EECE 5560 Robotics
Coordinate Transform Examples

Robot Moving Example Your robot starts at position (0, 0) facing along the x-axis. It turns 90° ($^W R_A$) left and moves ($^A p_1$) 10m forward (along its x-axis). It turns ($^A R_B$) 45° right and moves ($^B p_2$) 5m backward. See Figure 1 for a diagram of the movement. What is it's final position?

We will use $^y p_x$ to denote which position is in which coordinate system. The subscript (x in the previous sentence) will be used to denote which point we are talking about: p_0 is the initial position, p_1 is the position after the first movement and p_2 is the position after the second movement. The superscript (y) will be used to denote which coordinate frame is being used: $^W p$ is a point in world coordinate, $^A p$ after the first rotation and $^B p$ after the second. R will be used to denote rotations between coordinate systems $^W R_A$ is between the initial and first rotations, $^A R_B$ between the first and second).

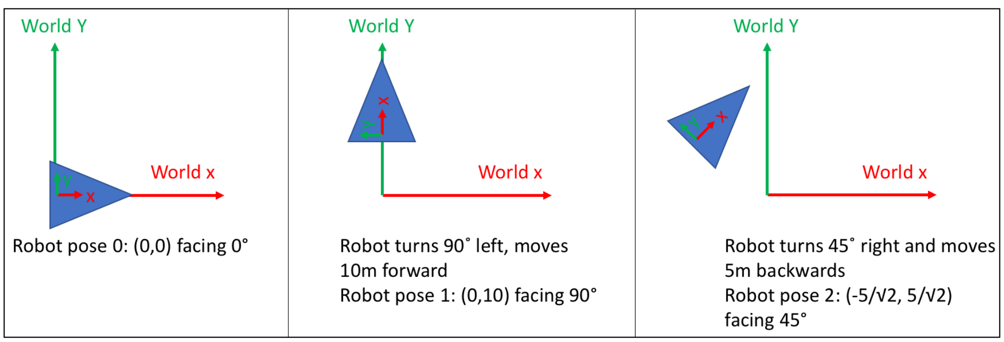

Figure 1: The figure above shows the robot movements. Note that objects are not to scale.

We start by defining the initial point:

$$^W p_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

And the rotation to the next point, as well as the definition of the next point in it's own reference frame:

$$^W R_A = \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$^A p_1 = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$$

Recall that the robot makes its first turn while at its initial point, so there is no offset between the first two coordinate frames. In this case, the transformation matrix between the initial and first coordinate frame is:

$$^{W}T_A = \begin{bmatrix} ^{W}R_A & ^{W}p_0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

To find the robot's position in world coordinates after it's first movement, we multiply the transformation matrix by the first point in its own coordinate system:

$$\begin{bmatrix} ^{W}p_1 \\ 1 \end{bmatrix} = {^{W}T_A} \begin{bmatrix} ^{A}p_1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \\ 1 \end{bmatrix}$$

So $^{W}p_1 = (0, 10)$.

Now define the terms for the second point, noting that the rotation is -45°relative to the coordinate frame after the first movement:

$$^{A}R_B = \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$^{B}p_2 = \begin{bmatrix} -5 \\ 0 \end{bmatrix}$$

Now we make the transformation matrix between robot frames A and B. Note that we turn AFTER the first movement, so we need to consider the displacement between the initial position and the first position here, in the first position's reference frame:

$$^{A}T_B = \begin{bmatrix} ^{A}R_B & ^{A}p_1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 10 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

To find the robot's coordinates in the first robot reference frame, we multiply the transformation matrix by $^{B}p_2$. To find the coordinates in the world frame, we multiply the transformation matrix $^{W}T_A$ by the result from the previous step.

$$\begin{bmatrix} ^{W}p_2 \\ 1 \end{bmatrix} = {^{W}T_A}\, {^{A}T_B} \begin{bmatrix} ^{B}p_2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 10 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix} =$$

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 10 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{5}{\sqrt{2}} \\ 10 - \frac{5}{\sqrt{2}} \\ 1 \end{bmatrix}$$

So ${}^{W}p_2 = (-\frac{5}{\sqrt{2}}, 10 - \frac{5}{\sqrt{2}})$

Sensors on a Robot Example Your robot is at position (10, 20) facing 45° to the left of the x-axis. It has a sensor 1 m forward of the center of the robot facing 90° to the right relative to robot x-axis. In robot and global coordinates, what is the position of the following obstacles (given in sensor, s coordinates):

