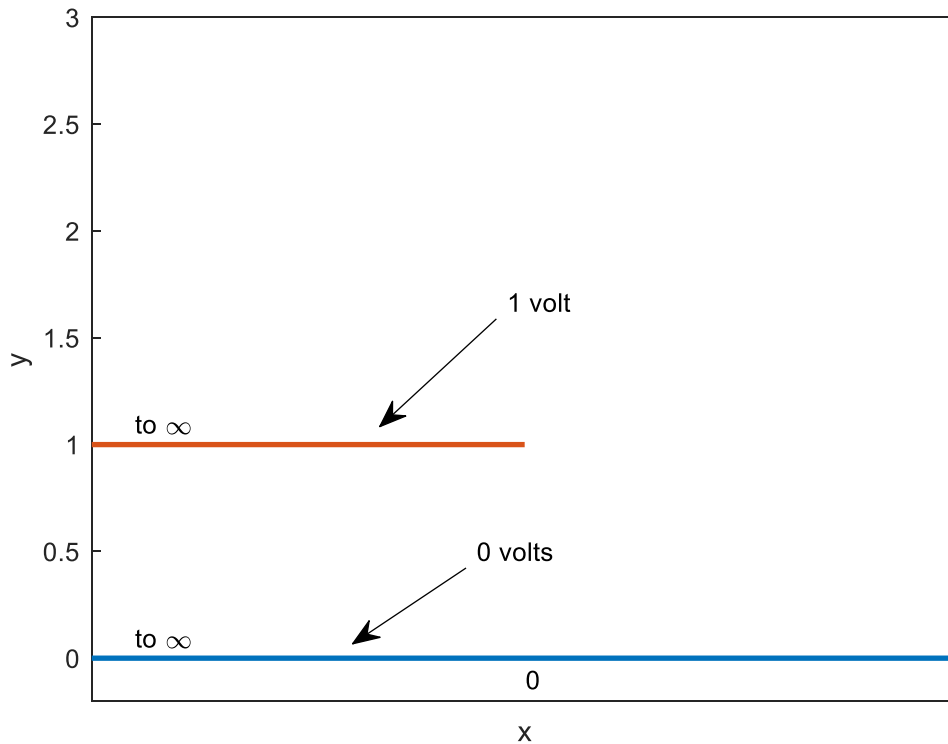
Figure 11.8 b

Example 8

Figure 11.9 a

Imagine that the above shows the cross section of two metallic conductors. The lower conductor is of infinite extent, lies along $y = 0$ and goes from $x = -\infty$ to $x = \infty$. The other conductor is semi-infinite in length and extends from $-\infty + i$ to i. The second conductor is 1 unit above the first and is maintained at a potential of 1 volt while the lower one is at ground (zero) potential. Our problem is to find the streamlines and equipotentials for this situation and find the voltage everywhere in space. We will apply the Schwarz-Christoffel transformation to the configuration shown below in Figure 11.9b and then let the angle α tend to zero to achieve the configuration shown above which will be mapped onto the real axis in the w plane with the exterior of the above figure mapped into the upper half of the w plane.
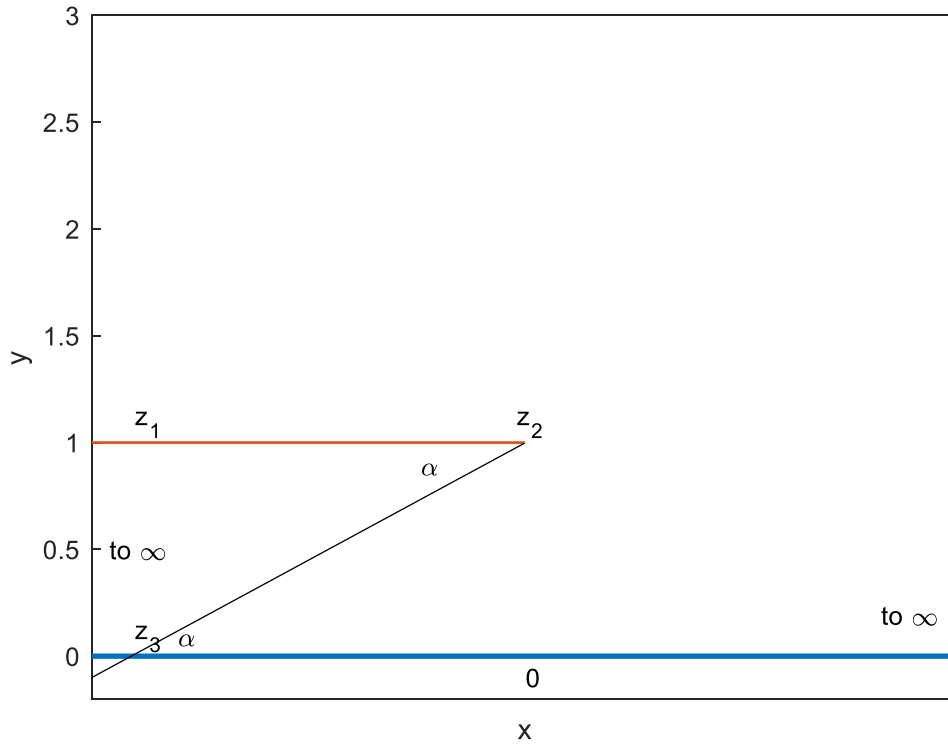
Figure 11.9 b

Following the notation of Figure 11.1(a) we have that $\alpha_2 = 2\pi - \alpha$ and $\alpha_3 = \alpha$. The point $z_2 = i$ will get mapped into $w_2 = -1$ and z_3 will get mapped into $w_3 = 0$. Notice that z_1, which is at infinity, will map into $w_1 = \infty$. We can explain that z_1 and z_3, which both lie at infinity, get mapped into different locations in the w plane. We imagine that z_1 and z_2 lie on different sides of a branch cut of $z = F(w)$ where $F(w)$ is the right side of Eq.(11.1). Notice for example that the function $w = e^{-\pi z}$ evaluated along the lines $z = x$ and $x + i$ have different limits as $x \to -\infty$ and they differ in sign. In this case, the inverse transformation is $z(w) = \dfrac{-1}{\pi} \operatorname{Log} w$ which has a branch cut. The above configuration in the figure is to be mapped into that shown in Figue11.10 below.

Notice too, for the above figure, that we have moved along the polygon (from z_1 to z_2 etc.)in such a way that the region above and outside the above broken line, which is to be mapped into the upper half of the w plane is on our left. We thus have from Eq.(11.1) .

$$z(w) = A \int^{w} (w+1)^{((2\pi - \alpha)/\pi) - 1)} (w)^{((\alpha/\pi) - 1)} dw + B$$

Passing to the limit $\alpha \to 0$ we have for the indefinite integral

$$z(w) = A\int^{w}(w+1)(w)^{-1}dw + B = A\operatorname{Log}w + Aw + B.$$

(11.20)

Let us take the lower limit of integration as -1, and we take $B = i$,

$$z = A\operatorname{Log}w + Aw - A\operatorname{Log}(-1) + A + i = A\operatorname{Log}w + A(w - i\pi + 1) + i$$

Note that we have that $z(-1) = z(w_2) = i$ as desired .

Suppose we move along the negative real axis in the w plane toward $w = 0-$. We are approaching $w=0$ (which is w_3, see figure below) from the left. Due to the logarithm, if A were complex then the imaginary part of this expression would become unbounded as $u \to 0-$. But we want the imaginary part to be bounded, as we see from Figure 11.9. Thus, A is real and positive so that the real part of the previous expression tends to $-\infty$ as $w \to 0$ on the real axis, .

The Schwarz Christoffel transformation maps the domain exterior to the broken line in the above figure into the upper half of the w plane with the boundary in the figure mapped into the real axis in the w plane. If that is the case, in the present problem, as we proceed along the line $\operatorname{Im}(w) = 0, 0 < \operatorname{Re}(w) < \infty$, using the transformation $z(w)$, the image points in the z plane must lie on the line Im(z)=0. Since A is known to be real, the expression

$z = A\operatorname{Log}w + A(w - i\pi + 1) + i$, where w is positive real, is real if $A = \dfrac{1}{\pi}$ since we want

$A(w - i\pi + 1) + i$ to be real. Thus finally:

$$z(w) = \frac{1}{\pi}\left(w + \operatorname{Log}w + 1\right)$$

(11.21)

There is no way to solve for w as a function of z. However, as before in this situation, we can plot the equipotentials and electric flux lines.

Note that the configuration of Figures 11.9 a and b, is mapped, in the limit, by Eq.(11.21) into the w plane as show below.
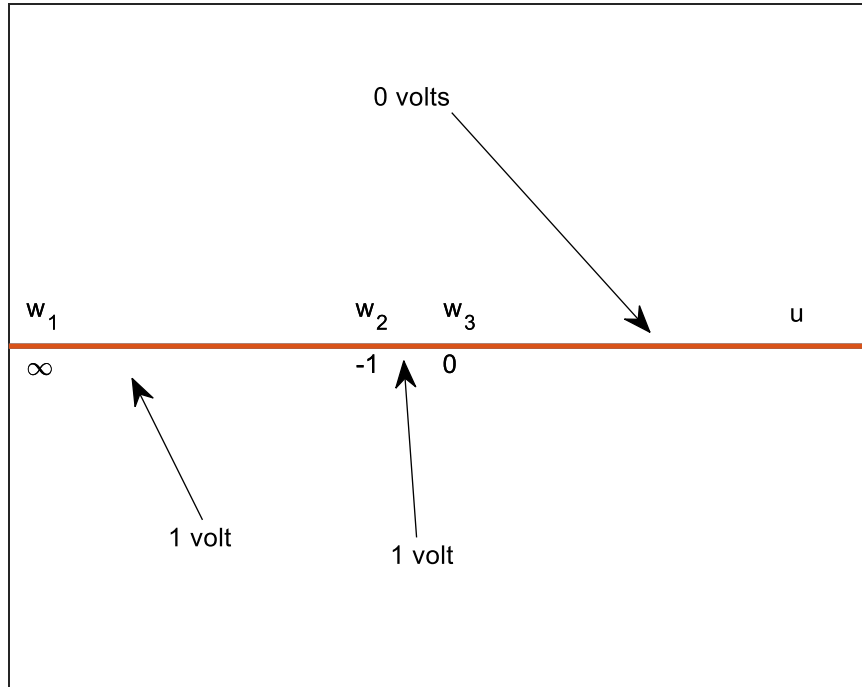
Figure 11.10

Observe that the potential $\phi(u,v)$ on the line Im(w)=v=0 is given by $\phi(u,0)=1$ for $-\infty < u < 0$ and $\phi(u,0)=0$ for $0 < u < \infty$. We can find the potential in the upper half space from the Poisson integral formula for the upper half plane and obtain

$\phi(u,v) = \dfrac{v}{\pi} \displaystyle\int_{-\infty}^{0} \dfrac{du'}{(u'-u)^2 + v^2} = \dfrac{1}{\pi} \arctan(v/u)$. The arctangent is to be interpreted as the imaginary part of $\text{Log}\, w$ (the principal branch). In this way we are assured at $0 \le \phi(u,v) \le 1$ in the upper half of the w plane. Thus, it should be apparent that the complex potential in the w plane is

$$\Phi(u,v) = \phi(u,v) + i\psi(u,v) = \dfrac{-i}{\pi}\text{Log}\, w = \dfrac{1}{\pi}\arctan(v/u) + \dfrac{-i}{\pi}\text{Log}\sqrt{u^2+v^2} \ (11.22)$$

The equipotentials in the *w* plane are lying on the surfaces for which $V_0 = \dfrac{1}{\pi}\arctan(v/u)$ so that

$$v = u\tan(\pi V_0) \quad 0 \le V_0 \le 1 \tag{11.23}$$

These are of course straight lines (rays) leaving the origin in the upper half of the *w=u+iv* plane. Note that as $V_0 \to 1/2$ we have $u \to 0$ for any finite *v*. On the equipotentials we have

$$w = u + iu \tan \pi V_0. \qquad (11.24)$$

We can substitute this expression into Eq.(11.21) to find the image of each equipotential in the z plane. We must be careful to use this expression in such a way that $\text{Im}\, w \geq 0$ because it is this region that forms the image of the domain whose boundary is shown in Figures 11.9 (a)and(b). Thus if $0 \leq V_o < 1/2$ we have that $0 \leq \tan(\pi V_0)$ and in using Eq.(11.24) we are restricted to $u \geq 0$. Conversely, if $1/2 \leq V_o < 1$ we have $\tan(\pi V_0) < 0$ and we are restricted to $u \leq 0$. Thus, the code we use to generate the images of the rays in the w plane, into the z plane, is divided into 2 parts dealing with $u \geq 0$ and $u \leq 0$ respectively.

The code to generate the equipotentials is given below.

```
%example 8 chap 11
clf
clear
%voltages of interest below for u>0, Re(w)>0
Vo=[.1 .2 .3];
beta=(tan(Vo*pi));
beta=beta.';
for j=1:length(Vo)
%the following allows for points near abs(w1)=0 which will yield very negative
%values of real(z1)without requiring Re(w1)<<0
u1=linspace(0,1,1000);
v1=(u1.*beta(j));
w1=u1+i*v1;
z1=1/pi*(w1+log(w1)+1);
u2=linspace(1,10,50);
v2=(u2*beta(j));
w2=u2+i*v2;
z2=1/pi*(w2+log(w2)+1);
plot(z1,'k');hold on
plot(z2,'k');hold on
end
% the following works for u=Re(w)<0
 % voltages of interest are below.
Vo=[ .51 .7 .9];
beta=(tan(Vo*pi));
beta=beta.';
for j=1:length(Vo)
u3=linspace(-10,-1,50);
v3=(u3.*beta(j));
w3=u3+i*v3;
z3=1/pi*(w3+log(w3)+1);
%the following allows for points near abs(w4)=0 which will yield very negative
%values of real(z4)without requiring Re(w4)<<0
u4=linspace(-1,0,10000);
v4=(u4.*beta(j));
```

```
w4=u4+i*v4;
z4=1/pi*(w4+log(w4)+1);
plot(z3,'k');hold on
plot(z4,'k');hold on
end
figure(1)
%the following plots the boundary conductors
 x=linspace(-5,-.01);
y=ones(length(x));
plot(x,y,'linewidth',2)
x=linspace(-5,5,10);
yo=0*x;
plot(x,yo,'linewidth',2)
xlim([-1.5 3]);ylim([0 4])
 And here is the result
```
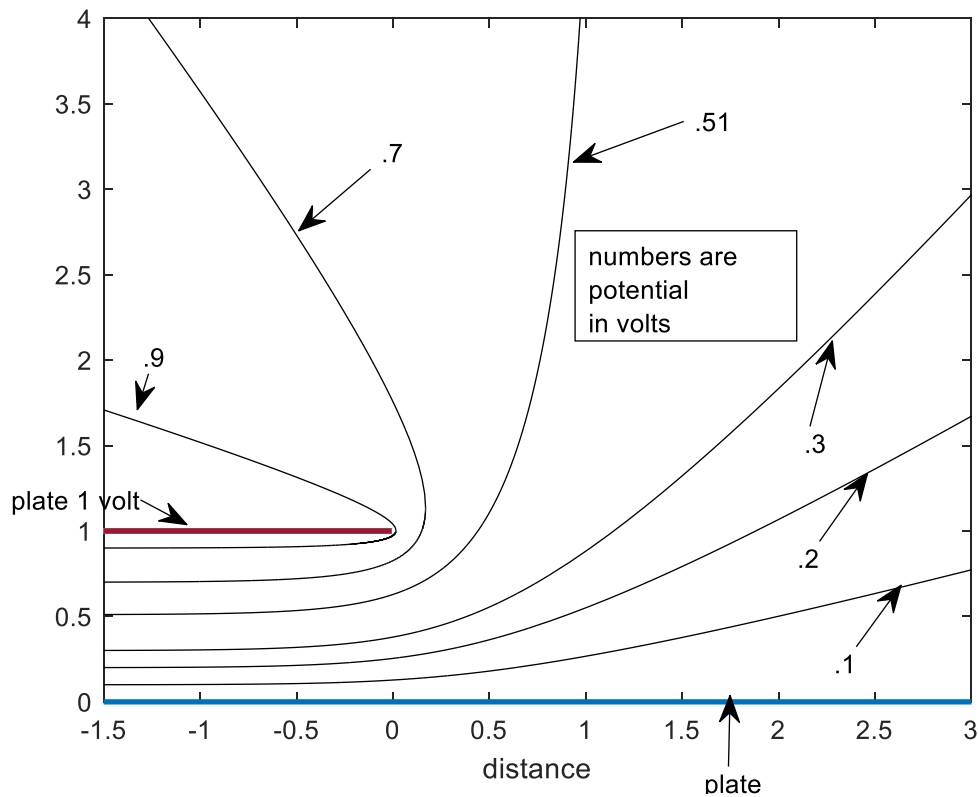

Figure 11.11

The streamlines for the above configuration are at each point tangent to the electric field and electric flux density vector. The streamlines are the lines along which the stream function $\psi(x,y)$ assumes constant values. For the present problem we have from Eq.(11.22) that the stream function in the w plane is

$$\psi(u,v) = \frac{-1}{\pi} \operatorname{Log} \sqrt{u^2 + v^2}. \qquad (11.25)$$

We can set this equal to a number of real values and plot the locus in the z plane for each value. By using uniformly spaced values of ψ we can see how tightly or sparsely spaced these loci are in the z plane. The more tightly spaced they are at the boundaries (the plates in the above figures) the higher the concentration of charge. The charge is positive at each point on the plate at the higher potential , $y = 1, x < 0$ and negative at the plate of lower potential

$y = 0, -\infty < x < \infty$. Let γ be a value of $\psi(u,v)$. We have from Eq.(11.25) that $u^2 + v^2 = e^{-2\pi\gamma}$. These are circles of radius $e^{-\pi\gamma}$, centered at the origin in the w plane. These circles are valid only in the space $v \geq 0$. In the following code we map circles, corresponding to uniformly spaced values of $\gamma = \psi$. The radii of the circles are not uniformly spaced because they are

equal to $e^{-\pi\gamma}$ which is not a linear function. The mapping is made from the w plane into the z plane so as to produce the streamlines (aka flux lines) for the above figure. Notice the

expression $z(w) = \dfrac{1}{\pi}(w + \text{Log } w + 1)$. If we wish to obtain values of z for the region between the

plates, which is where $\text{Re}(z) < 0, \ 0 < \text{Im}(z) < 1$, we will require that, $0 \leq \arg(w) + \text{Im}(w) < \pi$.

Also a necessary but not sufficient condition is that $0 < \text{Re}(w) \leq 1$ as can be seen from study of Eq.(11.21). In the following code and in the resulting plot we have taken the increments of ψ as

$\dfrac{1}{15}$. The reader might wish to experiment with other values.

```
%example 8  chap 11
clf
clear
gam=linspace(-10,30,60);
gam=gam/10;
%change in gam is 40/600=1/15;
beta=exp(-2*pi*gam);
beta=beta';

 for j=1:length(beta)

u1=linspace(-sqrt(beta(j)),sqrt(beta(j)),100000);
v1=sqrt(beta(j)-u1.^2);
w1=u1+i*v1;
z1=1/pi*(w1+log(w1)+1);hold on

plot(z1,'r');
end
ylim([0 5]);
xlim([-1 3.5]);
x=linspace(-2,5,100);
y=0*x;
plot(x,y,'k','linewidth',2)
xx=linspace(-2,0,1000);
yy=ones(length(xx));
plot(xx,yy,'k','linewidth',2)
```
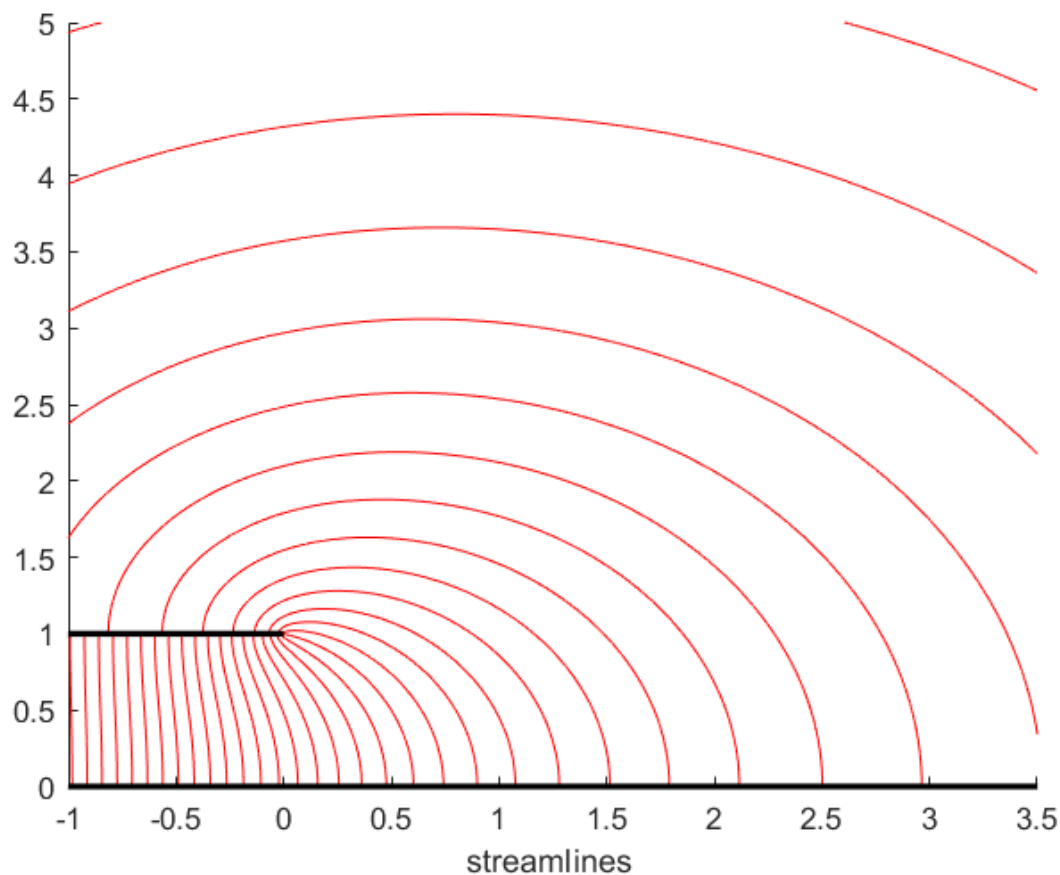
The output is

Figure 11.12

We see that charge tends to be concentrated at the edge of the upper plate. As we move to the left, between the plates, the field becomes uniform and has the behavior appropriate to an infinite parallel plate capacitor with no fringe field. The electric field is then directed downward, from the higher to the lower potential, and has a magnitude equal to the voltage difference between the plates divided by their separation, i.e. 1 . The field as a vector is simply *-i*.

Let us develop a program to give the electric field at any point of interest. Recall that the electric field, expressed as a complex number is given by $E = -\overline{\left(\dfrac{d\Phi}{dz} \right)}$. We have also that

$$\frac{d\Phi}{dz} = \frac{d\Phi}{dw}\frac{dw}{dz} = \frac{d\Phi}{dw} \div \frac{dz}{dw} \qquad (11.26)$$

Thus, using Eq.(11.21) and (11.22) we have

$$E(w) = \frac{-iw}{|w|^2 + w}$$

For every value of z of interest, we must find the corresponding value of *w* by using Eq. (11.21). We accomplish this with the following code . The input is a vector containing points of interest. Here is the code

```
z=input('z= vector')
for j=1:length(z)
syms w
digits(6)
warning('off')
syms w
w=solve(1./pi*(w+log(w)+1) ==z(j));
E_field=-i*w/(abs(w^2)+w);
E_field=eval(E_field);
result=[z(j), E_field]
end
```

And here is the output. We chose to investigate 3 points in Figure 11.8 , namely
z= vector[-.5+.5*i .5+.5*i .01+.99*i]

The results

result =

  z = -0.5000 + 0.5000i   field=0.0745 - 0.9887i

result =

  z= 0.5000 + 0.5000i   field =0.2275 - 0.5158i

result =

  z=0.0100 + 0.9900i   field = 3.0755 - 1.6257i

Notice that at -.5+.5*i , which is between the plates, the electric field is downward directed with a magnitude of roughly 1 and points very slightly to the right (as predicted by Fig. 11.12). At .5+.5*i, which is not between the plates, and is referred to as the region of "fringe field", the field has weakened in the downward direction from the previous value and shows fringing toward the right, while at the third location we are close to the upper edge where the field is quite strong, points downward but has an even stronger component toward the right. Of the three values of electric field we have considered the third is strongest owing to the singularity in field strength at the point $z = i$.

Example 9
It is possible to create a relatively simple configuration such that the required Schwarz-Christoffel transformation results in an integral appearing in Eq.(11.1) that must be evaluated numerically. We consider such an arrangement in Figure 11.13 and show how to deal with the integral by means of MATLAB
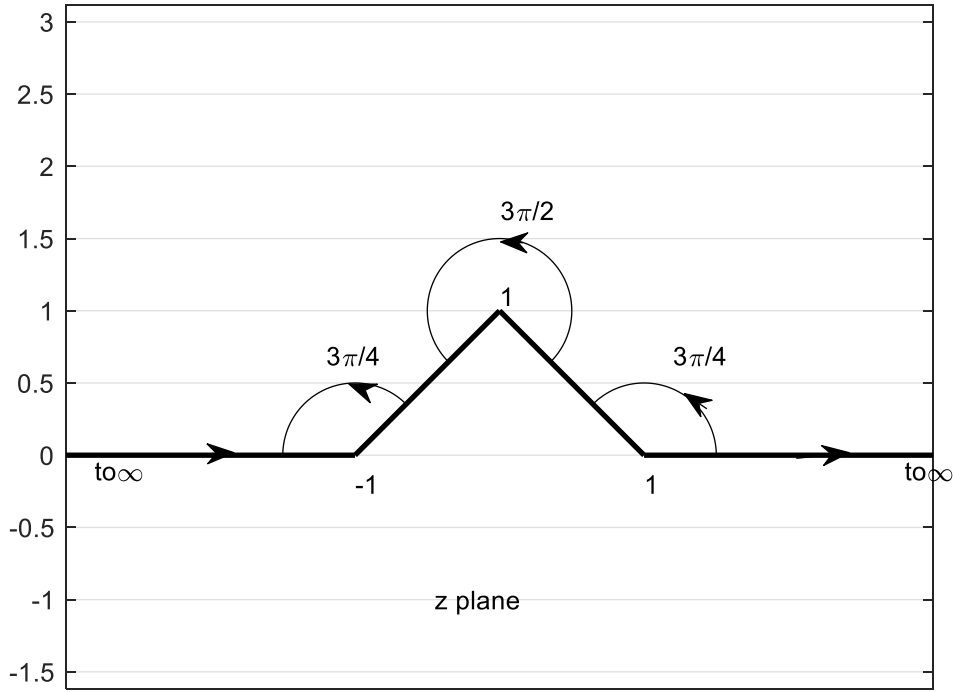
Figure 11.13

The above broken line, which begins and ends at infinity is to be mapped onto the real axis in the w plane so that the domain above the line is mapped into the upper half of the w plane. We want $z_1 = -1$ to have image $w_1 = -1$; $z_2 = i$ to have image $w_2 = 0$ and $z_3 = 1$ to have image $w_3 = 1$. The angles α_1, α_2, and α_3 to be used in Eq.(11.1) are respectively $\dfrac{3\pi}{4}, \dfrac{3\pi}{2}$, and $\dfrac{3\pi}{4}$.

Thus we have the transformation

$$z(w) = A\int^{w}(w+1\ )^{-1/4}(w^{1/2})(w-1)^{-1/4}\,dw + B \qquad (11.27)$$

The above integral will need to be evaluated numerically. We must *resist the temptation* to use $(w-1)^{-1/4}(w+1)^{-1/4} = \left(w^2-1\right)^{-1/4}$. The Schwarz-Christoffel transformation requires that the branch cuts for each of the factors on the preceding left side do not enter the upper half of the *w* plane. Principal values will be used. If we choose, for example, w=-2, we would find that each factor on the left has argument $-\pi/4$ so that the argument for their product is $-\pi/2$. The argument on the right is 0. This reminds us that the principal value of the argument (angle) of a product need not equal the sum of the principal values of each factor.

Since we require that $z(0) = i$ we can achieve this condition by taking the lower limit of integration as 0 and setting $B = i$. Thus Eq.(11.27) becomes

$$z(w) = A\int_{0}^{w}(w+1\ )^{-1/4}(w^{1/2})(w-1)^{-1/4}\,dw + i$$

We require $z(1) = 1$ which will allow us to find *A*. From MATLAB we obtain the above integration as follows:

```
>> syms w
>> int((w+1)^(-1/4)*w^(1/2)*(w-1)^(-1/4),0,1)

ans =

-(4*(-1)^(3/4)*2^(1/2)*gamma(3/4)^2*hypergeom([1/4, 3/2], 9/4, -1))/(5*pi^(1/2))

>> eval(ans)

ans =

  0.599070117367796 - 0.599070117367796i
```

And finally
```
>> A=(1-i)/ans

A =

  1.669253683348146 + 0.000000000000000i
```

We can check this result by seeing if $z(-1) = -1$. We use the MATLAB function **integral** rather than **int** as the latter will not perform the integration.

```
syms w
A=1.669253683348146;
check=A*integral(@(w)(w.^(1/2).*((w+1).^(-1/4)).*((w-1).^(-1/4)),0,-1)+i
```

which yields

```
check =

  -1.0000 - 0.0000i
```

We observe from Eq.(11.27) that as $w \to \infty$ along the positive real axis that its image tends to infinity along the positive real axis in the z plane. A corresponding statement applies to moving toward minus infinity in the z plane.

Having found A and B we now use Eq.(11.27) to map the streamlines from the w plane into the z plane. It should be apparent that in the w plane we can have uniform flow of a fluid from left to right by employing the complex velocity potential $\Phi(w) = w = u + iv$. The constant values of the stream function, $\operatorname{Im}\Phi(w)$, are the lines on which v assumes constant values. We have selected these values as elements of the vector v in the following code, which produces the resulting figure showing fluid flow around the obstruction

```
clear
tic
syms w
A=1.669253683348146;
u=linspace(-5,5,100);
```

```matlab
v=[.1 .3 .5  .9 1  1.3];%values for stream function

W=u+i*v';

for k=1:length(v);
    for j=1:length(u);
z(j,k)=A*integral(@(w)(w.^(1/2)).*((w+1).^(-1/4)).*((w-1).^(-1/4)),0,W(k,j))+i;

    end
   plot(z,'k');hold on
end
toc
grid
%the next 5 lines plot the boundary
x1=linspace(-8,-1,100);x2=linspace(1,8,100);
y1=0*x1;y2=0*x2;
plot(x1,y1,'k','linewidth',2);plot(x2,y2,'k','linewidth',2)
x3=linspace(-1,0); y3=(1+x3); x4=-x3; y4=(1-x4);
plot(x3,y3,'k','linewidth',2);plot(x4,y4,'k','linewidth',2)
xlim([-3 3]);
axis equal
psival=num2str(v');%creates text numbers for values of psi
   text(real(z(34,:)),imag(z(34,:)),psival)
   text(-2,1.5,'numbers on curves are values of{\psi}')
   text(.2,3,'flow')
   text(-2.5,.05,'0')
```

Fluid Flow Around the Obstruction in Figure 11.13

flow ⟶

1.3

1

0.9

numbers on curves are values of ψ

0.5

0.3

0.1

0
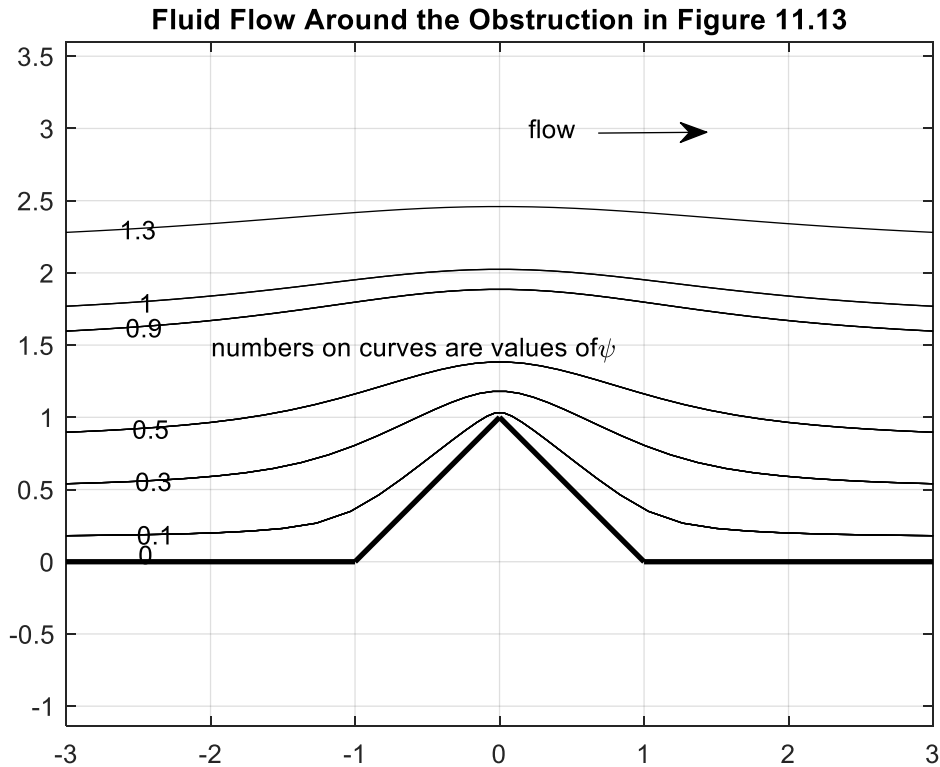
Figure 11.14

Note that it is possible to convert the integral in Eq.(11.27) to an elliptic integral through the substitution $w = \cos t$ and by combing the product $(w-1)^{-1/4}(w+1)^{-1/4} = (w^2-1)^{-1/4}$. Although MATLAB has a library function for the elliptic integral, the proper use of branches gets lost with this change of variable and multiplication. The figure above took only .4 seconds without our use of the elliptic functions.

Example 10
The Triangle .

All of the previous examples have treated the problem of mapping an unbounded region into the upper half of the *w* plane. In the present case we will map a region bounded, in the *z* plane, by a 30,60 90 degree triangle into the upper half of the w plane. The region and notation are shown below
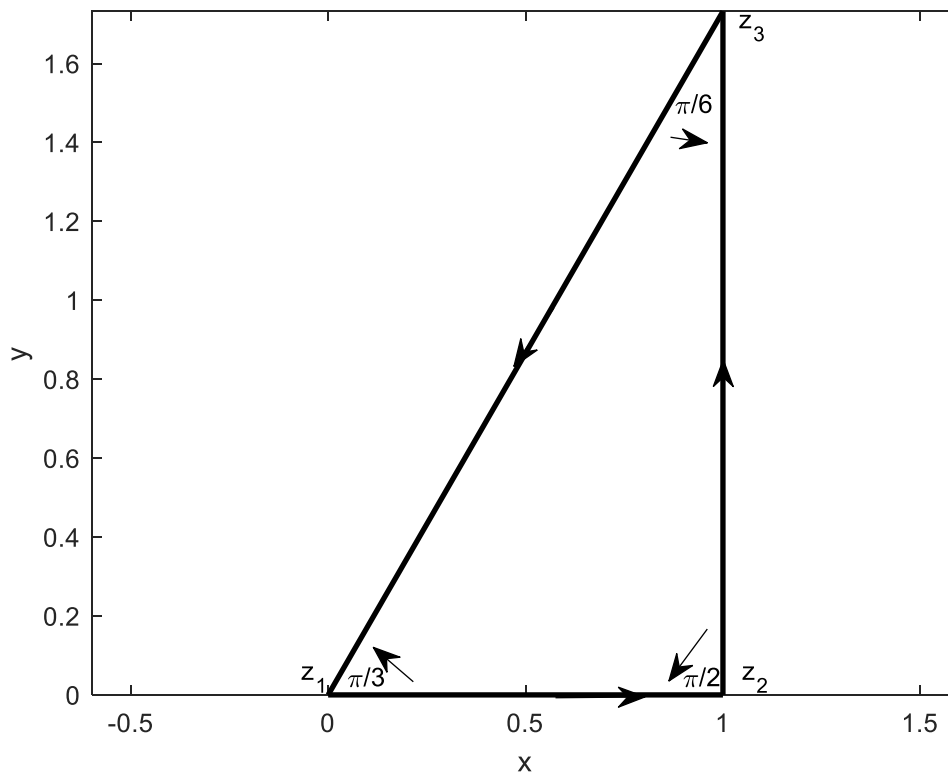
Figure 11.15

The mapping is that $z_1 = 0$ has image $w_1 = 0$, $z_2 = 1$ has image $w_2 = 1$, $z_3 = 1 + i\sqrt{3}$ has image $w_3 = \infty$. Thus applying Eq. (11.1) we have

$$z(w) = A\int^w (w)^{-2/3}(w-1)^{-1/2}\,dw + B \qquad (11.28)$$

If we take the lower limit of integration as zero we can take B=0 and ensure that $z(0) = 0$ as required. Notice that because z_3 is mapped to ∞ there is no corresponding term in the integral.

To find A we require that $z(w_2) - z(w_1) = z_2 - z_1 = 1$. From the following code we have

```
syms w f
f=@(w)(w).^(-2/3).*(w-1).^(-1/2);
Q=integral(f,0,1);
format long
A=1/Q
```

With the result:
A =

   0.0   + 0.237724776535395i

To check our work, we map the real axis in the w plane into the z plane to see if it results in the original triangle. We proceed as shown in the following code. The values chosen for *w* were the result of some experimentation. It is important to have some values of *w* near to, and on, both sides of, the branch points at $w=0$ and $w=1$. We have plotted the images of

$-\infty < w < 0,\ 0 < w < 1,$ and $1 < w < \infty$ in three different colors and have obtained the triangle following this program below. Another thing to note: we have plotted the values of the integral obtained from Eq. (11.28) . It is tempting to use the command **plot I** where I is the vector for the values of this integral at different values of *w*. If **I** has any complex elements the plot function plots the imaginary part of **I** vs. the real part. However, if **I** is a vector composed of real elements the plot is of the elements of **I** vs **the index** of each element, which we don't want. To prevent this we have used throughout here the command plot(real(I),imag(I)).

Here is our program

```
clear
clf
A= 0.237724776535395i;
syms w f;
f=@(w)(1./w.^(2/3)).*(1./(w-1).^(1/2));
%the following are points on the negative real w axis to the left of -.0001
w1=[-10^12 -10^10 -10^8 -10^7 -10^6 -100000 -10000 -1000 -100 -10 -1 -.01 -.001 -.0001 ];
wlim=w1;
for j=1:length(wlim)
I(j)=A*integral(f,0,wlim(j))
format long
end
plot(real(I),imag(I),'r','linewidth',2)
hold
clear I
%the following points lie to the left of the branch cut at w=0 and are near the branch point
w2=linspace(-.001, -eps,1000);
wlim=w2;
for j=1:length(wlim)
I(j)=A*integral(f,0,wlim(j));
format long
end
plot(real(I),imag(I),'r','linewidth',2)
% maps using color red

clear I

% the following points are close to and to the right of the branch point at
w=0

w3=linspace(eps, 1-eps,10);
wlim=w3;
for j=1:length(wlim)
I(j)=A*integral(f,0,wlim(j));
format long
end
plot(real(I),imag(I),'b','linewidth',2)
%maps using color blue
clear I
```

```
%the following points are to the right of the branch cut at w=1
w4= [1+eps 2 5 10 100 1000 10000 100000 10^6 10^10];

wlim=w4;
 syms w f
f=@(w)(w.^(-2/3)).*(w-1).^(-1/2);
for j=1:length(wlim)
I(j)=A*integral(f,0,wlim(j));
format long
grid
axis equal
end
plot(real(I),imag(I),'c','linewidth',2)
% maps using color cyan
```
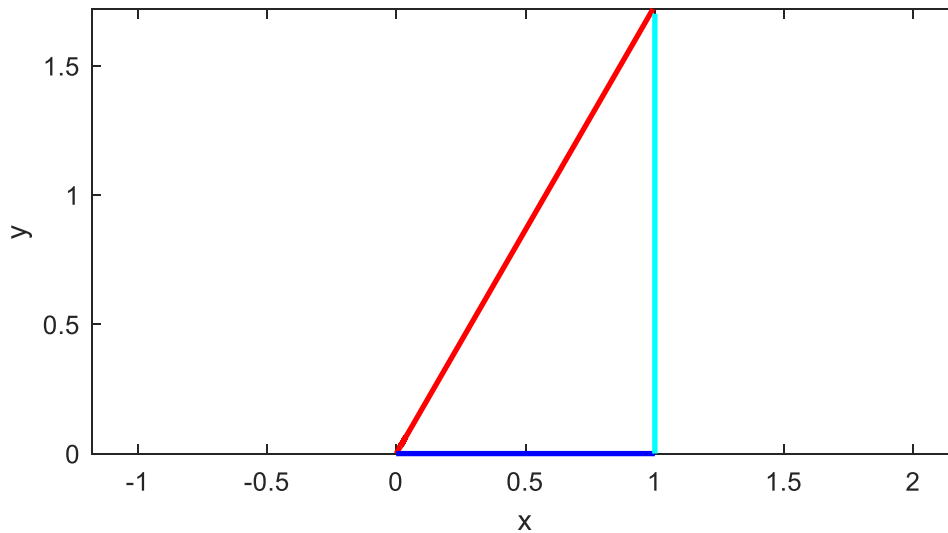
And here is our result

Figure 11.16

Example 11 Mapping of a Square
We have said that the Schwarz-Christoffel transformation permits specifying the mapping
between at most three points lying on the vertices of the polygon in the *z* plane into three
specified points in the *w* plane. The geometry of a square, or a rectangle, is determined by the
location of three of its corners. For this reason, we can map a square from the z plane into the w

plane in such a way that the image of the boundary is the real *w* axis and the interior of the square in the upper half of the w plane. The square we have chosen is shown below.
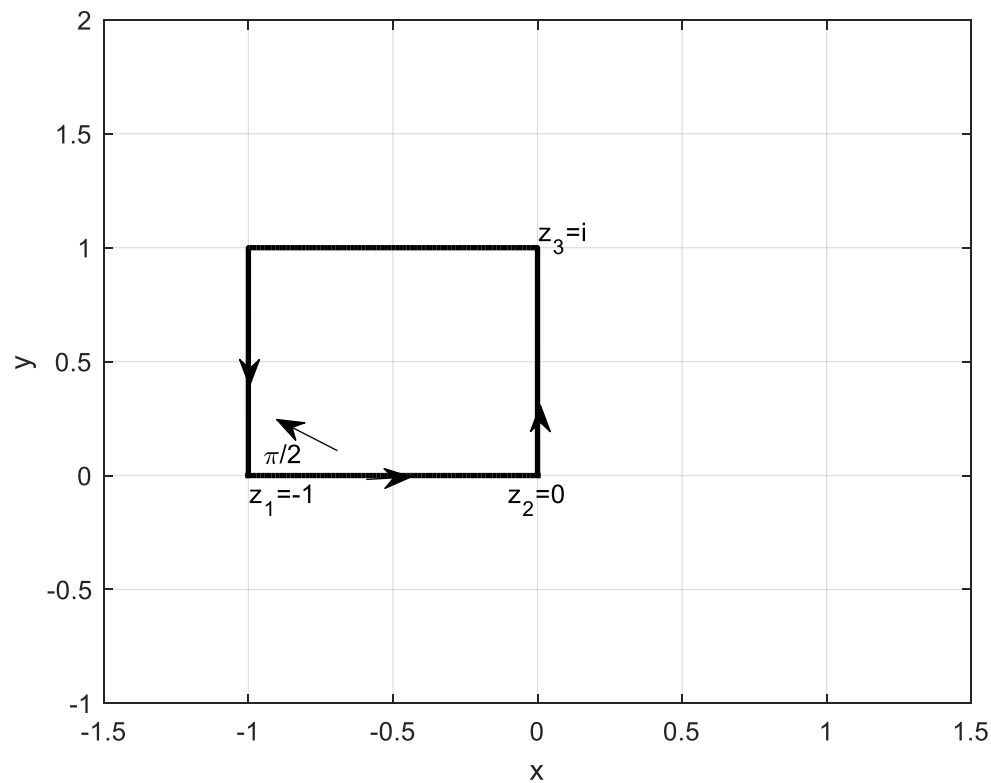

Figure 11.17

We will take $z_1 = -1$ has image $w_1 = -1$, $z_2 = 0$ has image $w_2 = 0$, and $z_3 = i$ has image $w_3 = 1$. Because $w_2 = 0$ is the image of 0 we can satisfy this condition by putting B=0 in Eq.(11.1) and taking the lower limit of integration as zero.

Our transformation becomes

$$z(w) = A\int_0^w (w+1)^{-1/2}(w)^{-1/2}(w-1)^{-1/2}dw \quad (11.29)$$

To find *A* we notice that $z(w_2) - z(w_1) = 1$. Thus,

$$A = \frac{1}{\int_0^1 (w)^{-1/2}(w-1)^{-1/2}(w+1)^{-1/2}dw} \quad (11.30)$$

Solving this with MATLAB we find A= -0.381379881750885
To check our work, let us map the real axis from the w plane into the z plane and see what occurs.

As before, we must resist the temptation to use $(w-1)^{1/2}(w+1)^{1/2} = \sqrt{w^2-1}$ in Eq.(11.29) and Eq.(11.30) even though it can lead to an elliptic integral which MATLAB can evaluate as a library function. The difficulty is that MATLAB will choose the wrong branch of the square root in parts of the w plane as for example if $\mathrm{Re}(w) < 1$

The following program computes A as described above and performs the required mapping.

```
clear
clf
syms w
tic
warning('off')
f=@(w)w.^(-1/2).*(w+1).^(-1/2).*(w-1).^(-1/2);
Q=integral(f,-1,0)
A=1/Q
W1=linspace(-2000,-1-eps,100);
W2=linspace(-1+eps,-eps,100);
W3=linspace(eps,1-eps,100);
W4=linspace(1+eps,3,100);
W5=linspace(3,100,100);
W6=linspace(100,2000,100);
W=[W1 W2 W3 W4 W5  W6 ];
for j=1:length(W)
z(j)=A*integral(f,0,W(j));
end
plot(z,'k','linewidth',2 );
toc
```
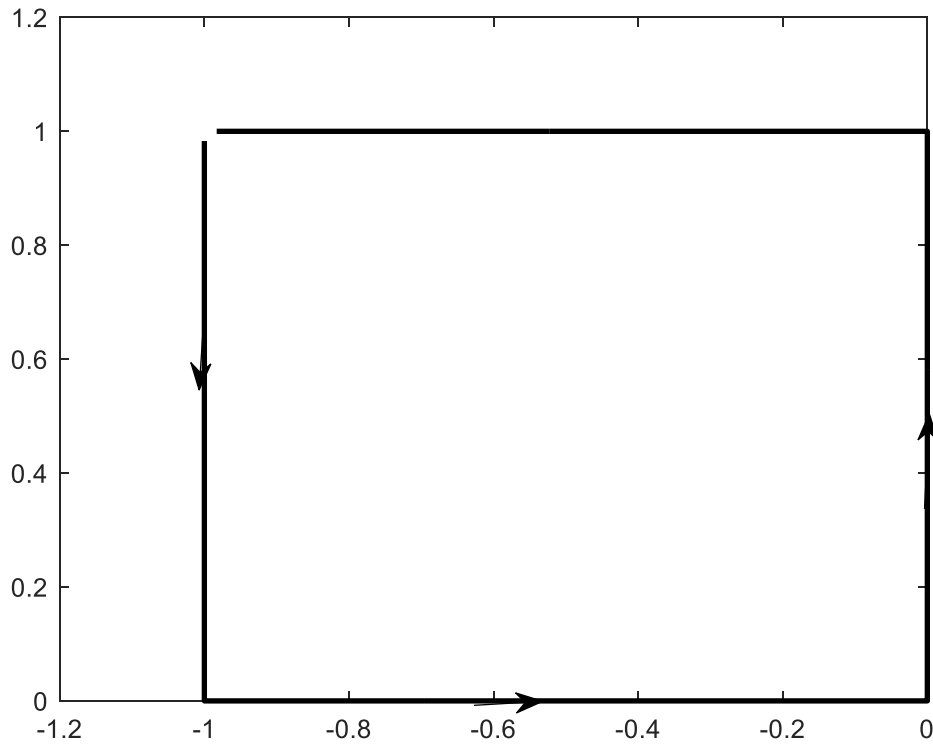
Figure 11.18
The arrow shows how the square is mapped as we move from left to right along the real axis in the *w* plane. Note the gap in the upper left-hand corner. The point $z = -1 + i$ is the image of $w = \infty$. As w tends to $\pm\infty$ along the real axis its image approaches but never equals $-1 + i$.

EXAMPLE 13 A Square, Grounded Tunnel, with the Floor at One Volt

We will assume that the above figure represents the cross section of a conducting tunnel or tube. All the walls are at zero potential except the bottom one is maintained at one volt. Our goal is to find the potential anywhere inside, and to plot the electric field lines (streamlines).
Solution :
The bottom extends from $z = -1$ to $z = 0$. This is mapped, by Eq.(11.28) into the line segment extending in the w plane from $w = -1$ to $w = 0$ which is maintained, like its image in the z plane, at 1 volt. . The remainder of the real axis in the w plane is kept at zero volts, like the remaining sides in the z plane .