${}^{S}p_a : (5,5)$
${}^{S}p_b : (10,25)$
${}^{S}p_c : (2,6)$

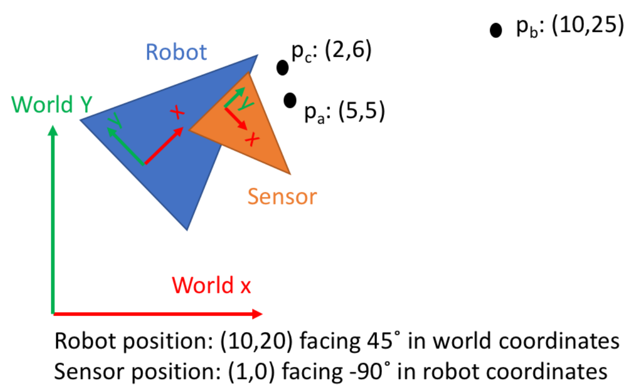

Figure 2: The figure above shows the objects detected by the robots sensors and the relative position of the sensor on the robot and the robot in the world. Note that objects are not to scale.

We start by defining the transformation matrix between the robot and the sensor. We denote the rotation from the sensor frame to the robot frame as ${}^{R}R_S$ and the position of the sensor on the robot as ${}^{R}p_S$ (e.g. the position of the sensor relative to the center of the robot coordinate frame).

$${}^{R}R_S = \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$${}^{R}p_S = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$
{}^{R}T_{S} = \begin{bmatrix} {}^{R}R_{S} & {}^{R}p_{S} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

And we define the transformation between the world and the robot:

$$
{}^{W}R_{R} = \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}
$$

$$
{}^{W}p_{R} = \begin{bmatrix} 10 \\ 20 \end{bmatrix}
$$

$$
{}^{W}T_{R} = \begin{bmatrix} {}^{W}R_{R} & {}^{W}p_{R} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 10 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 20 \\ 0 & 0 & 1 \end{bmatrix}
$$

Now, we calculate each point's position in robot coordinates:

$$
\begin{bmatrix} {}^{R}p_{a} \\ 1 \end{bmatrix} = {}^{R}T_{S} \begin{bmatrix} {}^{S}p_{a} \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ -5 \\ 1 \end{bmatrix}
$$

$$
\begin{bmatrix} {}^{R}p_{b} \\ 1 \end{bmatrix} = {}^{R}T_{S} \begin{bmatrix} {}^{S}p_{b} \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 25 \\ 1 \end{bmatrix} = \begin{bmatrix} 26 \\ -10 \\ 1 \end{bmatrix}
$$

$$
\begin{bmatrix} {}^{R}p_{c} \\ 1 \end{bmatrix} = {}^{R}T_{S} \begin{bmatrix} {}^{S}p_{c} \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 6 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ -2 \\ 1 \end{bmatrix}
$$

Now that we have the points in robot coordinates, they can be converted to world coordinates by multiplying ${}^{W}T_{R}$ by each:

$$
\begin{bmatrix} {}^{W}p_{a} \\ 1 \end{bmatrix} = {}^{W}T_{R} \begin{bmatrix} {}^{R}p_{a} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 10 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 20 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ -5 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{11}{\sqrt{2}} + 10 \\ \frac{1}{\sqrt{2}} + 20 \\ 1 \end{bmatrix} = \begin{bmatrix} 17.78 \\ 20.71 \\ 1 \end{bmatrix}
$$

$$\begin{bmatrix} {}^{W}p_b \\ 1 \end{bmatrix} = {}^{W}T_R \begin{bmatrix} {}^{R}p_b \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 10 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 20 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 26 \\ -10 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{36}{\sqrt{2}} + 10 \\ \frac{16}{\sqrt{2}} + 20 \\ 1 \end{bmatrix} = \begin{bmatrix} 35.46 \\ 31.31 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} {}^{W}p_c \\ 1 \end{bmatrix} = {}^{W}T_R \begin{bmatrix} {}^{R}p_c \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 10 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 20 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 7 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{9}{\sqrt{2}} + 10 \\ \frac{5}{\sqrt{2}} + 20 \\ 1 \end{bmatrix} = \begin{bmatrix} 16.36 \\ 23.53 \\ 1 \end{bmatrix}$$