As in Example 1 above, we find the potential in the space $\mathrm{Im}(w) \geq 0$ can be found from the Poisson integral formula. We have

$$\phi(u,v) = \frac{v}{\pi} \int_{-1}^{0} \frac{du'}{\left(u'-u\right)^2 + v^2}$$

We have

```
syms up u v
assume(u,'real')
assume(v,'real')
f=1./((up-u)^2+v.^2);
result=int(f,up,[-1  0]);
result=v/pi*result
```

The output is:

result =

-(atan(u/v) - atan((u + 1)/v))/pi

Which is more comfortably written as

$$\phi(u,v) = \frac{1}{\pi}\left[\arctan(\frac{u+1}{v}) - \arctan(\frac{u}{v})\right] = \frac{1}{\pi}\left[\arctan(\frac{v}{u}) - \arctan(\frac{v}{u+1})\right] = \frac{1}{\pi}\text{Re}\left[i\text{Log}(w+1) - i\text{Log}\,w\right]$$

Thus, the electric potential in the space $\text{Im}(w) \geq 0$ is the real part of the complex potential

$\Phi(w) = \dfrac{i}{\pi}\text{Log}\dfrac{w+1}{w}$. The stream function is the imaginary part of this expression, i.e.,

$$\psi(u,v) = \text{Im}\left[\frac{i}{\pi}\text{Log}\frac{w+1}{w}\right]$$

The streamlines in the *w* plane are those lines on which the real part of $\text{Log}\dfrac{w+1}{w}$ are constant

These are the lines on which $\left|\dfrac{w+1}{w}\right|$ are a real positive constant. Calling this real constant *k>0*,

and with $w = u + iv$ we have $k^2 u^2 + k^2 v^2 = (u+1)^2 + v^2$. With some algebra we find that these

are circles described by $\left(u - \dfrac{1}{(k^2-1)}\right)^2 + v^2 = \dfrac{k^2}{(k^2-1)^2}$. The centers of these circles are at

$u = \dfrac{1}{k^2-1}$ $v = 0$ and their radii are $\left|\dfrac{k}{k^2-1}\right|$ where *k* >0 is real. Notice that our mapping into the

w plane assumes $v \geq 0$.

We can plot these streamlines in the w plane, and verify that they are circles, with the following
code. Notice that we do not use a uniform separation for values of *k*. Fixed values of the stream

function ψ are values on which the logarithm of $\left|\dfrac{w+1}{w}\right|$ are equal to a constant value k. By choosing values of k such that the logarithm of these values are uniformly spaced we will be looking at uniformly spaced values of ψ . To accomplish this, each value of k chosen is equal to a fixed constant multiplied by its preceding value. Here is the code

```
k=1:10;
k=2.^k;
k=k/50;
clf
 for j=1:length(k)
    K=k(j);
u=linspace(-5,4,400);
v=sqrt(K^2/(K^2-1)^2-(u-1/(K^2-1)).^2);
plot(u,real(v),'k');axis equal
hold on
end
U=linspace(-1,0,10);
V=0*U;
plot(U,V,'k','linewidth',2)
xlim([-2 2]);
axis equal
```

The output is the following. We have added arrows to the streamlines which flow out of the image, in the w plane, of the bottom of the box in Figure 11.8. Notice the tendency of the streamlines to be crowded where the potential is discontinuous at $v=0, u=-1$ and $v=0, u=0.$ Here there is an accumulation of charge.

We see that the streamlines, which show the direction of the electric field, are semi-circles in the upper half plane. The interval on the u axis is not sufficiently large enough to show where two of these semicircles return to the u axis.
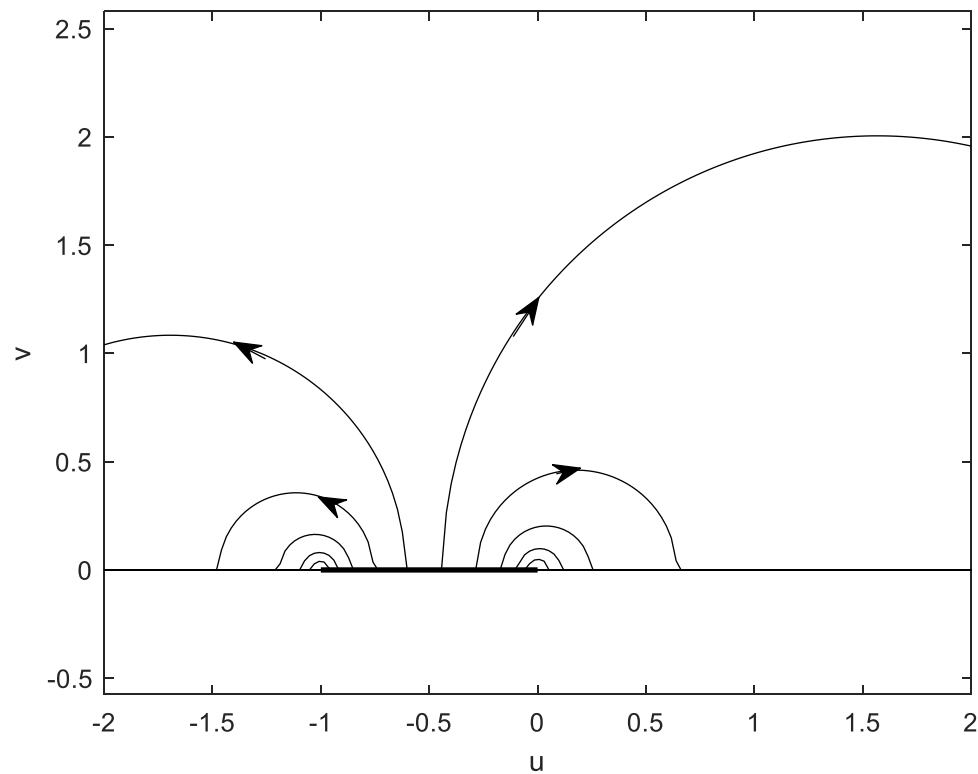
Figure 11.19

We can modify the preceding program and employ Eq.(11.28) to obtain the streamlines for Fig.(11.18) where the bottom of the box is maintained at 1 volt and the other sides are at zero potential. The code is as follows:

```
clf(figure(1));
k=linspace(log10(.1),log10(10),24);
k=10.^(k);
for j=1:length(k)
K=k(j);
u=linspace(-10,10,800);v=sqrt(K^2/(K^2-1)^2-(u-1/(K^2-1)).^2);
W=u+i*real(v);
syms w
warning('off')
for j=1:length(u)

    f=@(w)w.^(-1/2).*(w+1).^(-1/2).*(w-1).^(-1/2);
    A= -0.381379881750885;
    z(j)=A*integral(f,0,W(j));
end
figure(1)
plot(real(z),imag(z),'k'); hold on
```

```
end
grid

x=linspace(-1.5,.5,10);
y=0*x;
plot(x,y,'linewidth',2);
x=linspace(-1.5,.5,1000);
y=heaviside(x+1)-heaviside(x);
axis equal
plot(x,y,'linewidth',2)
```

The resulting figure is below, where arrows indicate the direction of some of the streamlines.
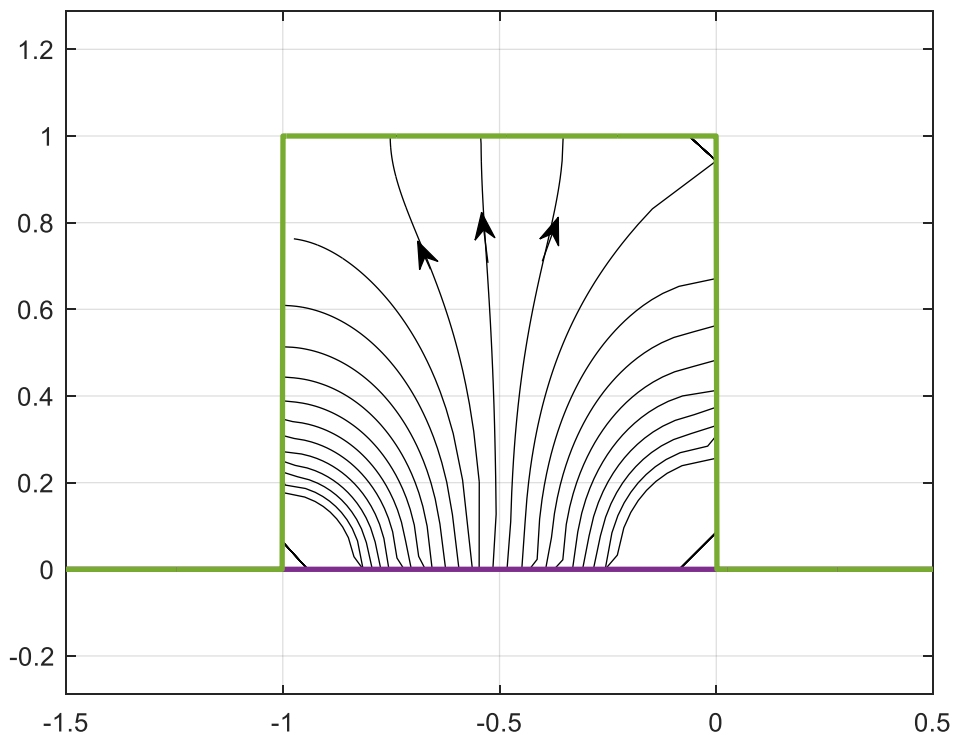

Figure 11.20

The short lines in each of three corners are not streamlines, and are due to roundoff errors, and can be eliminated if we choose more closely spaced values on u in the w plane with which to plot corresponding contours in the z plane, however the time involved for computation becomes excessive.

We can plot the equipotentials interior to the tunnel`. In the *w* plane the complex potential is given by $\Phi(w) = \dfrac{i}{\pi} \text{Log} \dfrac{w+1}{w} = \dfrac{-i}{\pi} \text{Log} \dfrac{w}{w+1}$. Its real part is $\phi(w) = \dfrac{1}{\pi} \arg \dfrac{w}{w+1}$ where we use the principal value. Note that $\dfrac{1}{\pi} \arg(w/(w+1)) = \dfrac{1}{\pi} \arg \dfrac{u+iv}{u+1+iv} = \dfrac{1}{\pi} \arg \dfrac{(u+iv)(u+1-iv)}{(u+1)^2 + v^2} =$

$\frac{1}{\pi}\arctan\frac{v}{u(u+1)+v^2}$. Thus if ϕ is a particular value of the potential we have

$$\tan\pi\phi = \frac{v}{u(u+1)+v^2} .$$

With some algebra this reduces to $(u+1/2)^2 + (v-\frac{1}{2}\cot(\pi\phi))^2 = \left(\frac{1}{2\sin(\pi\phi)}\right)^2$

It is easy to prove that if $v=0$ then $u=-1$ and 0 for all ϕ. All of the equipotentials must pass through these 2 points.

In general the equipotentials are circles in the w plane having centers at $u=-1/2, v=\frac{1}{2}\cot\pi\phi$

and radius $\left|\frac{1}{2\sin\pi\phi}\right|$ Notice that if the voltage is zero the equipotential degenerates into a circle

of infinite radius with center at $v=\infty$

Since boundaries in both the *z* and *w* plane are maintained at 0 and 1 volts, the potential inside the tunnel in the z plane must satisfy $0 < \phi < 1$. This also applies in the space $\mathrm{Im}(w) > 0$. Let us try the interval from .1 to .9 in steps of .1 . We will plot these equipotentials in the w plane. Solving the preceding equation for $v(u)$ we have

$$v = \frac{1}{2}\cot\pi\phi + \sqrt{\left(\frac{1}{2\sin\pi\phi}\right)^2 - (u+1/2)^2}$$. There is a requirement here, v must be a nonnegative

real. We must use points only in the upper half of the w plane. The potential is most easily plotted using polar coordinates and not the preceding equation. . We employ the transformation

$w = u + iv = re^{i\theta}$.

In the following we have plotted the equipotentials in the w plane and then mapped them Into the z plane (which contains the tunnel) using Eq.(11.28). The code follows the two figures.
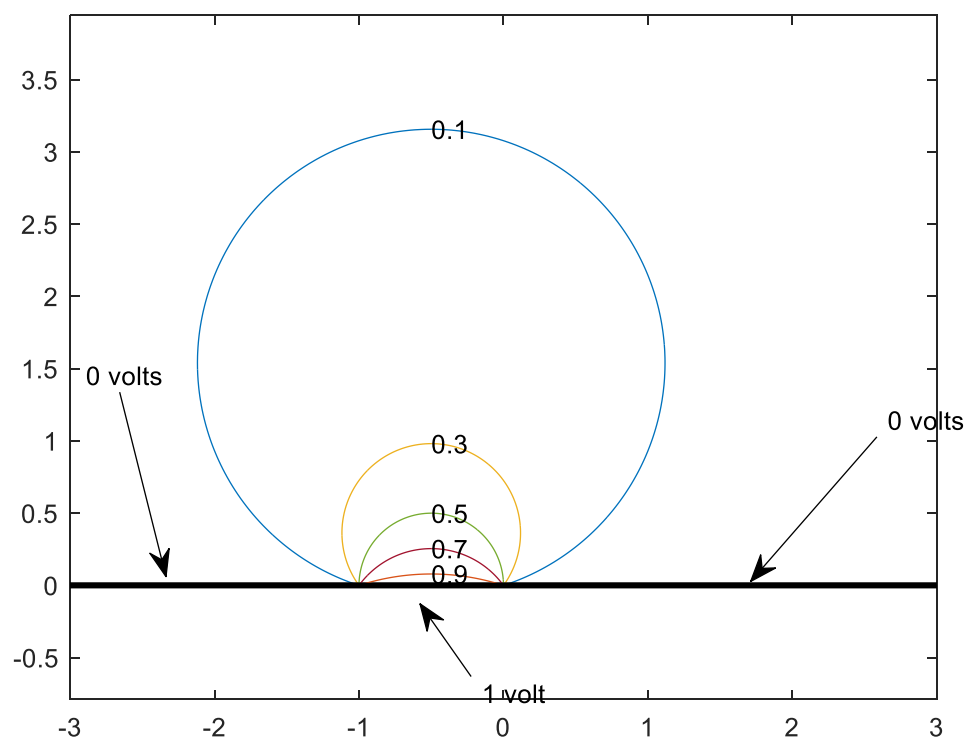
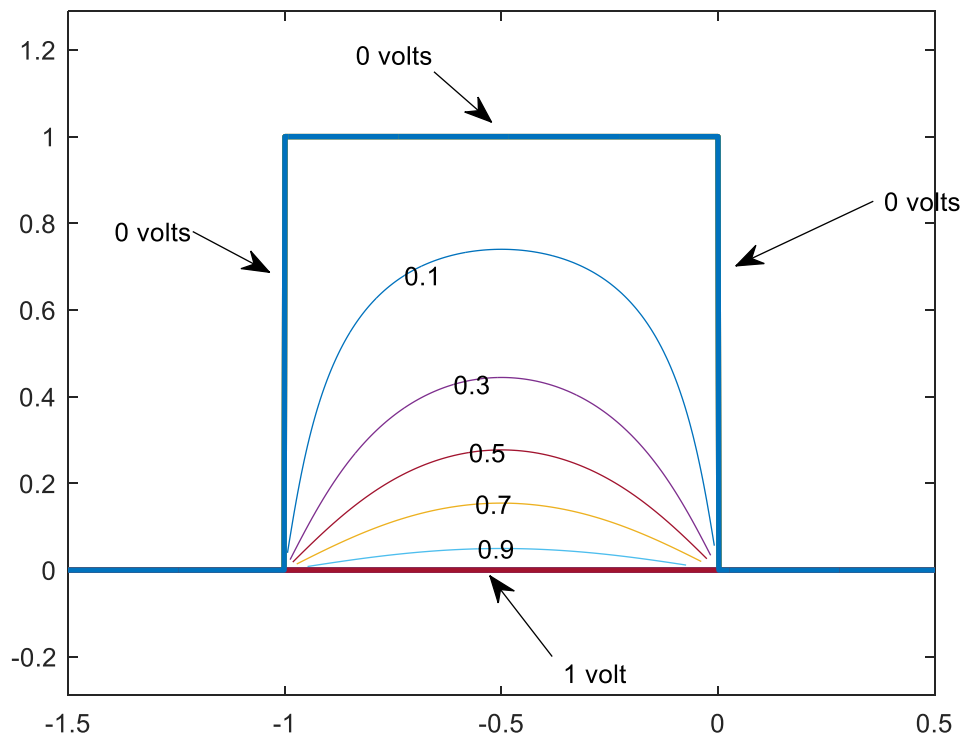Figure 11.21 Equipotentials in the w Plane

Figure 11.22 Equipotentials in the Tunnel

```
k=[1 3 5 7 9 ];
clf
phi=k/10;%these are the voltages to be considered
clf
 A= -0.381379881750885;
 warning off
 for j=1:length(k)
   center=-1/2+i/2*cot(pi*phi(j));
   Phi=phi(j);
   pphi=num2str(Phi)
   theta=linspace(pi/2,2*pi+pi/2,1000);
   r=1./(2*sin(pi*Phi));
   w=center +r.*exp(i*theta);
   u=imag (w)>0;
   wu=w.*u;
    volts=num2str(Phi);
    wr=real(w(1)); wi=imag(w(1));
  figure (1 );%potentials in w plane
plot(wu);ylim([0 4]);axis equal;hold on
text(wr,wi,volts)
U=linspace(-3,3,10);
V=0*U;
```

```
plot(U,V,'k','linewidth',2)
xlim([-2 2]);
axis equal

 for m=1:length(wu)

  f=@(W)W.^(-1/2).*(W+1).^(-1/2).*(W-1).^(-1/2);
  A= -0.381379881750885;
 z(m)=A*integral(f,0,wu(m));

 xlim([-3 3])

 end
figure (2);%potentials in the z plane
 plot(z); hold on
 zr=real(z(1)); zi=imag(z(1));
 text(zr,zi,volts);
 x=linspace(-1.5,.5,10);
y=0*x;
plot(x,y,'linewidth',2);
x=linspace(-1.5,.5,1000);
y=heaviside(x+1)-heaviside(x);
axis equal
plot(x,y,'linewidth',2)

 end
```

Example 14

Mapping of More Than Three Points.
We observed that the Schwarz-Christoffel transformation permits specifying the mapping
between at most three points lying on the vertices of the polygon in the *z* plane into three
specified points in the *w* plane. We also observed that infinity, if specified, counts as one of
these points.
Let us investigate a problem where we attempt to map 4 points from the z plane into the w plane
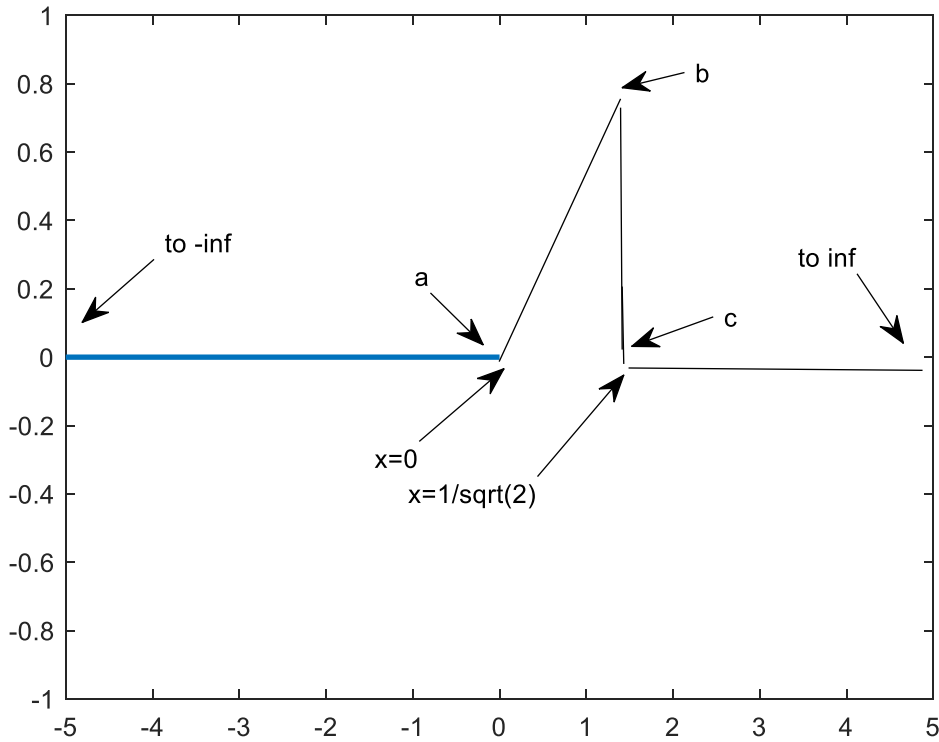
Boundary in the z Plane
Fig. 11.21

For the configuration shown above, the acute angle at the point b is $\pi/4$ and the intersection at point c is a right angle. The point b lies at $e^{i\pi/4}$ and the point c at $z = 1/\sqrt{2}$.

We want to find $z(w)$ such that z=0 has image w=0, $z = 1/\sqrt{2}$ has image w=1. We want $z = \infty$ to be mapped into $w = \infty$. The domain above the broken line is to be mapped into the space the domain $\text{Im}(w) > 0$ and the broken line is to be mapped into $\text{Im}(w) = 0$. Thus the image of $b = z = e^{i\pi/4}$ must be at a real point $w = u_2$ where $0 < u_2 < 1$. Verify your transformation by finding the image of $\text{Im}(w) = 0$, $-1 < \text{Re}(w) < 1.5$ using Eq.11.1. and compare your result with the above figure.

Solution:

Following Eq.(11.1), our transformation assumes the form

$$z(w) = A \int_0^w (w)^{((3\pi/4\pi)-1)} (w-u_2)^{((7\pi/4\pi)-1)} (w-1)^{((\pi/2\pi)-1)} dw =$$

$$A \int_0^w (w)^{(-1/4)} (w-u_2)^{(3/4)} (w-1)^{(-1/2)} dw \qquad (11.30)$$

We have taken the lower limit of the integral as 0 and the quantity B in Eq.(11.1) as zero in order to have $z = 0$ mapped into $w = 0$. Notice that the *integrand* grows as w^0 while $|w| \to \infty$ which means that the value of the integral grows as $|w|$ in the same limit, thus guaranteeing that

$z = \infty$ is mapped into $w = \infty$. We have thus specified the mapping between *three* points in the w plane into three points in the z plane, if we count infinity as a point. We *cannot specify the value of* u_2 as well. This is the image of z= $e^{i\pi/4}$. We will have to determine u_2 in such a way that the point b at z= $e^{i\pi/4}$. has an image on the real w axis located at u_2 where $0 < u_2 < 1$.

We will begin by trying a set of values for u_2, ranging from .2 to .9. We will use them to find A, for each value, by requiring the preceding integral (Eq. 11.30) when performed from 0 to 1, with each A, yield $z = 1/\sqrt{2}$. Having found A, we then perform the integration in Eq.(11.30) for each value of u_2 taking u_2 as the upper limit of integration, and select the one that yields the value for z that is closest to $e^{i\pi/4}$ or, in other words, the value for z such that the ratio $z/e^{i\pi/4}$ is closest to 1. Having found the correct value for u_2 we check our work by mapping the real axis in the w plane into the z plane and confirm that we have obtained the broken line shown in the above figure.

```
%u2=u
%take 0<u<1
u=linspace(.2,.9,100);
for k=1:length(u)



    f = @(w) ((w).^(-1/4).*(w-u(k)).^(3/4).*(w-1).^(-1/2));
        Q1 = integral(f,0,1);
        A=sqrt(1/2)/Q1;

        W =u(k);
        z=A* integral(f,0,W);
% in the following we take the value of the integral
% and divide by exp(i*pi/4) and try to find where the ratio
% is one.
        plot(u(k),real(z/(exp(i*pi/4))),'*'); hold on
        plot(u(k),imag(z/(exp(i*pi/4))),'o'); hold on
end
xlabel('values of u')
grid
```

The output is shown below. We see that if we take u2=.666...=2/3 then the point $w = u_2$ is mapped into $z = e^{i\pi/4}$.
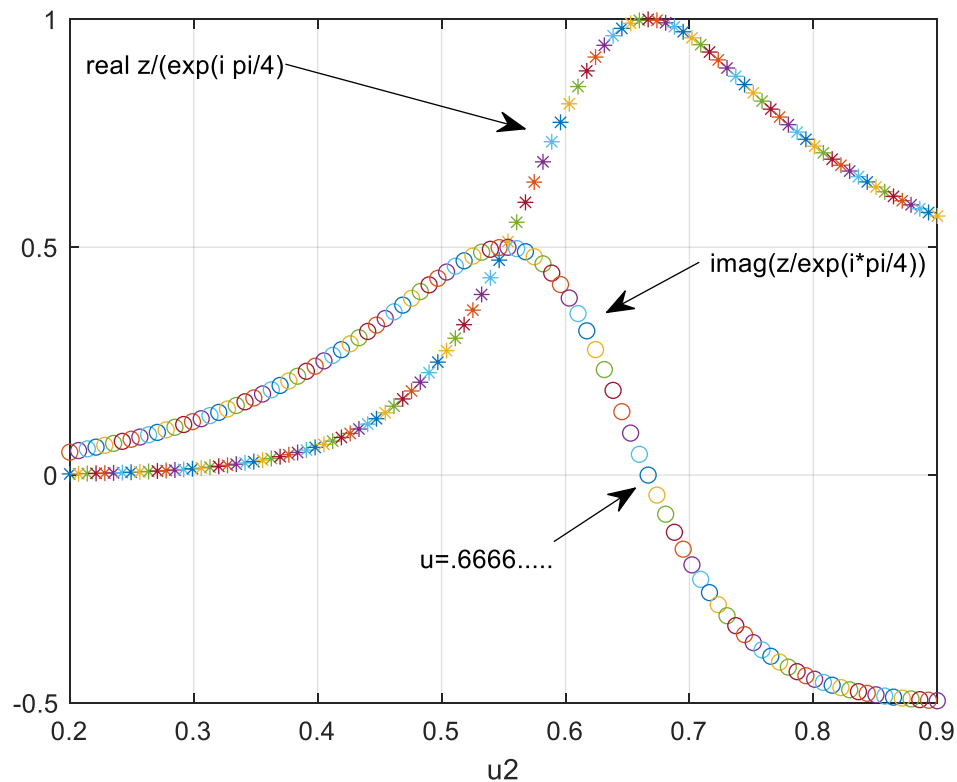
Figure11.22

We now verify this choice of u_2 by mapping the u axis from -1 to 1.5 using 5000 date points into the z plane using Eq. (11.1) .

We hope to get the broken line shown in Figure 11.21.
u=2/3;
f = @(w) ((w).^(-1/4).*(w-u).^(3/4).*(w-1).^(-1/2));
 Q1 = integral(f,0,1);
Format long
 A=sqrt(1/2)/Q1
     grid
u2=2/3;
%the following are the values of u to be considered
LU=linspace (-1,1.5,5000);
for j=1:length (LU)
f = @(w) ((w).^(-1/4).*(w-2/3).^(3/4).*(w-1).^(-1/2));
 Q1(j) =A* integral(f,0,LU(j));

end
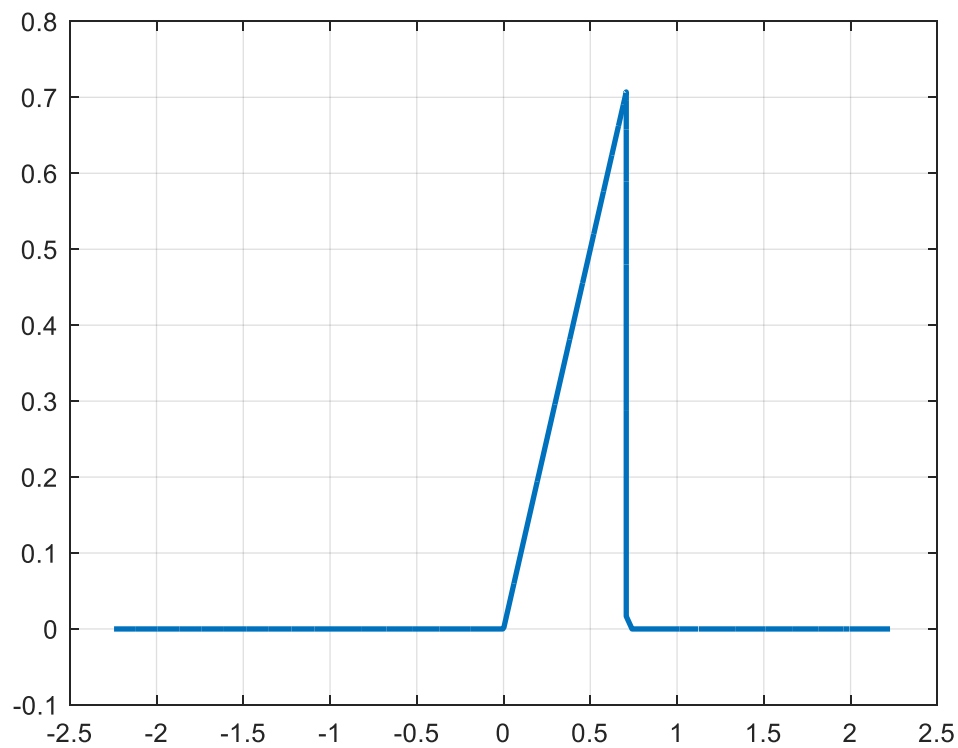plot(real(Q1),imag(Q1),'linewidth',2)
grid

The output is

Figure 11.23

And we learn A=1.902537610465034

;EXAMPLE 14

A Channel with an Abrupt Change in Width.
Fluid is flowing from left to right in the channel shown below. The width of the channel changes abruptly from 1 to 2 when the fluid reaches x=0. Our problem is to map the streamlines and to develop a formula for the velocity everywhere between the plates assuming that the flow as $x \to -\infty$ is 1 and is in the positive x direction.
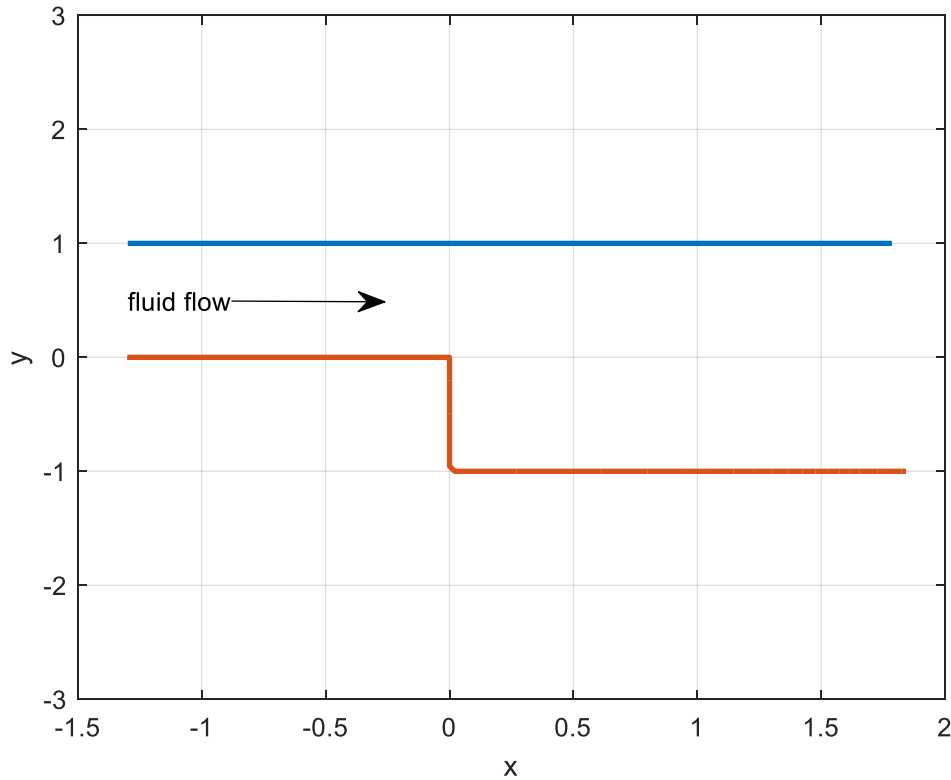
Figure 11.24

To solve this moderately difficult problem, we regard the configuration in the above figure as being the limit of the one shown below (Fig. 11.25) as the angle α tends to zero (at infinity). We proceed along the upper line in that figure in the direction of the arrow. The intersection of the two lines (at angle α) which lies at $z = -\infty$ on the real axis will have image $w = 0$. The point $z = 0$ will have image $w = 1$. The point $z = -i$ will have image $w = a$ where a is a positive real greater than 1. Notice that in the limit (for $\alpha = 0$) that the image of the upper boundary in Figure 11.24 is the negative real w axis while the image of the lower boundary in the same figure is the positive real w axis .

Thus Eq. (11.1) assumes this form

$$z(w) = A\int_{1}^{w} (w)^{(-1)}(w-1)^{(1/2)}(w-a)^{(-1/2)}\,dw \qquad (11.31)$$

where B has been absorbed into the lower limit of integration and $w = 1$ is automatically the image of $z = 0$. The angle α_1 which is α is zero , $\alpha_2 = 3\pi/2$ (which is the obtuse angle at z=0) and $\alpha_3 = \pi/2$, which is the right angle at $z = -i$.

Now suppose that we require that $z(a) = -i$ in Eq.(11.31). We have

$$-i = A\int_{1}^{a} (w)^{(-1)}(w-1)^{(1/2)}(w-a)^{(-1/2)}\,dw \cdot$$

The integrand is imaginary over the interval of integration since $a > 1$. Since the left side of this equation is imaginary this means that A is real.
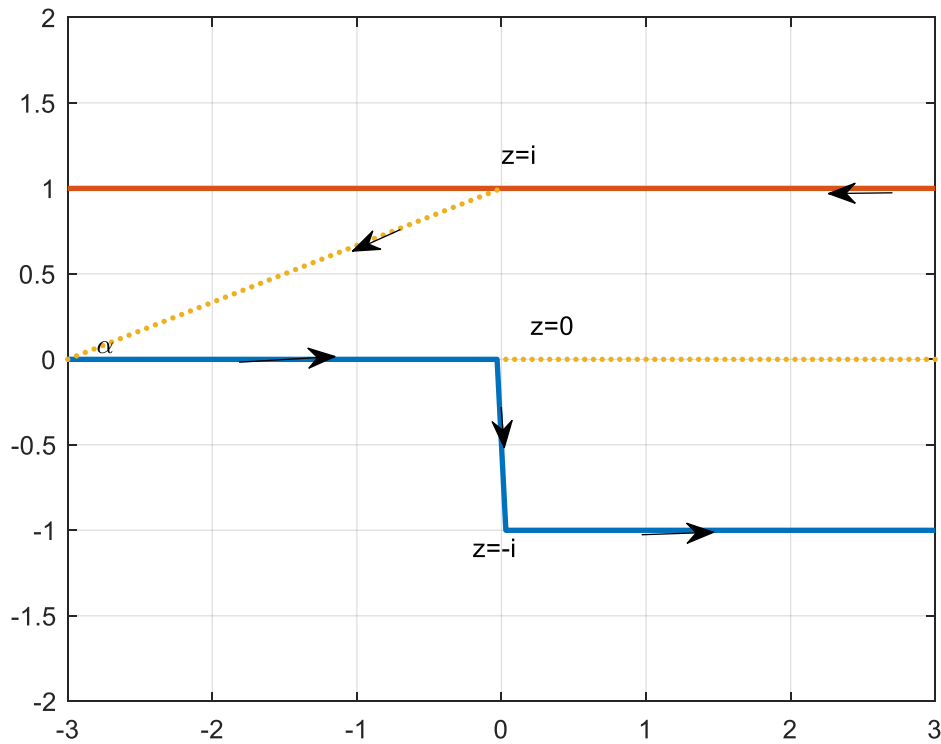
Figure 11.25

Now consider Eq.(11.31) and assume that the upper limit w is a negative real number. Observe that if the path of integration is taken along the real axis from 1 to this negative real value that the integrand is imaginary (recall that $a > 1$). However, the path of integration would take us through the pole of the integrand at $w = 0$. We must not evaluate this integral as a Cauchy principal value. Since the function $z(w)$ that we obtain must be analytic in the upper half of the *w* plane we must evaluate the integral by employing a semicircle in the upper half of the *w plane that avoids the pole*. The path is shown below. We use a semi-circle of radius *d* in the contour to go around the pole. Note that in Example 8 we also faced the problem of having a pole at $w = 0$. However, we were able to find a closed form expression for the value of the integral; it contained $\text{Log } w$ and by using the principal branch of this function we were able to avoid a path of integration through the origin.
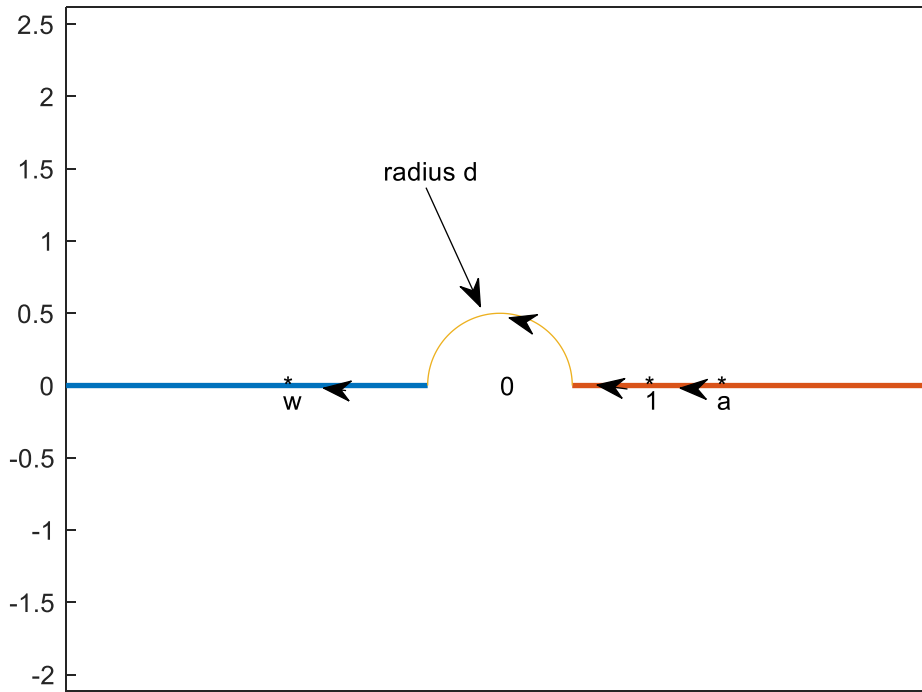
Figure 11.26
Path of Integration for the Integral in Eq. 11.31 for $w < 0$

The integral around the semi-circle in the above can be evaluated using residue calculus in the limit as $d \to 0+$. Its value is $A2\pi i \frac{1}{2} \operatorname{Res} \frac{1}{w}(w-1)^{(1/2)}(w-a)^{(-1/2)} @ w=0$. The use of ½ of the residue is explained in W p. 389. It arises because we are going ½ way around the pole (in the positive sense) on a shrinking path.

The value of the preceding quantity is $A\pi i / \sqrt{a}$ In our actual calculation along the *straight portion* of the path shown above we choose for d a small positive number. It was found that $d=$ .0001 was satisfactory. Thus, for the integral in Eq.(11.31) for $w < 0$ we have

$$z(w) = A\left[\int_{1}^{d}(w)^{(-1)}(w-1)^{(1/2)}(w-a)^{(-1/2)}dw + \int_{-d}^{w}(w)^{(-1)}(w-1)^{(1/2)}(w-a)^{(-1/2)}dw + \pi i / \sqrt{a} \right]$$

Notice since $a > 1$ the integrands are real in the above two integrals. Recall that the negative real w axis has an image in the z plane whose imaginary part is 1. The only imaginary quantity within the above brackets is $\pi i / \sqrt{a}$. Since A is real, we see that $A\pi / \sqrt{a} = 1$ or $a = A^2\pi^2$.

We know that with the Schwarz-Christoffel transformation we cannot freely assign mappings between assigned points in the w and z planes if these correspondences exceed three. We are requiring here that w=0 has two possible images: one with an imaginary part of 1 and the other with an imaginary part of zero depending on whether we use $w = 0+$ or $w = 0-$. Also, w=1 has image z=0. Thus, we are not free to assign in advance the image of $z = -1$ and we have assigned it the symbolic value a and we know $a = A^2\pi^2$. Thus, if we know A we know a. Eq.(11.31) becomes

$$z(w) = A \int_1^w (w)^{(-1)} (w-1)^{(1/2)} (w - \pi^2 A^2)^{(-1/2)} dw. \qquad (11.32)$$

If we choose the upper limit as $a = \pi^2 A^2$ the integral must yield the value *-i* which is the image of $w = a$. Notice that with this limit the integrand is imaginary over the interval of integration because $a = \pi^2 A^2 > 1$. Thus, we could choose a range of values for A in the preceding equation, taking the upper limit of the integral as $a = \pi^2 A^2 > 1$, and plot these values against A to determine the value of A that yields $-i$. Since $A = \dfrac{a}{\sqrt{\pi}}$ and $a > 1$ we know that

$A > 1/\sqrt{\pi} \approx .5642$. We evaluate the right side of Eq.(11.32) taking the upper limit as $\pi^2 A^2$ and plot the imaginary plot of result as a function of A. If we begin the plot at *A*=.6 and extend it to *A*=2 we find that the right side of Eq.(11.32) is about equal to *-i* for A lying between . 6 and .7. Thus, we make a new plot using 1000 data points between these two numbers. The result with the code is shown below. It indicates that the imaginary part of the right side of Eq.(11.32) has the value -1 for $A \approx .636$ which is a close approximation of $2/\pi$. We shall use the latter for our value of *A*. Since $a = \pi^2 A^2$, this means that for our particular geometry $a = 4$. Thus the image of $z = -i$ in the *w* plane is *w*=4.

```
clear;clf
 A0=linspace(.6,.7,1000);
 for j=1:length(A0)
     A=A0(j);
f=@(w)((w-1).^(1/2)).*(w.^(-1)).*((w-pi^2*A^2).^(-1/2));
z(j)=A*integral(@(w)f(w),1,pi^2*A^2);
end

 plot(A0,imag(z));grid
```
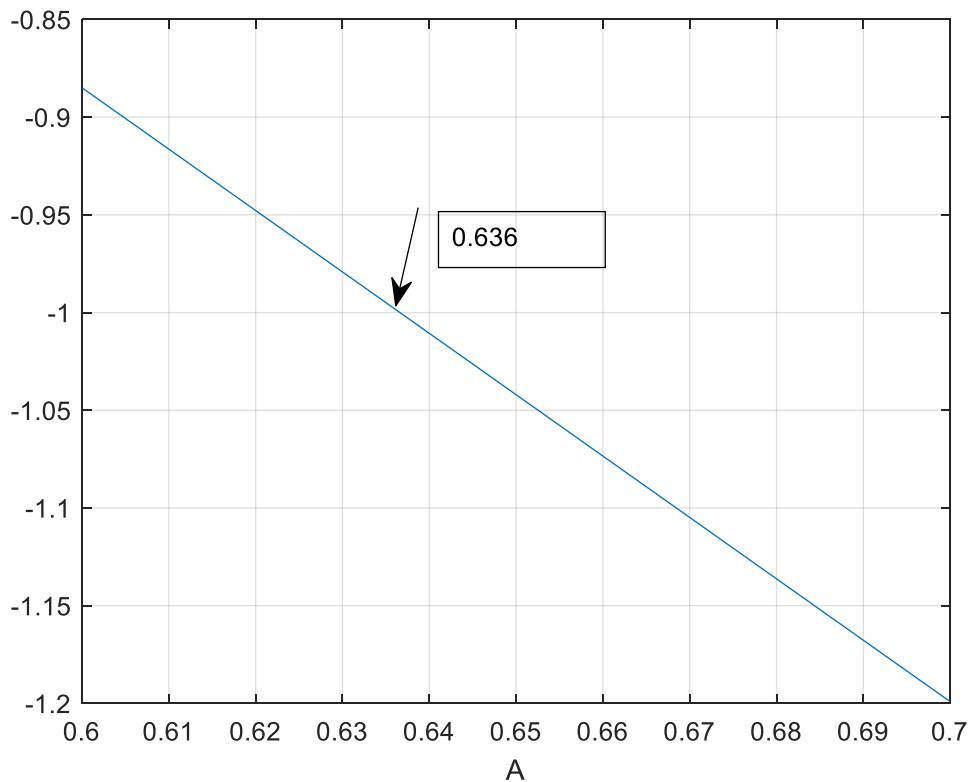
Fig 11.27

Having found A, we can check our work by mapping the images of the boundaries (as they appear in the w plane) into the z plane and see if we obtain the channel shown in Figure 11.24. The code and the results follow:

```
clf;clear
  A=2/pi;
 WW=linspace(-.0001,-20,100);
a=(pi*A)^2;
a=4;
d=.0001;
for j=1:length(WW)
   W=WW(j);
f=@(w)((w-1).^(1/2)).*(w.^(-1)).*((w-a).^(-1/2));
residue=i;
z(j)=A*integral(@(w)f(w),1,d)+A*integral(@(w)f(w),-d,W);
z(j)=z(j)+residue;
end
figure(1)
plot(real(z),imag(z),'linewidth',2); ylim([-3 3]);hold on

WW=linspace(.0001,20,3000);
a=(pi*A)^2;
%note in the following we do not encounter the pole %because WW is positive
for j=1:length(WW)
   W=WW(j);
```

```
f=@(w)((w-1).^(1/2)).*(w.^(-1)).*((w-a).^(-1/2));

z(j)=A*integral(@(w)f(w),1,W);
end

plot(real(z),imag(z),'linewidth',2); grid
```

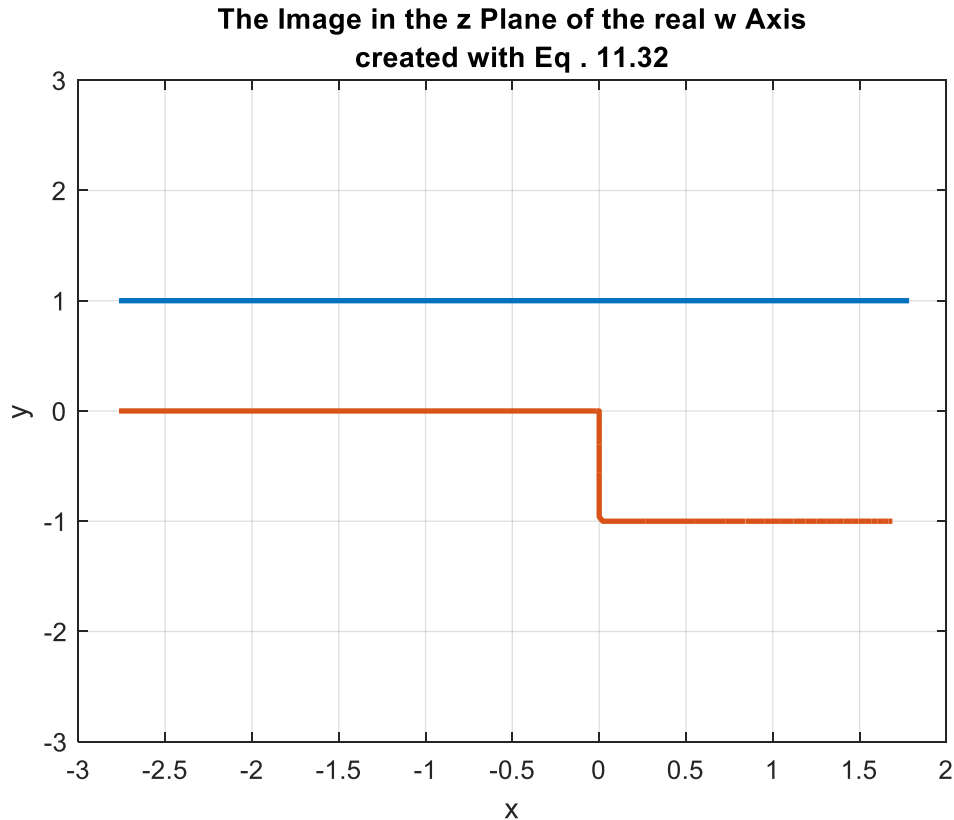**The Image in the z Plane of the real w Axis**
created with Eq . 11.32

Figure 11.28

Let us assume that fluid flows from left to right between the boundaries in the above. Thus, the upper and lower boundaries must each correspond to constant values of the stream function $\psi(x,y)$ or $\psi(z)$. Let us say that on the lower boundary we have $\psi = 0$ while on the upper we have $\psi = S$ where S is real. Now in the w plane we know that $\psi(w)$ or $\psi(u,v)$ satisfies Laplace's equation.

The image of the upper boundary in the $w = u + iv$ w plane is the line $v = 0, -\infty < u < 0$. Here we assign the value $\psi(u,0) = S$. The image of the lower boundary is the positive u axis and here we assign the value $\psi(u,0) = 0$. We can find $\psi(u,v)$ in the upper half plane from the Poisson

Integral formula as follows $\psi(u,v) = \dfrac{v}{\pi} \displaystyle\int_{-\infty}^{0} \dfrac{S\,du'}{(u'-u)^2 + v^2} = \dfrac{S}{\pi}\arctan(v/u) = \dfrac{S}{\pi}\arg(w)$. This is the

imaginary part of the function

$$\Phi(w) = \frac{S}{\pi}\operatorname{Log} w = \frac{S}{\pi}\Big[\operatorname{Log}|w| + i\arg(w)\Big] = \phi(w) + i\psi(w) \qquad\qquad (11.32)$$

where the principal branch of the log is used. Thus, the streamlines in the w plane are those

lines satisfying $\psi = \dfrac{S}{\pi}\arg(w) = \dfrac{S}{\pi}\arctan(v/u)$ is constant. We will take $S=1$. If v/u is a constant

say m then the streamlines in the w plane satisfy $v = mu$ and the value of ψ on any streamline is

given by $\psi = \dfrac{1}{\pi}\arctan(m)$. Here m is real. Since the arctangent lies between 0 and π in the

upper half of the w plane we see that $0 \le \psi \le 1$. The streamlines are of course rays emanating

from the origin in the w plane and extending upward. Suppose we take $\psi = (0.1, 0.2, 0.5, 1, 2, 5)$.

Multiplying each of the values by π and taking their tangent we get the corresponding values of

m. We plot them in the z plane by using the transformation in Eq.(11.32) with $A = \dfrac{2}{\pi}$. The code

and the results follow :

```
clear;clf
 A=2/pi;

 psi=[.2 .3 .45];
 m=tan(pi*psi);

    u=linspace (.000001,10,1000);
    v=m'*u;
    W=u+i*v;

    for j=1:length(m)
       for k=1:length(u)
          Upper=W(j,k);
f=@(w)((w-1).^(1/2)).*(w.^(-1)).*((w-pi^2*A^2).^(-1/2));
z(j,k)=A*integral(@(w)f(w),1,Upper);
       end
       plot(z(j,:),'k');hold on
       a=real(z(j,length(u)));b=imag(z(j,length(u)));

       text(a,b,num2str(psi(j)),'FontSize', 8)
       clear z
       end
    psi=[.51 .7 .8 .9];
 m=tan(pi*psi);

    u=linspace (-.00001,-10,1000);
    v=m'*u;
    W=u+i*v;

    for j=1:length(m)
       for k=1:length(u)
          Upper=W(j,k);
f=@(w)((w-1).^(1/2)).*(w.^(-1)).*((w-pi^2*A^2).^(-1/2));
z(j,k)=A*integral(@(w)f(w),1,Upper);
```

```
            end
        plot(z(j,:),'k');hold on
        a=real(z(j,length(u)));b=imag(z(j,length(u)));

        text(a,b,num2str(psi(j)),'FontSize', 8)
        clear z
    end
grid
u=linspace(-3,2,100);
v=u./u;
plot(u,v,'linewidth',2)

v=-heaviside(u);
plot(u,v,'linewidth',2)
```
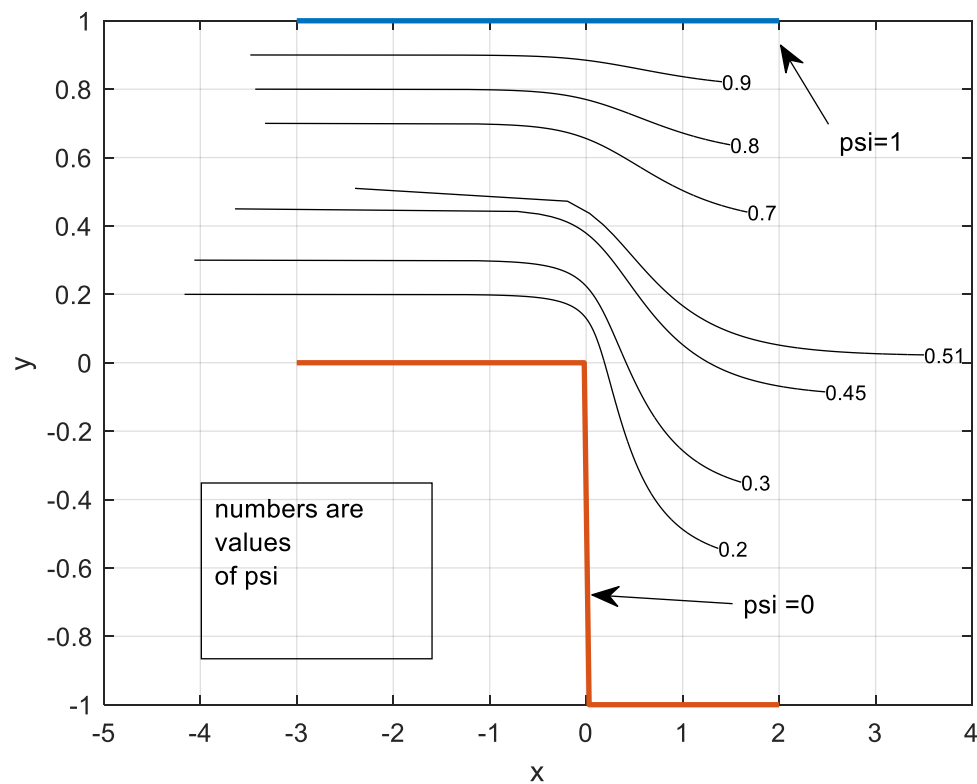

Figure 11.29

From Eq.(11.32) we have that the (real) potential function is $\phi(w) = \dfrac{1}{\pi}\Big[\text{Log}|w|\Big]$ where we take

S=1. If ϕ_1 is a certain value of the potential , we see that it must lie on the semicircle in the upper w plane given by $|w| = e^{\pi \phi_1}$ where ϕ_1 is any real number including zero. The following plot shows the equipotentials between the boundaries of the fluid as the real potential varies in 40 steps from -1 to 1.
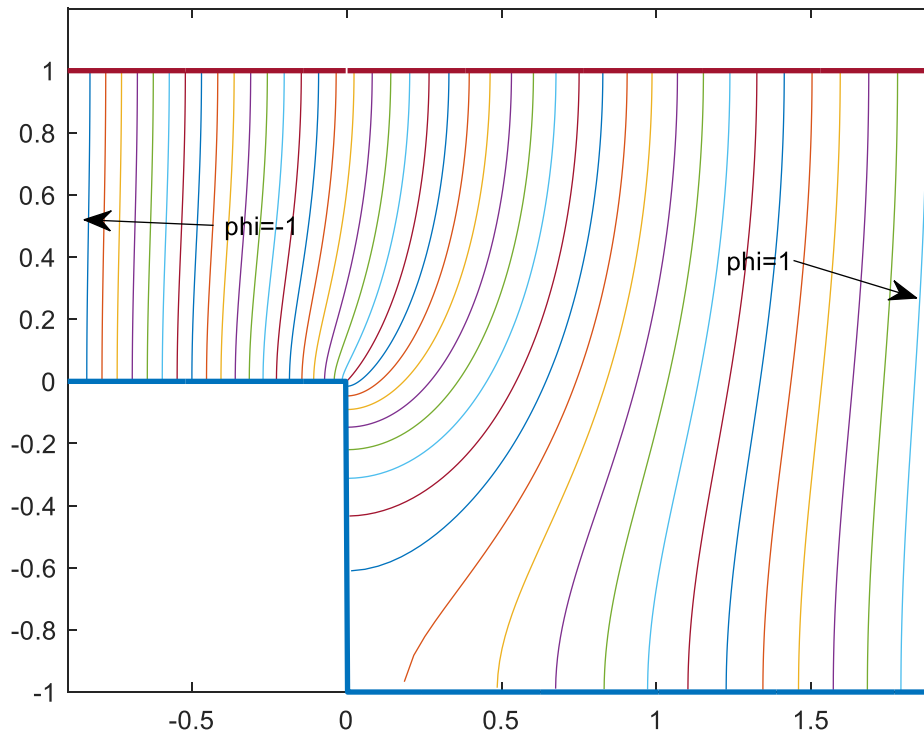

Figure 11.30

If we regard this as a problem in electrostatics where the upper boundary is maintained at some positive voltage and the lower is at zero this above curves show the shapes of the electric field lines.

The code for this figure follows

```
clear;clf
 A=2/pi;
 phi=-20:20;
 phi=phi/20;
  R=exp(pi*phi);
 theta=linspace(.01,pi-.01,100);
 %for j=10
 for j=1:length(phi);
  W=R(j)*exp(i*theta);
 for k=1:length(theta)
        Upper=W(k);
f=@(w)((w-1).^(1/2)).*(w.^(-1)).*((w-pi^2*A^2).^(-1/2));
z(k)=A*integral(@(w)f(w),1,Upper);
V(j,k)=z(k);
     end
```

```
     z;
      figure(1)
      plot(real(z),imag(z));hold on
 end
 x=linspace(-1,2,1000);
 y=x./(x);
 figure(1)
 plot(x,y,'linewidth',2);ylim([-1 1.2])
 yy=heaviside(-x)-1;
 plot(x,yy,'linewidth',2);xlim([-.9 1.9])
```

Bilinear Transformations:

The bilinear (or Mobius) transformation is of the form

$$Z = \frac{az+b}{cz+d} \quad \text{where } ad \neq bc \quad \text{Z=X+iY} \qquad (11.33)$$

The transformation is discussed in W (p.537) and in S (p.245) . Its most interesting property is that it will map a circle in the *z* plane into a either a circle in the *Z* plane or into a straight line and it will map an infinite line in the *z* plane into either a circle or a straight line in the Z plane. Note that a *portion* of a circle or a straight line is mapped into either a portion of a circle or a straight line.   It is useful to transform some of the configurations in the z plane in the previous problems into the Z plane. These can be of useful shapes . The solutions to the configurations in the Z plane can then be found in the already solved configurations in the *w* plane .

Example 15

Apply the transformation $Z = 1/z$ to the configuration in Example 1, and designate the voltages at the boundaries in the Z plane that are on their images in the *z* plane. Find the voltages and electric field for the configuration in the Z plane.

Solution: The transformation $Z = 1/z$ applied to the vertical line segment $x = 0, 0 \leq y \leq 1$ lead to a line along the negative imaginary axis in the Z plane, $X = 0, -1 \leq Y < -\infty$. The line segment $Y = 0 \ -\infty < X < 0$, becomes the same line in the Z plane . The semi-infinite line $Y = 1, -\infty < X < 0$ must have an image passing through $Z = -i$ and Z=0. It cannot extend to

$Z = \infty$ because the image in the z plane is never zero. The arguments (angles) of $Z = \frac{1}{z}$ must

lie between $-\pi / 2$ and 0. Thus, the image in the z plane is a semicircle in the 3$^{\text{rd}}$ quadrant of radius 1/2 with diameter 1, and center at $-i / 2$. The entire image in the Z plane of the configuration in the z plane, is shown below.
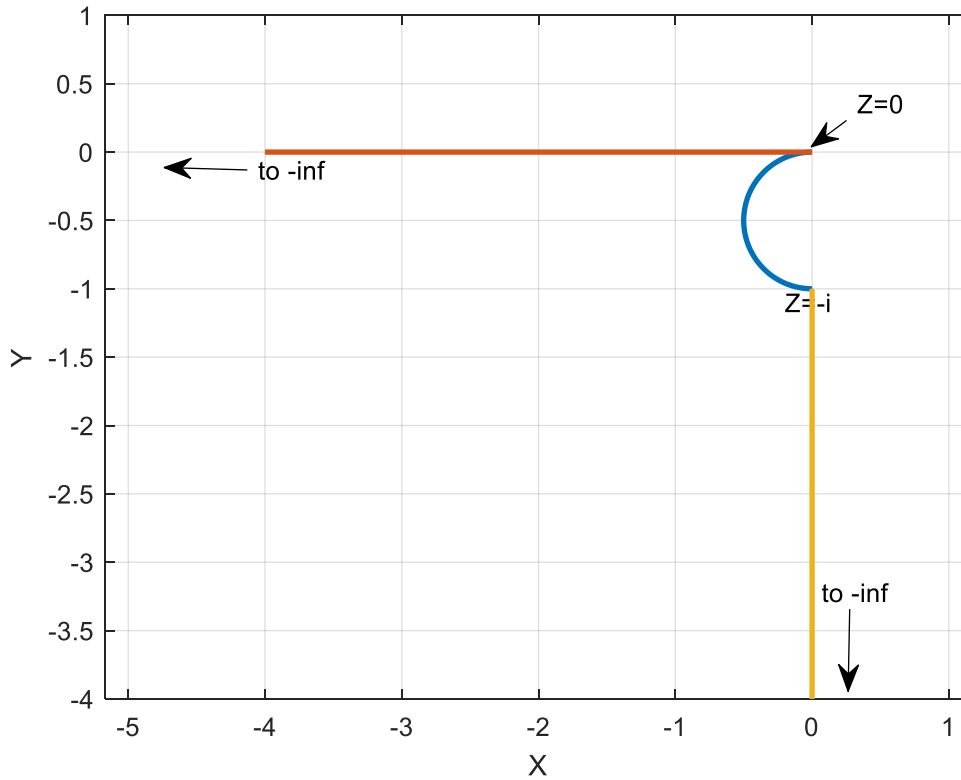
Figure 11.31

Let the above configuration represent the cross section of a metallic sheet conductor. Now suppose that the curved portion of the above boundary is maintained at 1 volt while the straight-line boundary. on the negative real axis is at zero potential. The straight- line boundary along $X = 0, Y \le -1$ is also at zero potential. If we apply our inverse transformation $z = 1/Z$ to the above configuration and transform the voltages to the corresponding boundaries we have exactly the configuration in Example 1. In that configuration we found the complex

potential to be $\Phi(z) = \dfrac{-i}{\pi}\left[\mathrm{Log}\,\dfrac{\cosh(\pi z)+1}{\cosh(\pi z)-1} \right]$. In the present problem we have

$$\Phi(Z) = \frac{-i}{\pi}\left[\mathrm{Log}\,\frac{\cosh(\pi / Z)+1}{\cosh(\pi / Z)-1} \right] . \qquad (11.34)$$

Let us plot the contour lines on which the real part of this expression has some constant values. We will use a grid in the space $X \le 0, Y \le 0$. $X^2 +(Y+1/2)^2 \ge 1/4$ which describes the space within the boundaries of the above figures.

Equipotentials

Z=0

0.05

0.05

0.1

0.1

zero volts

zero volts

Z=-i

0.3

0.3

0.5

0.5

0.7

0.9

one volt

Streamlines

0

-0.2

0.1

0

-0.1

0.7

0.5

0.3

0.2

numbers are
values
of psi
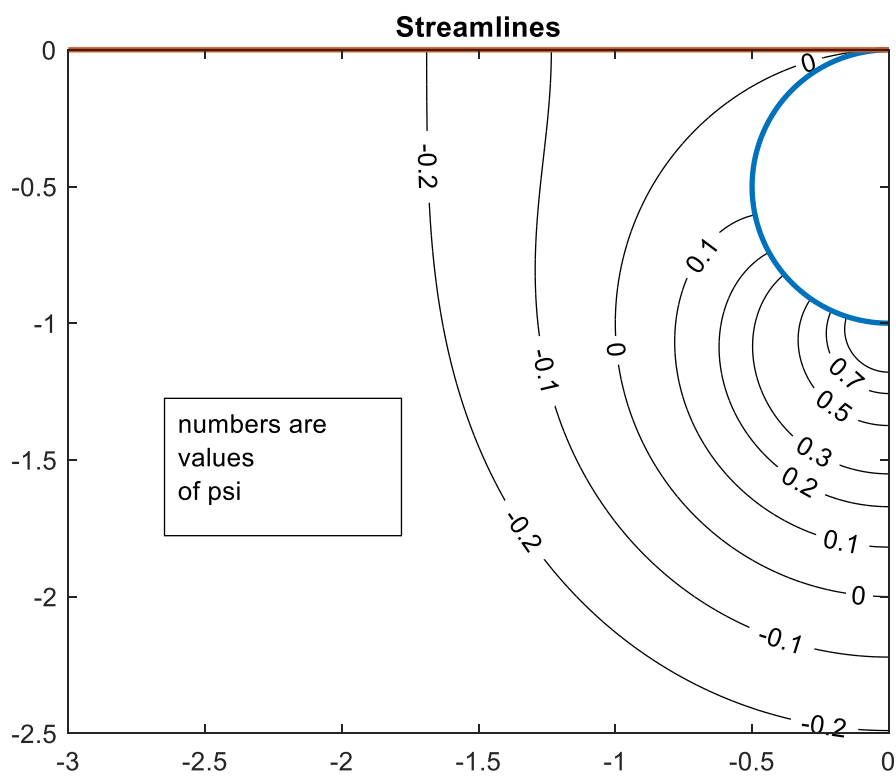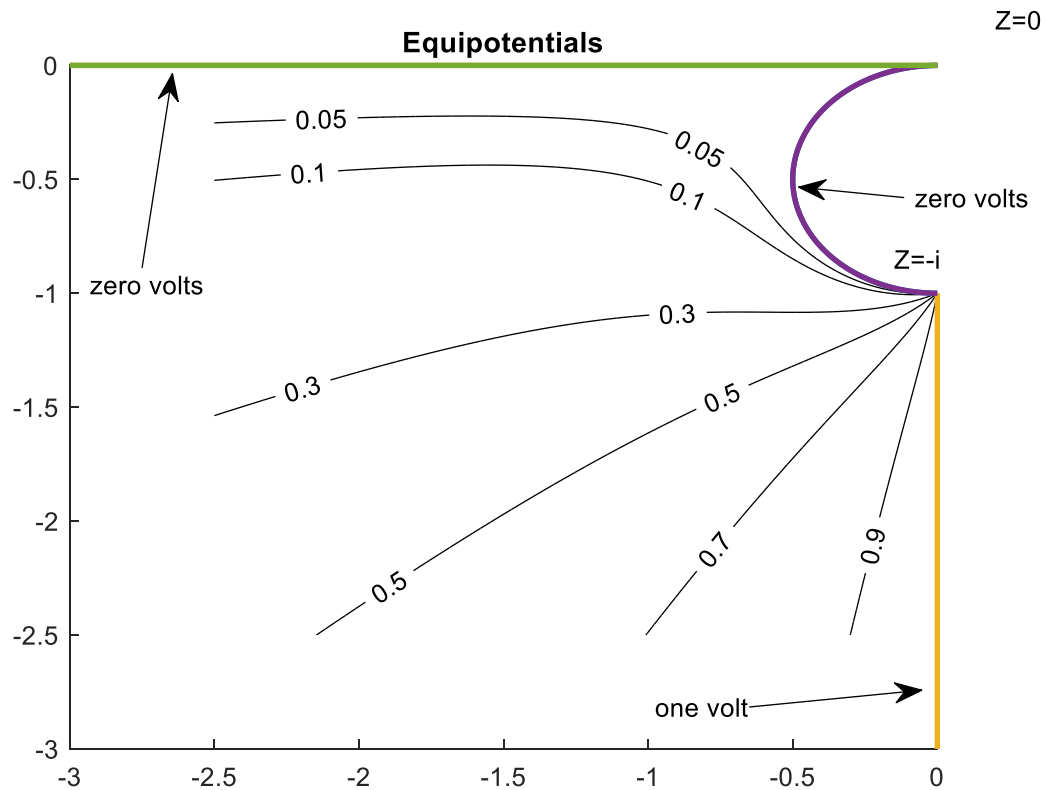
0.1

0

-0.1

-0.2

-0.2

Figure 11.32

We have used the following code

```
X=linspace(-2,-10*eps,5000);
Y=linspace (-2,0,5000);
[X Y]=meshgrid(X,Y);
Z=X+i*Y;
clf
%the following ensures that we will
%not consider points within the semicircular
%boundary
ZZ=abs(Z+i/2)>1/2;
Z=ZZ.*Z;
Z(Z==0)=nan;
z=1./Z;
Phi=-i/pi*log((cosh(pi*z)+1)./(cosh(pi*z)-1));
phi=real(Phi);
figure(1);hold on
[c,h]=contour (X,Y,phi,[ .05 .1 .3 .5 .7 .9]);
clabel(c,h);colormap([0 0 0]);
grid;ylim([-2 0])
x=linspace (-2,0,100);
y1=0*x;
theta=linspace (pi/2,3*pi/2,100);
z=(-i/2+1/2*exp(i*theta));
plot(z,'linewidth',2);hold on
plot(x,y1,'linewidth',2);%ylim([-5 2])
grid
text(.2,.2,'Z=0')
text(-.15,-.95,'Z=-i')
y3=linspace(-1,-5,100);
x3=0*y3;
plot(x3,y3,'linewidth',2)
xlim([-2 0])
plot(z,'linewidth',2);hold on
plot(x,y1,'linewidth',2);%ylim([-5 2])
title('Equipotentials')
psi=imag(Phi);
figure(2)
n=0:10;
n=n*.1;
n=[-n n];
[p,q]=contour (X,Y,psi,[. .01 .1 .2 .3 .5 .7 .9]);colormap([0 0 0])
hold on
plot(z,'linewidth',2);hold on
plot(x,y1,'linewidth',2);xlim([-2 0]); hold on
axis equal
title ('Streamlines')
clabel(p,q)
```

We can also write a program that will yield the electric field everywhere between the 2 conductors. In the Z plane the electric field is given by $E = -\overline{\left(\dfrac{d\Phi}{dZ}\right)}$. We rewrite Eq(11.34) as

$$\Phi(Z) = \frac{i}{\pi}\left[\text{Log}(\cosh(\pi/Z)-1)-\text{Log}(\cosh(\pi/Z)+1)\right]$$ and for the fun of it differentiate it using

MATLAB to obtain $\dfrac{d\Phi}{dZ}$.

The code is shown below
```
syms z pi
my_result1=-i/pi*diff(log(cosh(pi/z)+1))
my_result2=i/pi*diff(log(cosh(pi/z)-1))
result=my_result1+my_result2
simplify(result)
```

whose output is
-2i/(z^2*sinh(pi/z))

The electric field is the negative of the conjugate of that expression. Hence

$$E = \frac{2}{\pi}\overline{\left[\frac{i}{Z^2\sinh(\pi/Z)}\right]}$$

for the electric field. Let us write a program such that if you supply a complex location Z the program will yield the electric field as a complex vector. If the location is such that the point chosen is not in the space $x\le0, y\le0, \text{and} |z+i/2|>1/2$ you will be given a false result.

Here is the code
```
Z=input('Z=')
phi_Z=-2*i/(Z^2*sinh(pi/Z));
E=-conj(phi_Z)
```

Example 16

Suppose we apply the transformation

$$Z =1/(z-i) \tag{11.35}$$

to the configuration in the z plane shown in Example 8. The upper boundary in the z plane is given by $z=x+i$ $x\le0$. With this used in the above equation we have $Z=1/x$ $x\le0$ from which it becomes evident that that upper semi-infinite boundary is mapped into the negative real axis in the Z plane. The lower boundary in the z plane, the infinite line $y=0$ $-\infty<x<\infty$ maps into $Z =1/(x-i)$. The denominator in this expression is never zero which means that that as x varies all the corresponding values in the Z plane lie on a circle. If

$x = 0$ $Z = i$ which maximizes $|Z|$. Thus, the image of the real axis in the Z plane is the circle having radius ½ and center at $Z = i/2$. The arrangement is shown below
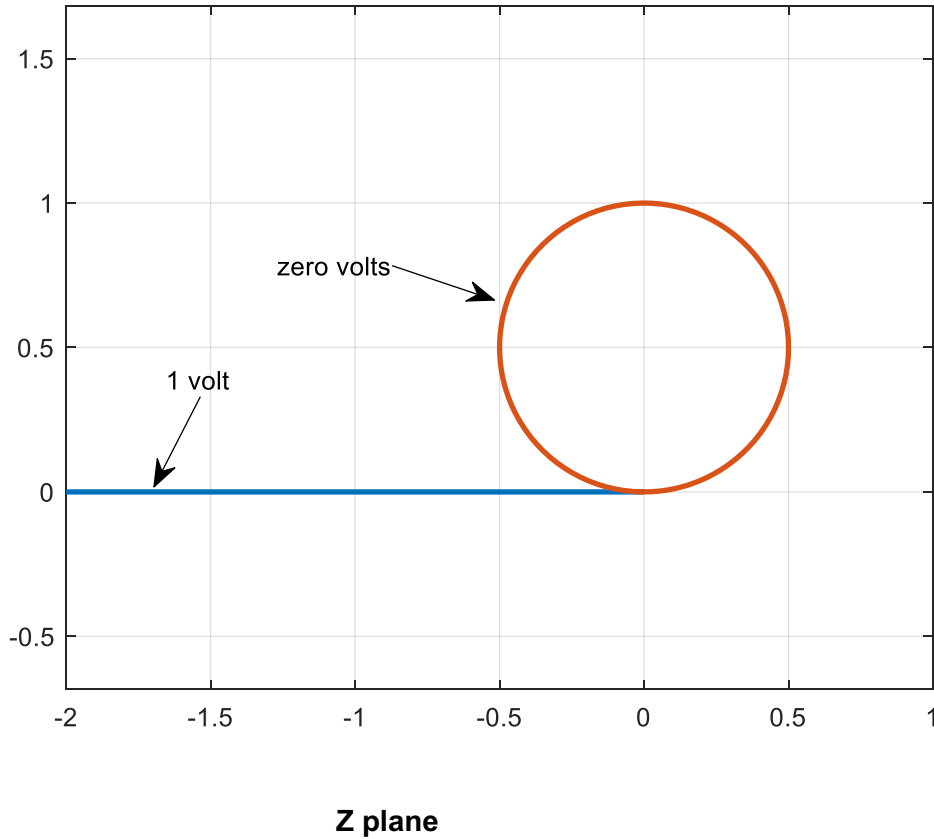

Z plane

Figure 11.33

Suppose now that the in the Z plane that the boundary on the negative real axis is maintained at 1 volt while the circular boundary described above is maintained at zero volts. We want an expression for the potential everywhere between these 2 conductors. Let us turn our attention to the w plane. Note that Eq.(11.21)

$$z(w) = \frac{1}{\pi}(w + \text{Log } w + 1)$$ maps the real axis in the w plane into the configuration of

Figure 11.9 a in the z plane. We apply the additional transformation $Z = 1/(z-i)$ to that transformation and we have

$$Z = \frac{\pi}{(w + \text{Log } w + 1) - i\pi} \tag{11.36}$$

The boundary conditions in the w plane are given in Fig. (11.10). We have that

in the *w* plane the complex potential was found above to be

$$\Phi(u,v) = \phi(u,v) + i\psi(u,v) = \frac{-i}{\pi}\operatorname{Log} w = \frac{1}{\pi}\arctan(v/u) + \frac{-i}{\pi}\operatorname{Log}\sqrt{u^2+v^2}$$

We can map the equipotentials into the Z plane from the w plane using
the above, where the surfaces of constant voltage are of the form $v = u\tan(\pi V_0)\ 0 \le V_0 \le 1$
as expressed in Eq.(11.23). The following code is used

```
clf
clear
%voltages of interest below for u>0, Re(w)>0
Vo=[.25 .45];
beta=(tan(Vo*pi));
beta=beta.';
for j=1:length(Vo)
%the following allows for points near abs(w1)=0
%note that z1 and z2 use the same voltages
u1=linspace(0,2,10^6);
v1=(u1.*beta(j));
w1=u1+i*v1;
z1=pi./(w1+log(w1)+1-i*pi);
u2=linspace(2,100,500);
v2=(u2*beta(j));
w2=u2+i*v2;
z2=pi./(w2+log(w2)+1-i*pi);
plot(z1,'k');hold on;axis equal
plot(z2,'r');hold on
end
% the following works for u=Re(w)<0
% voltages of interest are below.
Vo=[.6 .7];
beta=(tan(Vo*pi));
beta=beta.';
for j=1:length(Vo)
u3=linspace(-10,-1,1000);
v3=(u3.*beta(j));
w3=u3+i*v3;
z3=pi./(w3+log(w3)+1-i*pi);
%the following allows for points near abs(w4)=0
u4=linspace(-1,0,1000);
v4=(u4.*beta(j));
w4=u4+i*v4;
z4=pi./(w4+log(w4)+1-i*pi);
[u4' w4']
plot(z3,'r');hold on
plot(z4,'k');hold on
end
figure(1);
%the following plots the boundary conductors
x=linspace(-5,-.01);
y=0*x;
theta=linspace(0,2*pi,100);
z=i/2+1/2*exp(i*theta);
```

```
plot(z,'linewidth',3)
x=linspace(-4,0,100);
y=0*x;
plot(x,y,'linewidth',2)

axis equal
grid
```
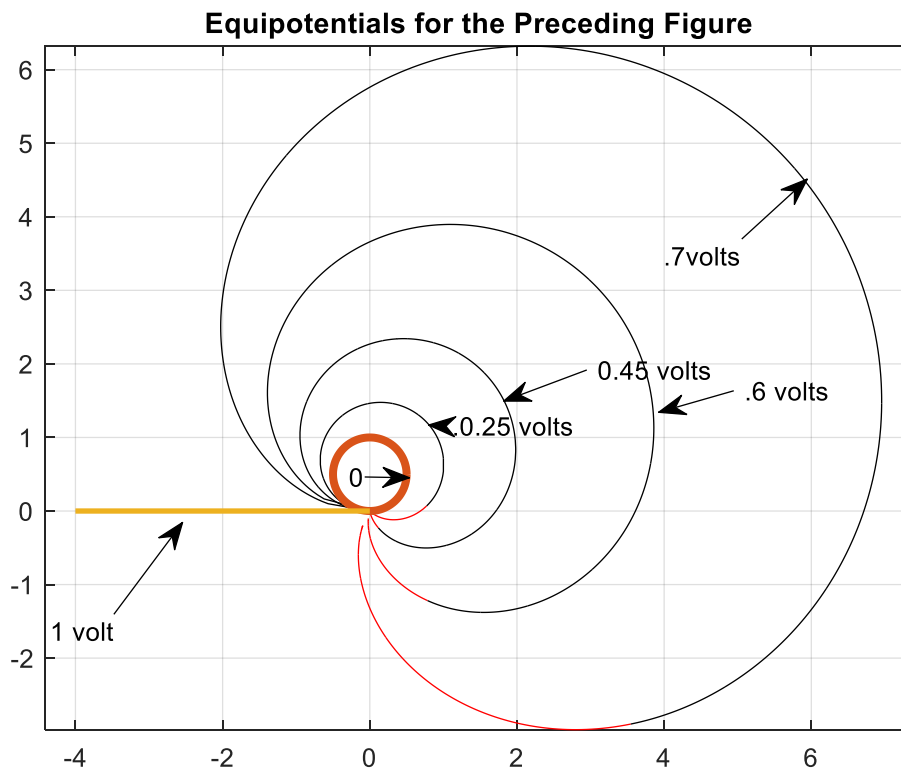
Equipotentials for the Preceding Figure

Figure 11.34

Now we wish to find the streamlines for the above configuration. In Example 8 we found that in the w plane the streamlines were the lines on which $\psi(u,v) = \dfrac{-1}{\pi} \mathrm{Log} \sqrt{u^2 + v^2}$

is constant. These are circles satisfying $u^2 + v^2 = e^{-2\pi\psi}$. We proceed as in Example 8 but add the additional transformation z1=1./(z1-i)and find:
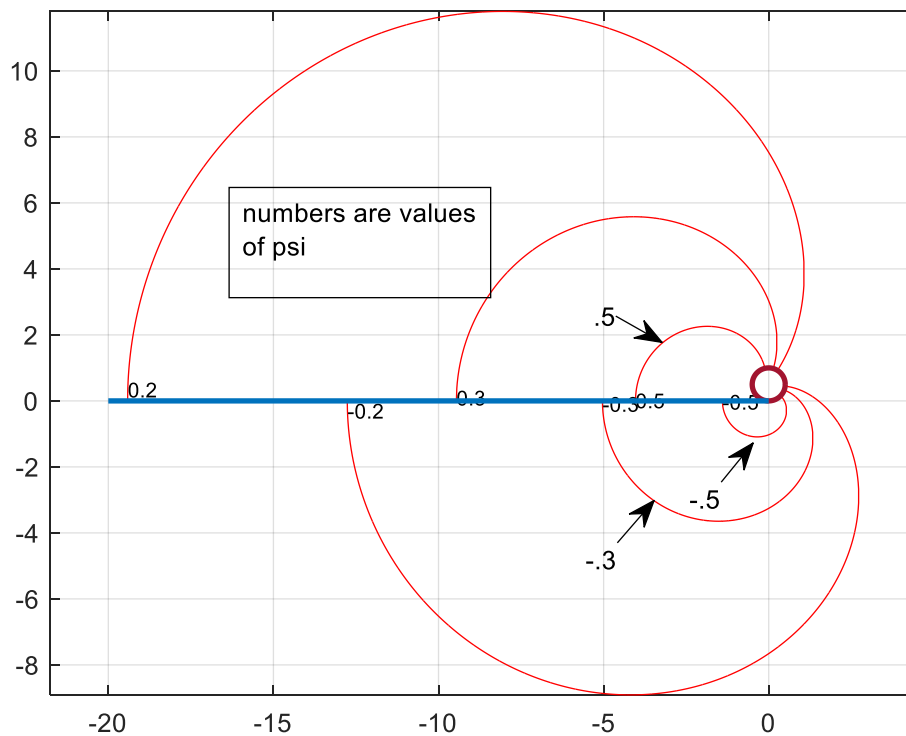
Figure 11.35

Which was generated from the following code:

```
clf
clear
%gam=linspace(-10,10,60);
gam=[ 2 3 5  -2 -3 -5 ];
gam=gam/10;
% note that gam are the values taken for psi
beta=exp(-2*pi*gam);
beta=beta';

 for j=1:length(beta)
    psi=gam(j);
    psi=num2str(psi)

u1=linspace(-sqrt(beta(j)),sqrt(beta(j)),100000);
v1=sqrt(beta(j)-u1.^2);
w1=u1+i*v1;
z1=1/pi*(w1+log(w1)+1);
z1=1./(z1-i);

plot(z1,'r');hold on
axis equal
Z1=z1(1/3*length(beta));
text (real(Z1),imag(Z1),psi,'FontSize',8)
```

Line Charges:

Suppose that in our original configuration in the z plane there resides a line charge of infinite extent perpendicular to the plane of the paper. If we try to obtain the potential and stream function for this setup by means of the Schwarz-Christoffel transformation we will be mapping this line charge into the w plane. The strength of the line charge will be preserved and it will be perpendicular to the w plane and pass through the point u_0, v_0. Let us assume that we will be doing problems in which the potential of the conductors in the z plane are at the value 0. Thus, we will assign the potential zero to the conductor along the real axis in the $w = u + iv$ plane. The problem of finding the potential of a line charge a distance v_0 above a grounded plane lying along the real axis can be found using the method of images. See W sec. 8. 7. Suppose the line charge carries ρ_L coulombs per meter. If we take $Q = \rho_L / 2\pi\varepsilon_0$ where ε_0 is the permittivity of free space, then we can say that the complex potential a distance from a line change passing

through w_0 is $\Phi(w) = \phi(w) + i\psi(w) = Q \, \text{Log} \dfrac{a}{w - w_0}$ where a>0 is a real constant and is chosen

to yield the distance from the line charge at which $\phi(w)$ vanishes. Suppose the line charge is placed parallel to the perfectly conducting plane defined by $v = 0$ and is perpendicular to the w (or u,v) plane.

The line charge passes through u_0, v_0. The conducting plane is assumed to be at potential zero. The complex electrostatic potential above the plane is now found using the method of images (see W p.593). We remove the plane and compute the potential for $v > 0$ created by a line charge of strength Q passing through u_0, v_0 and a line charge of strength -Q passing through $u_0, -v_0$. Thus, we now have

$$\Phi(r) = Q \, \text{Log} \dfrac{1}{u + iv - u_0 - iv_0} - Q \, \text{Log} \dfrac{1}{u + iv - u_0 + iv_0} = Q \, \text{Log} \dfrac{u + iv - u_0 + iv_0}{u + iv - u_0 - iv_0} = Q \, \text{Log} \dfrac{w - u_0 + iv_0}{w - u_0 - iv_0}$$

$v_0 > 0, \ \text{Im}(w) \geq 0$ (11.37)

The equipotential surfaces created by this arrangement in the space $v > 0$ are of interest. The electrostatic potential (the voltage) created by the line charge and its image is the real part of the preceding expression.

Thus $\phi(u,v) = V = Q \log \dfrac{\sqrt{(u-u_0)^2 + (v+v_0)^2}}{\sqrt{(u-u_0)^2 + (v-v_0)^2}}$ Let V be the potential on a surface. We then

require $V = Q \log \dfrac{\sqrt{(u-u_0)^2 + (v+v_0)^2}}{\sqrt{(u-u_0)^2 + (v-v_0)^2}}$ so that

$e^{V/Q} = \dfrac{\sqrt{(u-u_0)^2 + (v+v_0)^2}}{\sqrt{(u-u_0)^2 + (v-v_0)^2}}$. Letting $e^{V/Q} = K$ we have the equation

$K^2(u-u_0)^2 + K^2(v-v_0)^2 = (u-u_0)^2 + (v+v_0)^2$.

After completing the square we arrive at

$$(u-u_0)^2 + \left(v - v_0\left(\frac{K^2+1}{K^2-1}\right)\right)^2 = v_0^2 \frac{4K^2}{(K^2-1)^2}.$$ (11.38)

Thus, in the w plane the equipotentials in the space $\mathrm{Im}(w) \geq 0$ are circles. The centers

are at $u_0, v_0\left(\dfrac{K^2+1}{K^2-1}\right)$ and the radii are $v_0 \dfrac{2K}{\left(\left|K^2-1\right|\right)}$. Recall that]t $e^{V/Q} = K$

where V is the voltage on the equipotential and $Q = \rho_L / 2\pi\varepsilon_0$.

As a check we see that if K>>1 (a high voltage) the equipotentials are centered close to u_0, v_0

and have radius $v_0 \dfrac{2}{K}$. For a low voltage we have K is positive and close to 1. Thus, for low

voltage, the center of the equipotential is at a large positive number on the v axis and the radius is also large.

We will also need the stream function created by the same line charge above a conducting plane as for the previous configuration. This would be the imaginary part of the complex potential given in Eq. (11.37). Calling this $\psi(u,v)$ we have

$$\psi(u,v) = Q \,\mathrm{Im}\left(\log \frac{u+iv-u_0+iv_0}{u+iv-u_0-iv_0}\right) = Q\left(\arctan\frac{v+v_0}{u-u_0} - \arctan\frac{v-v_0}{u-u_0}\right)$$ (11.39)

Now

Taking $A = \dfrac{v+v_0}{u-u_0}$ and $B = \dfrac{v-v_0}{u-u_0}$, $1 + AB = 1 + \dfrac{v^2 - v_0^2}{(u-u_0)^2}$, $A - B = \dfrac{2v_0}{(u-u_0)}$

$\arctan(A) - \arctan(B) = \arg\left(\dfrac{1+iA}{1+iB}\right) = \arg\dfrac{1+iA-iB+AB}{1+B^2} = \arctan\left(\dfrac{A-B}{1+AB}\right)$

Taking $Q = \rho_L / 2\pi\varepsilon_0$ =1 we have

$\psi = \arctan \dfrac{2v_0(u-u_0)}{(u-u_0)^2 + (v^2 - v_0^2)}$. If a streamline is to produce a particular value ψ , and if we

take $\tan\psi = 1/k$ so that $\psi = \arctan(1/k)$ we have

$$(u - u_0 - kv_o)^2 + v^2 = (k^2 + 1)v_o^2 \qquad (11.40)$$

The locus of this in the $w = u + iv$ plane is a set of circles of radius $v_0\sqrt{k^2 + 1}$ with centers on the u axis at $u_0 + kv_0 = u_0 + v_0/\sin\psi$. We can write a typical circle in MATLAB as.

w=uo+k(j)*vo + vo sqrt(k(j)^2+1)*exp(i*theta);where $0 \le \theta \le 2\pi$

Let us plot some of these streamlines in the upper half of the w plane. Notice that all streamlines pass through the point $w = u_0 + iv_0$ which is where the line charge is located. This can be verified by noticing that if we take $u = u_0$ in Eq.(11.40) then the equation is satisfied if we put $v = v_0$ The center of each circle has horizontal coordinate $u_0 + v_0/\tan\psi$ and the vertical coordinate 0. To plot these streamlines, it is best to use the polar representation of each circle. Letting *w* be the variable we
put
psi=[.2 .6   1 -.2 -.6 -1] along with Q=1. We take uo=0 and
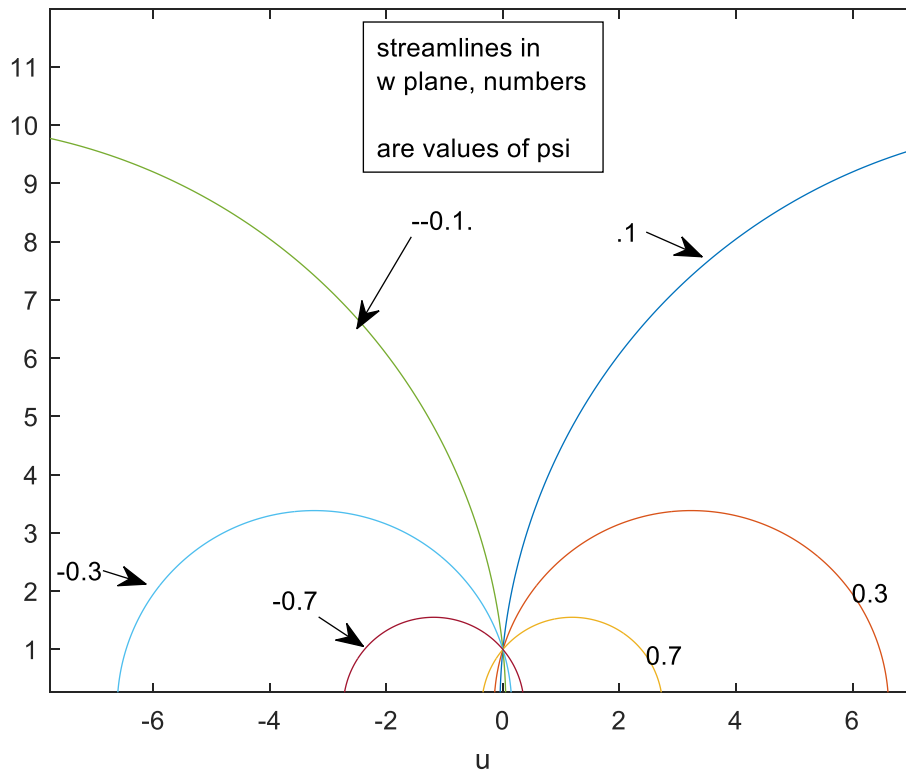vo=1;
The code is as follows

Figure 11.36
Streamlines of a Line Charge One Unit Above a Perfectly Conducting Plane

$Q = \rho_L / 2\pi\varepsilon_0 = 1$ charge per unit length . Numbers are values of ψ .

Example 17 Line Charge above a Semi-infinite Conducting Plane at Zero Potential (Grounded Plane)
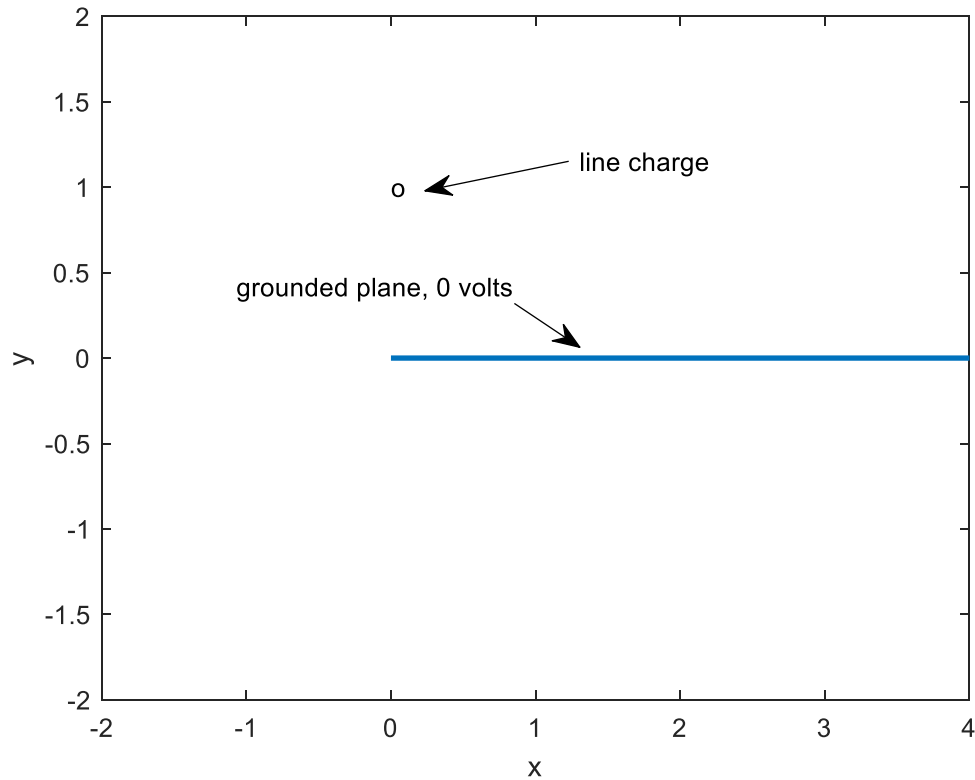

Figure 11.37 z plane

A semi-infinite conducting plane kept at zero potential fills the positive real axis in the z plane. Above it is a line charge of infinite length carrying ρ_L coulombs per meter perpendicular to the plane of the paper . The line charge is one unit from the edge of the plane in the y direction. We will take $\rho_L = 2\pi\varepsilon_0$ so that Q=1. We want to find the electrostatic potential and stream function being created. We would also like to know the surface charge density of the top and bottom of the plane.

We can solve this problem with the Schwarz-Christoffel transformation. We regard the half plane in the above figure as the limit as $\beta \to 0$ of the wedge shown below.
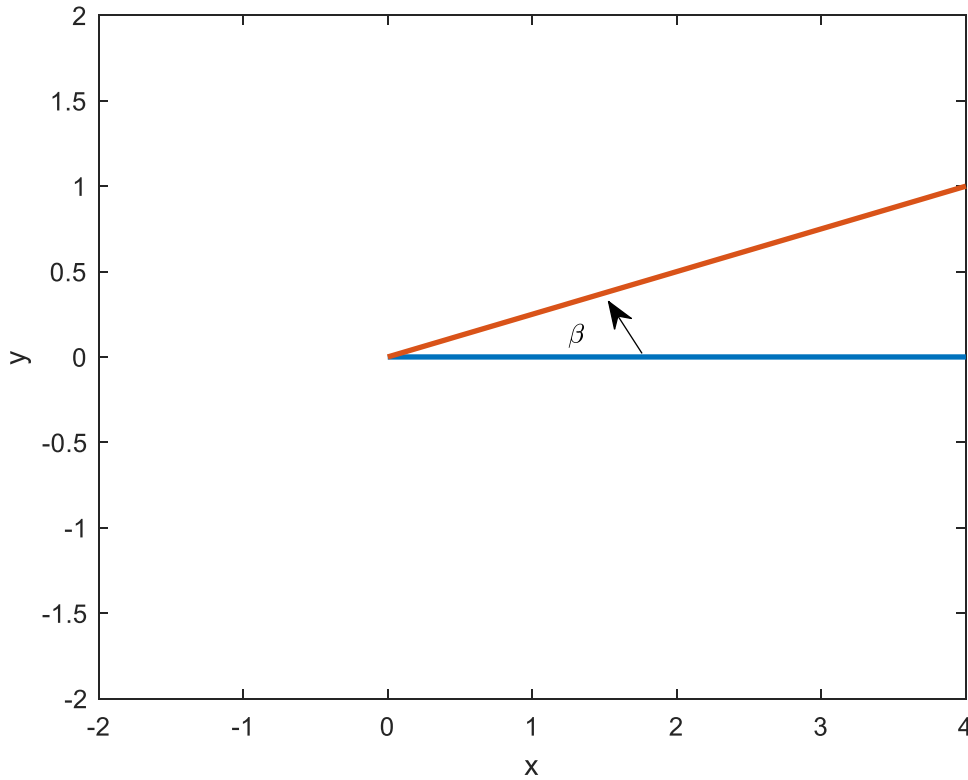
Figure 11.38

We will assume that for the transformation $z(w)$ the point $w = 0$ is mapped into $z = 0$. Following the notation of Eq.(11.1) and putting $\beta \to 0$ we have that our equation reduces to

$z(w) = A \int^{w} (w - u_1)^{((\alpha_1/\pi)-1)} dw + B$. In the limit as $\beta \to 0$, the angle $\alpha_1 = 2\pi$. We will take $u_1 = 0$

B=0 and the lower limit of the integration as zero so that the apex of the wedge in the z plane has an image of w=0. Thus $z(w) = A \int_{0}^{w} w^1 dw = Aw^2 / 2$.. Now suppose we place a line charge

perpendicular to the w plane. It passes through the point $w = e^{i\pi/4}$. Its image will be a line charge in the z plane at $z = Ai/2$. We will take $A = 2$ and $z = w^2$ and $w = z^{1/2}$. Thus, in the z plane the line charge passes through $z = i$. We will take as before $\rho_L = 2\pi\varepsilon_0$ so that Q=1. The stream function in the w plane is found in a manner similar to that in the previous problem. Because the line charge passes through $w = u_0 + iv_0 = e^{i\pi/4}$ we take $u_0 = v_0 = 1/\sqrt{2}$. Thus, we plot the streamlines as follows in the w plane using the transformation $z = w^2$ and $w = z^{1/2}$. The complex potential in the w plane (see Eq. 11.37) is

$$\Phi(w) = \operatorname{Log} \frac{w - \dfrac{1}{\sqrt{2}} + i\dfrac{1}{\sqrt{2}}}{w - \dfrac{1}{\sqrt{2}} - i\dfrac{1}{\sqrt{2}}}.$$ In the *z* plane, -where we are seeking our solution, this becomes

$$\Phi(z) = \operatorname{Log} \frac{z^{1/2} - \dfrac{1}{\sqrt{2}} + i\dfrac{1}{\sqrt{2}}}{z^{1/2} - \dfrac{1}{\sqrt{2}} - i\dfrac{1}{\sqrt{2}}}.$$ (Recall that we are using the transformation $w = z^{1/2}$.) We are

employing the Schwarz-Christoffel transformation which means that all points in the *z* plane must have images in the space $\operatorname{Im}(w) \geq 0$. However, given a value of z, MATLAB takes as its

square root the *principal value* of $z^{1/2}$. Thus, using MATLAB we will obtain for *w*
values in the space $\operatorname{Im}(w) < 0$ whenever $\operatorname{Im}(z) < 0$. Since this is not what we want, under these

circumstances ($\operatorname{Im}(w) < 0$) when using MATLAB we take the value of $z^{1/2}$ produced by
MATLAB and multiply it by -1, *which is still a valid value* for the square root. This is incorporated into the code that produces the following figure for the equipotentials. The values chosen for the voltage resulted after some experimentation and we are seeking the lines on which

$$\phi(z) = \operatorname{Re}\left[\operatorname{Log} \frac{z^{1/2} - \dfrac{1}{\sqrt{2}} + i\dfrac{1}{\sqrt{2}}}{z^{1/2} - \dfrac{1}{\sqrt{2}} - i\dfrac{1}{\sqrt{2}}}\right] = \operatorname{Log}\left|\frac{z^{1/2} - \dfrac{1}{\sqrt{2}} + i\dfrac{1}{\sqrt{2}}}{z^{1/2} - \dfrac{1}{\sqrt{2}} - i\dfrac{1}{\sqrt{2}}}\right|$$

is constant. Note that the equipotentials near the line charge are nearly circular on account of the diminished effect of the plane.
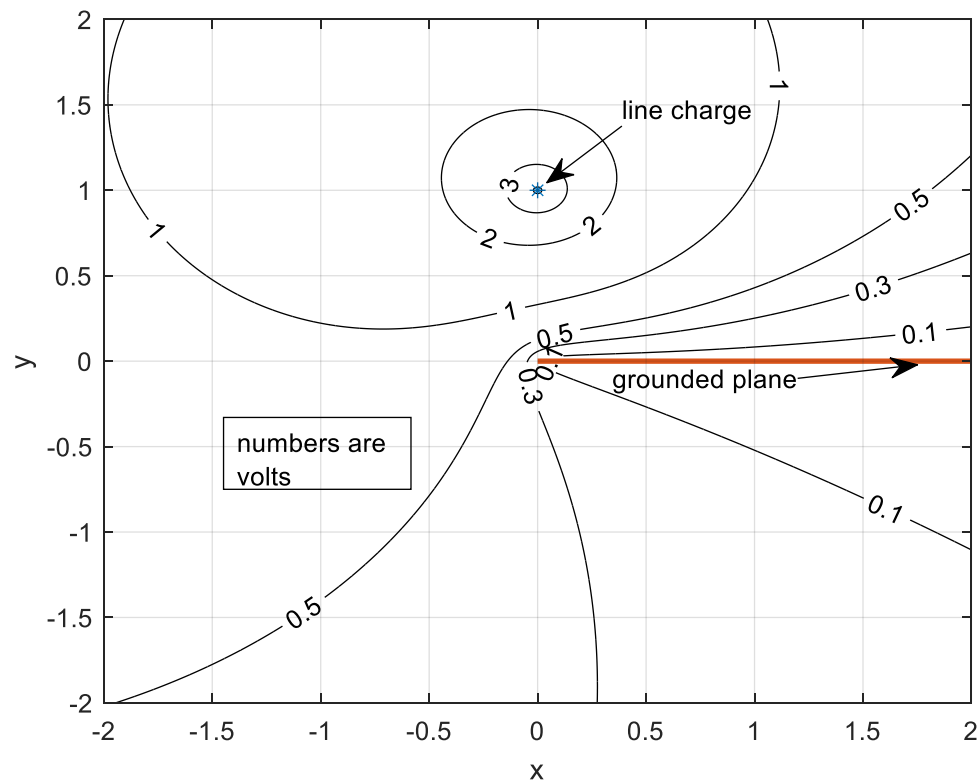
Figure 11.39

```
x=-2:0.001:2;
y=-2:0.001:2;

[X Y]=meshgrid(x,y);
z=X+i*Y;
a1=length(x);0
a2=length(y);
%the following ensures the correct value of the square root
for n1=1:a1
for n2=1:a2
if imag (sqrt(z(n1,n2)))<=0
b(n1,n2)=-sqrt((z(n1,n2)));
else
b(n1,n2)=sqrt(z(n1,n2));
end
end
end
num=b-1/sqrt(2)+i/sqrt(2);
 deno=b-1/sqrt(2)-i/sqrt(2);
Phi=log(num./deno);
phi=real(Phi);
figure(1)
[c,h]=contour (X,Y,phi,[ .1 .3 .5 1 2 3 5]);
 clabel(c,h);colormap([0 0 0])
```

```
grid;hold on
figure(1)
plot(0,1,'*')
x=linspace(0,2,100);
y=0*x;
plot(x,y,'linewidth',2)
```

Example 18 Line Charge Near a 60 Degree Wedge.
A line charge sits next to a 60 degree infinite conducting wedge at zero potential as shown in
the figure below. As before we will take the charge per unit length as
$\rho_L = 2\pi\varepsilon_0$. Our problem is to find the complex potential outside the wedge. We also want to use
this potential to write a program that will yield the electric field at any point outside the wedge.
Recall that the angles α are measured inside the domain of space in which we are interested in
finding the field . The angle measured inside the domain is 300 degrees or
$5\pi/3 = \alpha_1$ radians. In our Schwarz-Christoffel transformation we will assume that $z = 0$ has
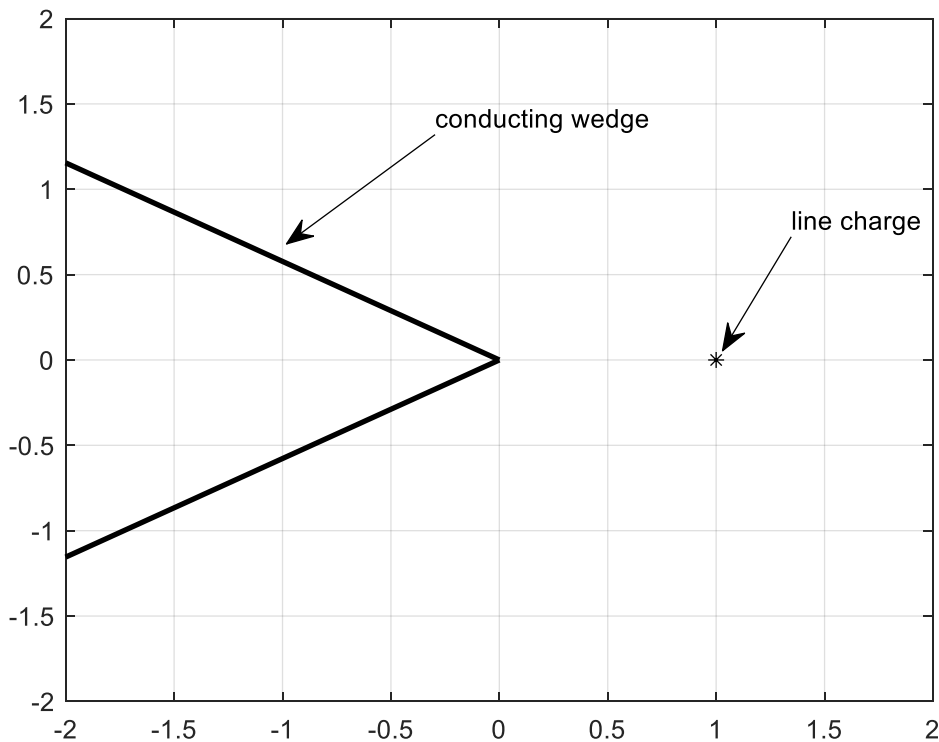image w=0.

Figure 11.40 60 Degree Wedge with Line Charge

Thus Eq.(11.1) becomes $z(w) = A\int_0^w (w)^{(2/3)}dw + B = C'w^{5/3}$. We have taken B=0 so that w=0 is

mapped into z=0. We have now $w = Cz^{3/5}$ where C is a constant to be determined. We would

like the line charge in the z plane at $z = 1$ to be mapped into $w = i$ in the w plane. We can do this by taking $C = i$ so that $w = iz^{3/5}$. Notice that the portion of the perfect conductor in the 2nd quadrant is mapped into the negative axis in the w plane while the portion in the 3rd quadrant is mapped into the positive real axis.

To accomplish this we must take $z^{3/5}$ as a continuous function in the space outside the wedge. The branch cut of this function must lie within the wedge. MATLAB chooses the negative real axis, which is fine. Taking $\rho_L = 2\pi\varepsilon_0$ as before we have from Eq.(11.37) in the w plane that the complex potential is

$\Phi(w) = \text{Log} \dfrac{w+i}{w-i}$. Thus, in the z plane the complex potential becomes

$\Phi(z) = \text{Log} \dfrac{iz^{3/5}+i}{iz^{3/5}-i} = \text{Log} \dfrac{z^{3/5}+1}{z^{3/5}-1}$. The physically measurable potential is

$\phi(z) = \text{Re}(\Phi(z)) = \text{Log} \left| \dfrac{z^{3/5}+1}{z^{3/5}-1} \right|$. We generate the equipotentials for this function and display
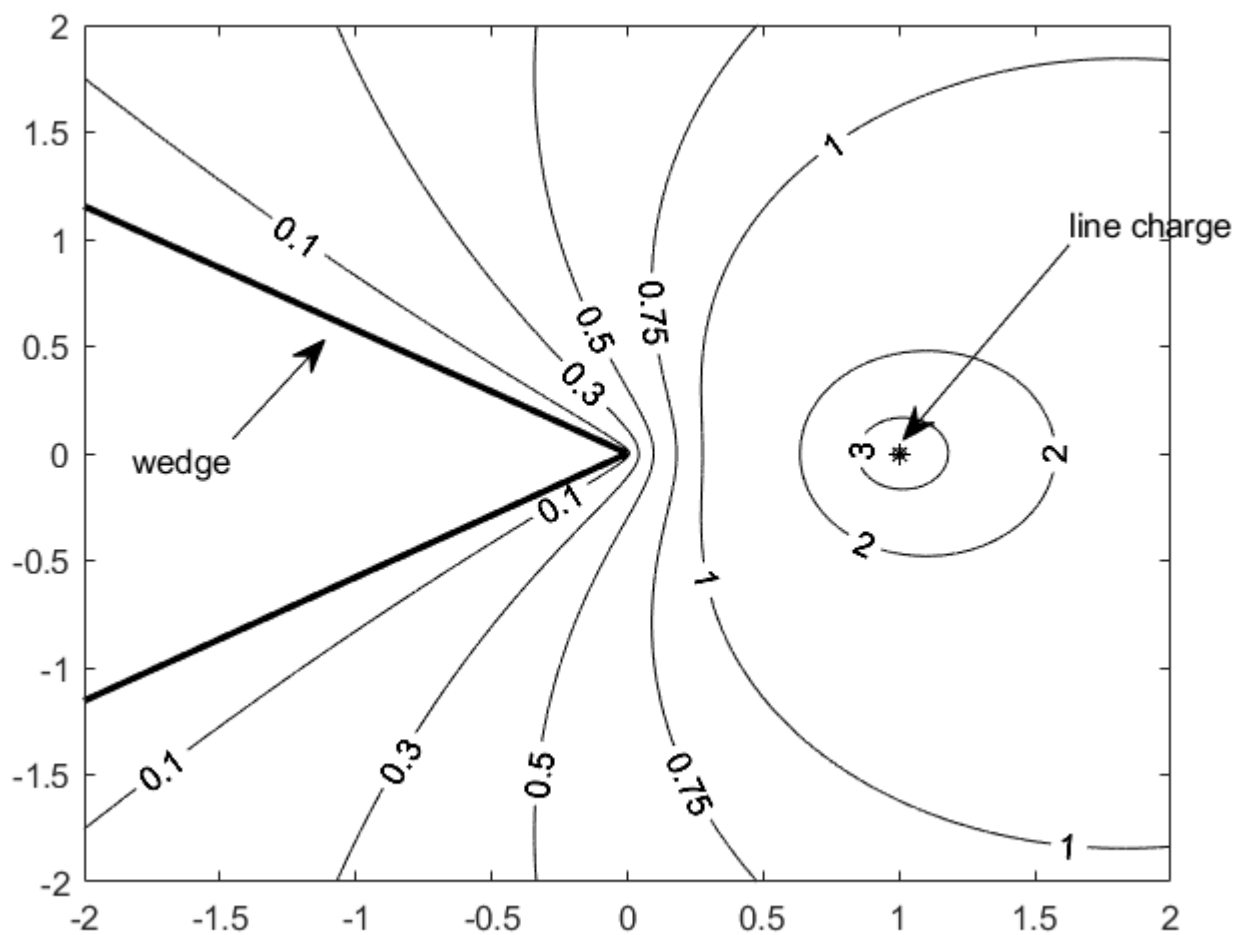
them below.
The code is under the figure .

Figure 11.41, Equipotentials for Figure 11.40

Code for the above

```
x=-2:0.001:2;
y=-2:0.001:2;
 [X Y]=meshgrid(x,y);
z=X+i*Y;
a1=length(x);
a2=length(y);
for n1=1:a1
for n2=1:a2
b(n1,n2)=i*z(n1,n2)^(3/5);
end
end
num=b+i;
 deno=b-i;
Phi=log(num./deno);
phi=real(Phi);
figure(1)
[c,h]=contour (X,Y,phi,[ .1 .3 .5 .75  1 2 3 5]);
 clabel(c,h);colormap([0 0 0])
```

```
hold on
figure(1)
plot(1,0,'*')
%figure for problem 18 , chap 11
x=linspace(0,-5,100);
y=x/sqrt(3);
plot(x,y,'k','linewidth',2);hold on
y=-x/sqrt(3);
plot(x,y,'k','linewidth',2)
plot(1,0,'*k')
xlim([-2 2]);ylim([-2 2])
grid
```

The stream function is the imaginary part of the complex potential. Thus

$$\psi(z) = \arg \frac{z^{3/5}+1}{z^{3/5}-1} = \arctan \frac{\mathrm{Im}\left(\dfrac{z^{3/5}+1}{z^{3/5}-1}\right)}{\mathrm{Re}\left(\dfrac{z^{3/5}+1}{z^{3/5}-1}\right)}.$$

Using the gradient and quiver functions of MATLAB we can plot arrows showing the negative of the gradient of the potential $\phi(x, y)$ external to the wedge.

These arrows show the direction of the electric field and their lengths are proportional to the electric field. Any arrows that might appear within the wedge have no physical meaning – there is no electric field there and we have blocked them out with the command *fill*. Here is the code and the plot. Notice that the electric field is strongest near the line charge and at the apex of the wedge. In theory the vectors should strike the wedge at right angles but the coarseness of the grid used to evaluate E has prevented that ,
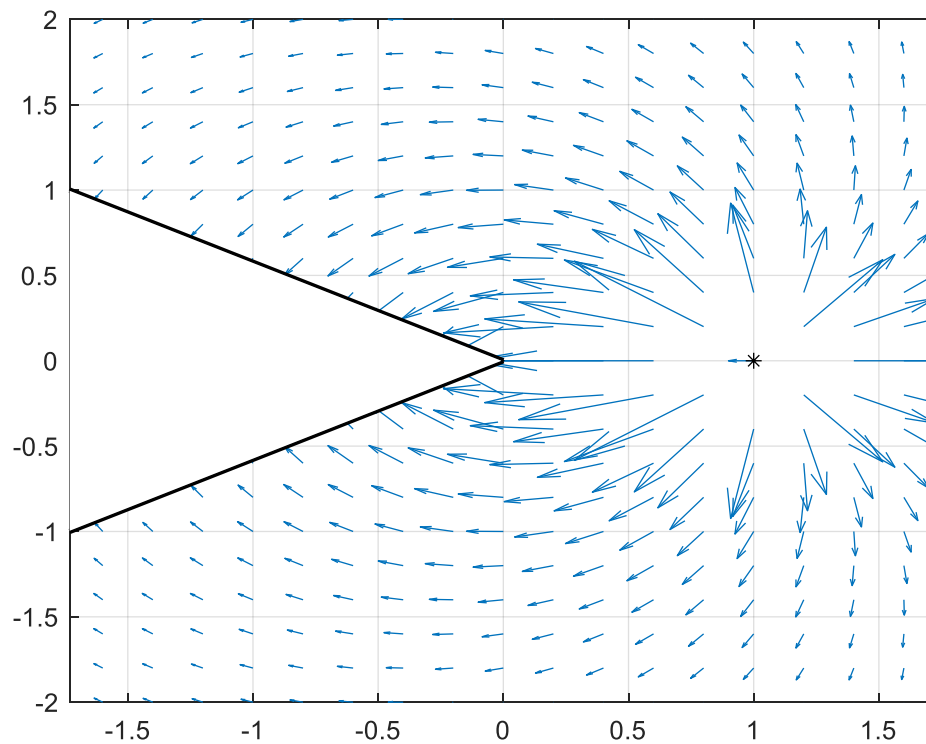
Figure 11.42 Quiver plot for Gradient for Figure 11.40

```
clear;
close all;
x=-2:0.2:2;
y=-2:0.2:2; % decrease size of x and y
[X,Y]=meshgrid(x,y);
z=X+1i*Y;
a1=length(x);
a2=length(y);
for n1=1:a1
    for n2=1:a2
        b(n1,n2)=1i*z(n1,n2)^(3/5);
    end
end
num=b+1i;
deno=b-1i;
Phi=log(num./deno);
phi=real(Phi);
[gx,gy]=gradient(-phi);
figure
quiver(X,Y, gx,gy,0); % setting the scale argument to 0
hold on
x=linspace(0,-5,100);
y=x/sqrt(3);
plot(x,y,'k','linewidth',2);hold on
y=-x/sqrt(3);
```

```matlab
plot(x,y,'k','linewidth',2)
plot(1,0,'*k')
xlim([-2 2]);ylim([-2 2])
grid

x=[-1.5*sqrt(3) 0 -1.5*sqrt(3)];y=[-1.5 0 1.5];

fill(x,y,'w')
xlim([-sqrt(3) sqrt(3)])
```

The following will provide the streamlines outside the wedge
```matlab
%upper half plane
clear
clf
x=linspace(-2,2,1000);
y=linspace(.01,3,1000);
 [X Y]=meshgrid(x,y);
z=X+i*Y;
a1=length(x);
a2=length(y);
for n1=1:a1
for n2=1:a2
b(n1,n2)=i*z(n1,n2)^(3/5);
end
end
num=b+i;
 deno=b-i;
Phi=log(num./deno);
Z=imag(Phi);
figure(1)
[c,h]=contour (X,Y,Z,[  .5 .75  1 1.5 2  -.5 -.75 -1 -1.5 -2]);
clabel(c,h);colormap([0 0 0])
hold on
figure(1)
plot(1,0,'*')
x=linspace(0,-5,100);
y=x/sqrt(3);
plot(x,y,'k','linewidth',2);hold on
y=-x/sqrt(3);
plot(x,y,'k','linewidth',2)
plot(1,0,'*k')
xlim([-2 2]);ylim([-3 3])
grid
figure(1)
%lower half plane
x=linspace(-2,2,1000);
y=linspace(-.01,-3,1000);
 [X Y]=meshgrid(x,y);
z=X+i*Y;
a1=length(x);
a2=length(y);
```

```matlab
for n1=1:a1
for n2=1:a2
b(n1,n2)=i*z(n1,n2)^(3/5);
end
end
num=b+i;
 deno=b-i;
Phi=log(num./deno);
Z=imag(Phi);
figure(1)
[c,h]=contour (X,Y,Z,[  .5 .75  1 1.5 2  -.5 -.75 -1 -1.5 -2]);
clabel(c,h);colormap([0 0 0])
hold on
figure(1)
plot(1,0,'*')
%figure for problem 18 , chap 11
x=linspace(0,-5,100);
y=x/sqrt(3);
plot(x,y,'k','linewidth',2);hold on
y=-x/sqrt(3);
plot(x,y,'k','linewidth',2)
plot(1,0,'*k')
xlim([-2 2]);ylim([-2 2])
grid
figure(1)
x=[-1.5*sqrt(3) 0 -1.5*sqrt(3)];y=[-1.5 0 1.5];
fill(x,y,'w');%this eliminates streamlines in wedge
```

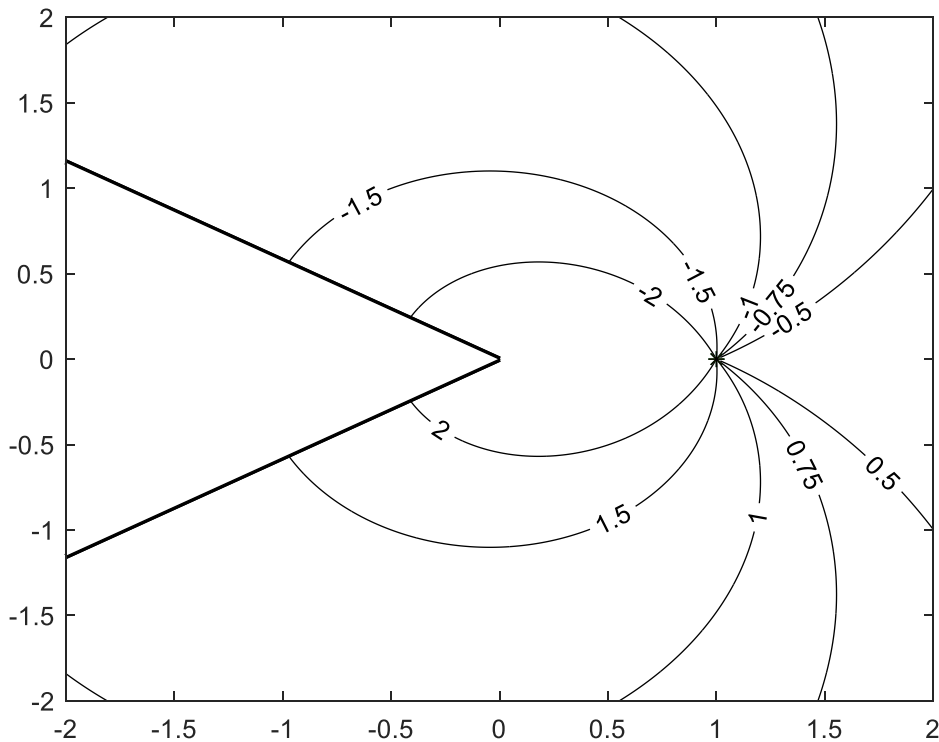This results in the following plot where the numbers are the values of ψ

Figure 11.43   Streamlines for Figure 11.40

To find the electric field at just one point we recall from Eq.(11.26) that the electric field in the z plane is given by the conjugate of the negative of the derivative of complex potential with respect to z.

Since $\Phi(z) = \mathrm{Log}\,\dfrac{z^{3/5}+1}{z^{3/5}-1}$ we have $\dfrac{d\Phi}{dz} = \dfrac{6}{5}z^{-2/5}\left[\dfrac{1}{z^{6/5}-1}\right]$. Thus $E = -\dfrac{6}{5}\overline{z^{-2/5}\left[\dfrac{1}{z^{6/5}-1}\right]}$. We

observe that the electric field is infinite at $z=1$, where the line charge is and at z=0 which is the edge of the bent plane . In our program we supply a value for z and get the electric field as a complex number. We must design our program so that if the value of z supplied lies within the wedge we will get a value of zero since the electric field cannot penetrate the interior of the wedge.

```
z=input('z=')
while abs(z)>0
if( -5*pi/6<=angle (z))&&(angle(z)<=5*pi/6)
   phi_z=6/5*z^(-2/5)/(1-z^(6/5));
   E=-conj(phi_z)
   angleE_degrees=angle(E)/pi*180
else
   E=0
end
z=input('z=')
end
```

Let us verify that the electric field strikes the wedge at right angles. We will evaluate the electric field at the point z=exp(-i*5*pi/6)
which is on the lower surface of the wedge (in the 3rd quadrant). We have
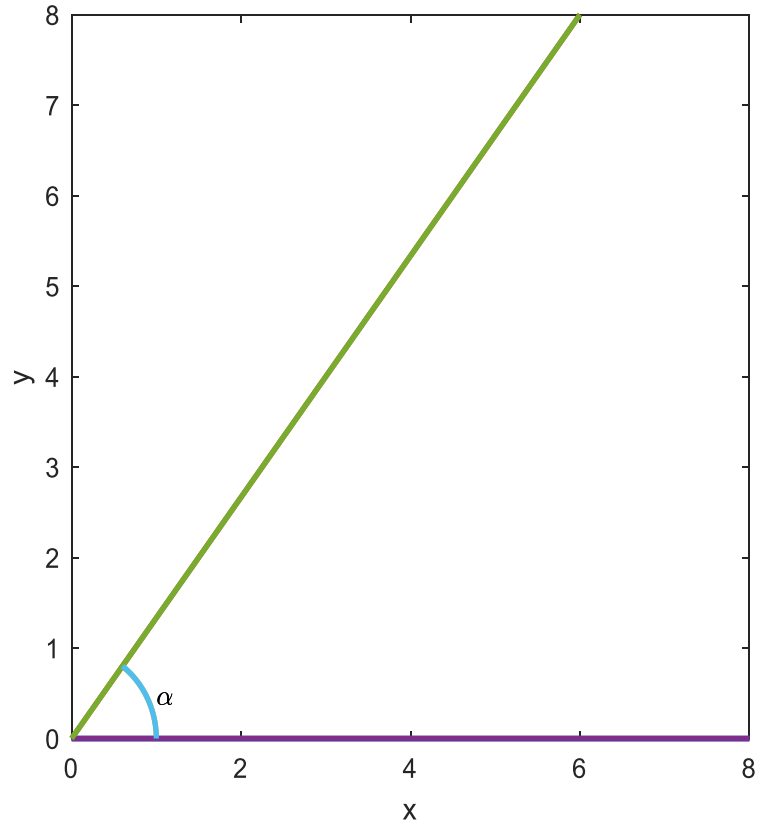z=exp(-i*5*pi/6)

z =

 -0.8660 - 0.5000i

E =

 -0.3000 + 0.5196i

angleE_degrees =

 120.0000

Some simple geometry shows that this vector does strike the bottom of the plane at right angles since the angle of E is 120 degrees.

Exercises

1. a) Using Eq.(11.1) map the interior of the above infinite wedge, in the z plane, into the upper half of the w plane . Assume that z=0 has image w=0 and that z=1 has image w=1and that infinity in the z plane has an image of infinity in the w plane. Find $z(w)$ and $w(z)$. Assume $0 < \alpha < \pi$.

b) Assume that fluid flows from left to right in the upper half of the w plane and is given by the complex velocity potential $\Phi(w) = w$. Thus, the fluid flow is given by the vector $V = 1$. What is the corresponding velocity potential in the z plane ? And what is the fluid velocity vector in the z plane ?
Identify any places where the velocity is infinite, including infinity. And state where the velocity is zero.

c) Write a program such that if you enter the coordinates of a point you will obtain the velocity at that point. Test the program by taking $\alpha = \pi / 4$ and obtain the velocities at the points $e^{i\pi/5}$ and also $.1e^{i\pi/8}$

d)Plot the streamlines on which ψ =0 and .01, 0.1, 1 in the z plane.
For each ψ , use 10^7 points in the w plane, where $-2 \le \text{Re}(w) \le 2$ and map them into the z plane.
e) Repeat part a) but map the *exterior* of the above wedge into the upper half of the *w* plane. The arrangements of the images cannot be the same as in part a), i.e., we cannot map $w = 0$ into $z = 0$ and $w = 1$ into $z = 1$. This is because as we move along the wedge in such a way that the exterior is on our left, we encounter the point $z = 1$ before we encounter the point $z = 0$. Thus, in moving along the real axis in the *w* plane, from left to right, the images of these points must be encountered in the same order. This can be satisfied by taking $w = -1$ as the image of $z = 1$ and $w = 0$ as the image of $z = 0$. Use these the conditions in your solution.

f) Take $\alpha = \pi / 2$, and check your result by finding $z(w)$ and map the real axis of the w plane $-3 < \text{Re}(w) < 3$ into the z plane. Now repeat parts b), c) and d). Take $\Phi(w) = w$ so that in the z plane the fluid approaches the corner of the wedge by coming from the right. Test the program for the velocity by using points outside the wedge. Notice that unlike in the first part of this problem the velocity is zero at infinity but is infinite at the corner in the wedge.
Take ψ = 0 .1 .2 .3 .4 .5 .7 1 in plotting the streamlines. They should look as follows

numbers are values of ψ
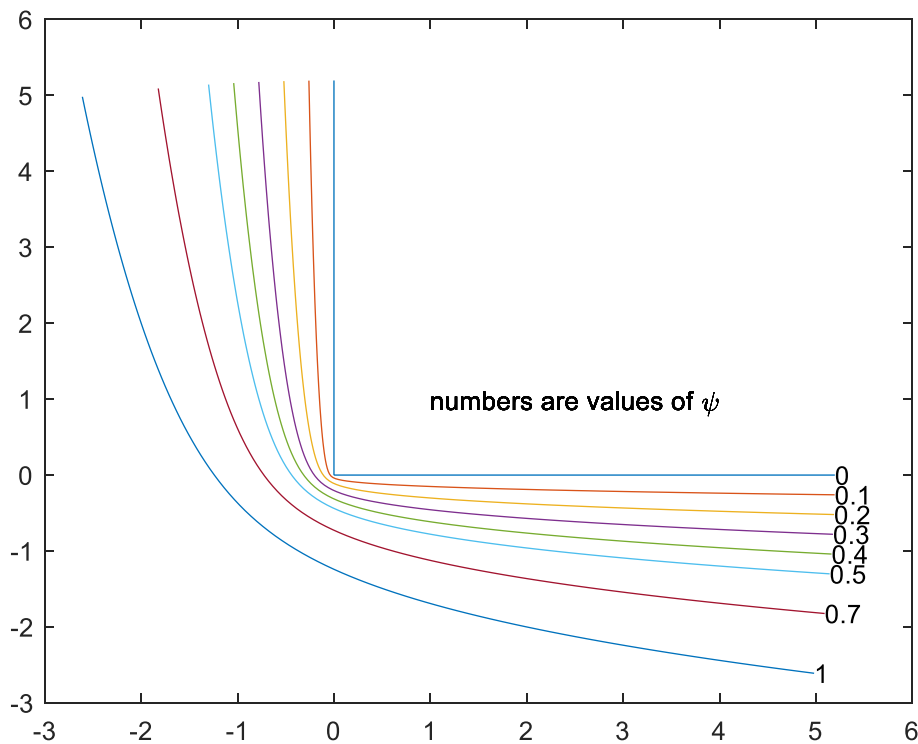
The flow of is from right to left and then upward .

2. For the configuration shown in worked Example 1, suppose that the upper surface of this structure is maintained at 1 volt while the vertical and bottom portions are at zero potential. Write a program that will display the equipotentials within the structure. Use the voltages from 0 to 1 as follows
[.01 .05 .1 .3 .5 .7 .9]
And also have your program plot the streamlines for these values
n=-1:30;
n=n*.2;

3. a)For the configuration shown in Figure 11.2(a) assume that we want to map the region outside the U shaped boundary into the upper half of the w plane so that
$z_1 = i$ has image $w_1 = -1$ and $z_2 = 0$ has image $w_2 = 0$ and that infinity has image infinity. Note the change in notation from the figure.
Find $z(w)$. Note: the interior of interest is on our left as we negotiate this shape.
Thus the angles α_1 and α_2 are now each $3\pi / 2$

b) Check your work by mapping the line segment $\mathrm{Im}(w) = 0$ -3<Re(w)<3 from the w plane into the z plane and verify that you get a portion of the U shaped boundary.

c) Suppose the vertical portion of the boundary is maintained at one volt. Plot the equipotentials outside the U shaped boundary in steps of .2 volt going from .2 to 1. Label the curves and show

the line where the voltage is zero. Also, before doing that, plot the corresponding equipotentials in the w plane and verify they are circles.

d) Plot the streamlines in both the planes described in part c). Suggestion: choose $\psi = \pm[.1 \quad .2 \quad .4]$. Use MATLAB to label these streamlines in the z plane.

---

4. The charge density at a point on the surface of a perfect conductor can be computed by finding the component of the electric field normal to the surface at that point, where the component is considered positive if it points away from the conductor. This component is called E_n. Multiplying this by a constant $\varepsilon_0 = 8.85 \times 10^{-12}$ one obtains the surface charge density in coulombs per square meter. Thus, a plot of E_n along the surface of a conductor will indicate the distribution of electric charge along that conductor divided by ε_0.

   a) Refer to Figure 11.12. Obtain a plot of $-E_y = E_n$ along the surface of the upper conductor. This should be the side closest to the bottom conductor, i.e., the inner surface of the upper conductor. . The plot should be along $-1 < x < -10^{-2}$ $y = 1-$and should involve 100 points. Note that we must avoid getting close to the edge of the upper conductor where the electric field has a singularity. Hint: begin with the code presented toward the end of Example 8 which will yield the electric field anywhere in space. Evaluate the value of the field along a locus $x + i0.9999$.

   b) Repeat the above but choose the outer surface of the upper conductor. Thus, the outward normal here is $E_y = E_n$. Take as the locus $x + i1.0001$

5.. a)Consider the transformation used in Example 2 , $z = \left(w^2 - 1\right)^{1/2}$. Suppose we were to take the negative of the reciprocal of this expression. Thus $z = -\left(w^2 - 1\right)^{-1/2}$. With this transformation, if we keep the same branches in the w plane as before, then verify that the boundary shown in Figure 11.5(b) is transformed into that shown below:
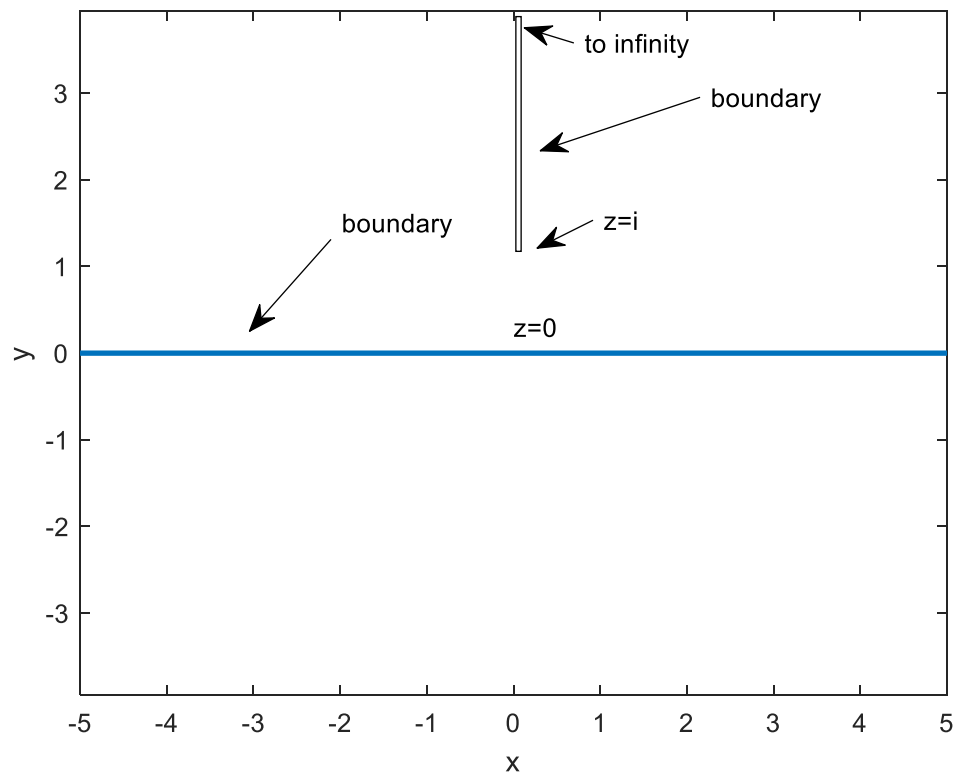
Now assume that there is a conductor maintained at 1 volt whose cross section extends along the y axis from $y = 1$ to ∞ and that the x axis represents the cross-section of a conductor maintained at zero volts.

b) Obtain a plot of the streamlines for the above configuration. Consider

ψ going from -1 to 1 in steps of $.25$. Use the function **contour.** Note that this function is undefined at the vertical portion of the barrier where ψ changes sign and is discontinuous.

c) Write a program such that if you enter a coordinate of a point in the space $y \geq 0$ that you will obtain the components of the velocity as a complex number. What is its numerical value at (1,1)? And at x=0, y=0+?

FLUID FLOW

y

z=0

π/4

x
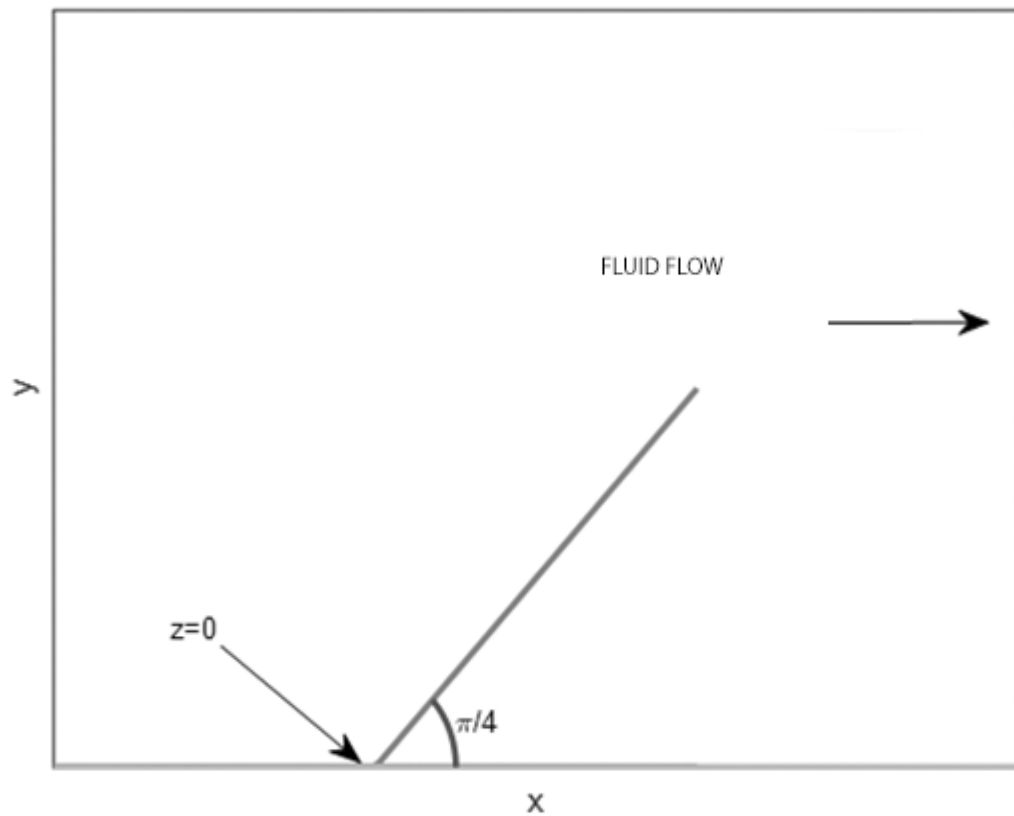
6. Fluid is flowing from left to right above a horizontal plane and encounters a barrier consisting of a vane whose corner is at the origin as shown above. The vane has angle $\pi/4$. The length of the above vane is one. The fluid velocity at infinity is in the x direction and has magnitude 1. The x axis defines another barrier guiding the flow. Our goal is to plot the streamlines for the above configuration and to find the velocity everywhere. To do this we approximate the ray using two rays as shown below.
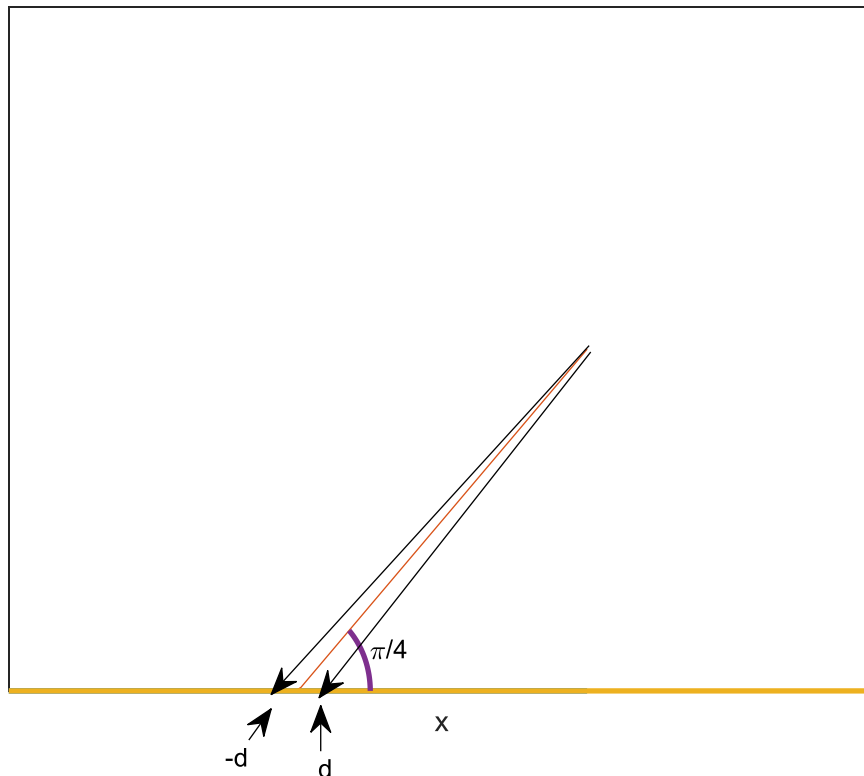
We must do a Schwarz Christoffel transformation of the above configuration into the w plane in the limit as $d \to 0+$ We will take the image of $z = 0-$ as $w = -1$ and the image of $z = e^{i\pi/4}$ as $w = u$ (where u, a real, will be determined) and the image of $z = 0+$ as $w = 1$. Derive the following result, for the transformation]

$$z(w) = A\int^{w} (w+1)^{-1/4}(w-1)^{-3/4}(w-u')dw + B.$$ Explain why if we take the lower limit of integration as -1 then B=0

b) Using the above lower limit, show numerically that $u' = .5$. Hint; with the upper limit set at 1 what value must the integral be ?

c) Show that A=0.877382647087441
by doing the above integration, in MATLAB from -1 to u' found above

d) With the values of A and B derived above, approximately map the real w axis from -2 to 2 into the z plane and verify that you get the topmost figure given in this problem. Take $w = u + i.001$ in order not to evaluate the integral at or near the singularities.

e) Taking the velocity at infinity as 1 in the positive x direction, plot the streamlines on which $\psi = .05, 0.1, 0.2, 0.5$ assuming that the boundary corresponds to $\psi = 0$

7. Consider the configuration shown in Example 1. Assume that what is shown is the cross-section of a conductor maintained at 0 volts. There is an infinite line

charge perpendicular to the plane of the page. Assume that the charge density on the line charge is given by $\rho_L = 2\pi\varepsilon_0$. The line charge passes through
x=-1, y=.5. Find the complex potential $\Phi(z) = \phi(z) + i\psi(z)$ inside the channel and plot the equipotentials for these values of ϕ
.01 .05 0.1 0.2 0.5 0.9 1.5. Your solution should look like this
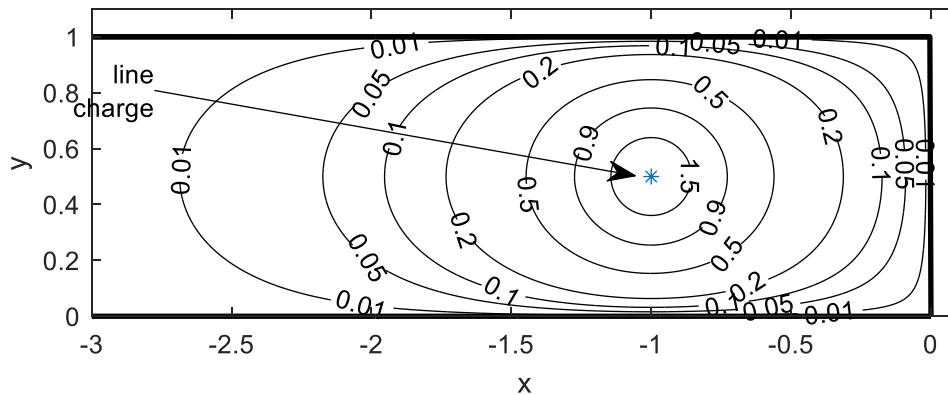

8.Consider the configuration in the X-Y plane shown in Fig. 11.33. Assume that all conductors are at zero potential and that there is a line charge $\rho_L = 2\pi\varepsilon_0$ coulombs per meter passing through   X=-1, Y=1/2 as shown below
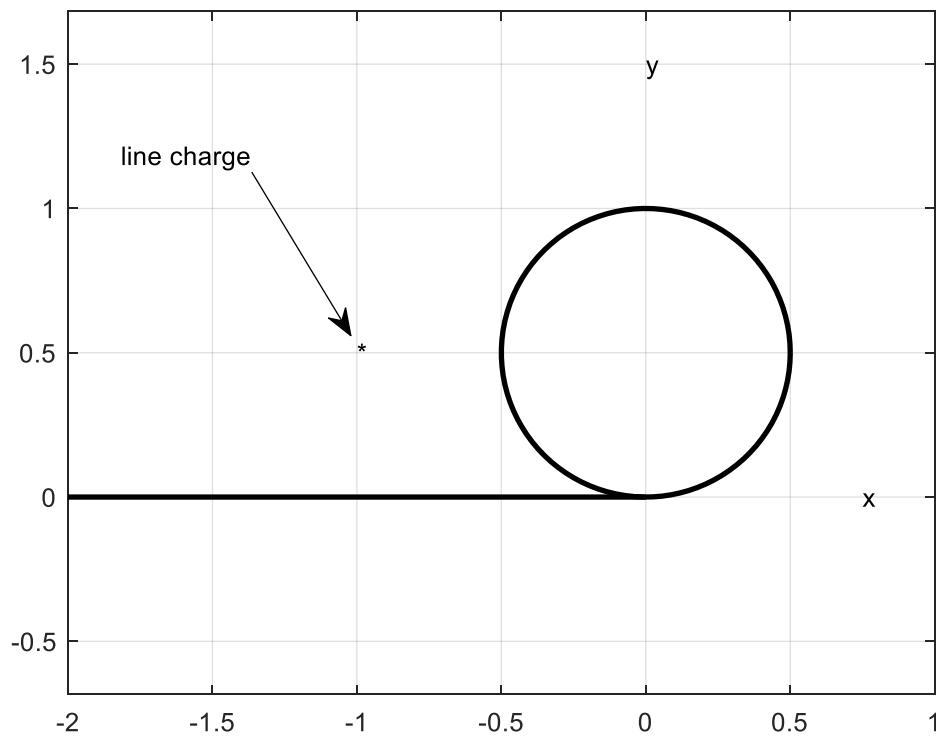
Z plane

a) Show that the complex transformation will map the upper half of the w plane onto the space defined by the boundaries shown above . $Z = \dfrac{\pi}{\left(w + \mathrm{Log}\,w + 1\right) - i\pi}$ Hint see Example 16.

b) Find the image in the w plane of the line charge shown above.

c) Plot the (real) equipotentials in the above plane for these values of $\phi = .1\ .3\ .5\ 1.0$ Your results should be as shown below x marks the location of the line charge.
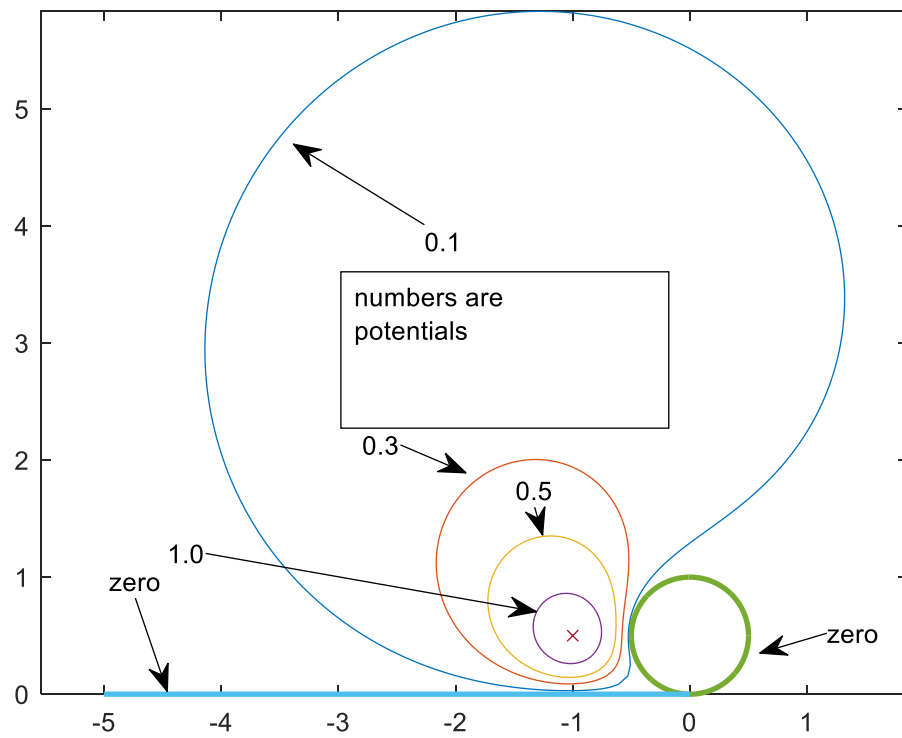
c) Plot the locus of the following streamlines $\psi = .2, .4, .6, .9$ You should get the following plot.