Homework 2

Chapter 3. Linear/Angular Momentum.

Problem 3.4. (J. Taylor "Classical Mechanics") (10 points)

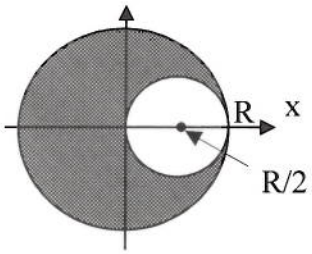
Problem 3.11. (J. Taylor "Classical Mechanics") (10 points)

Problem 3.21. (J. Taylor "Classical Mechanics") (10 points)

Problem 3.36. (J. Taylor "Classical Mechanics") (10 points)

Problem A. (10 points)

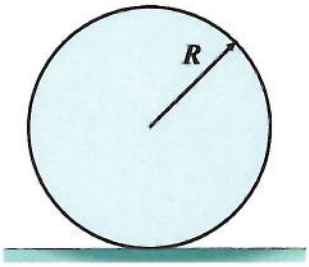
A spherical hole of radius R/2 is cut out in a uniform circular disc of surface density σ and radius R in such a way as to be internally tangent to the disc. Find the center of mass of the composite body.



Problem B. (10 points)

A ball of mass m and radius R is thrown on the floor so that its initial velocity is v_0 along the surface. The ball moves in a straight line and initially slips without rolling. As it slides, it begins to spin, and eventually rolls without slipping. The coefficient of kinetic friction is μ_k. $I_{CM} = \frac{2}{5}mR^2$

a. Draw a free body diagram.
b. Before it starts experiencing a pure rotation, find the velocity of the center of mass (CM) and the angular velocity about the CM.
c. Apply the condition for rolling without slipping to find the time it takes to begin rolling without slipping.

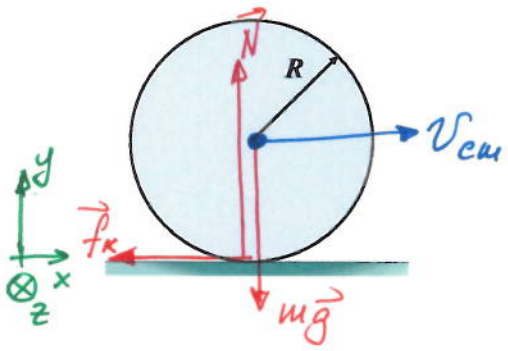


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Translational N. 2nd law

(y) $N - mg = 0 \Rightarrow N = mg \Rightarrow f_k = \mu_k \cdot mg$

(x) $m \cdot a_{CM} = -f_k \Rightarrow a_{CM} = -\frac{\mu_k \cdot mg}{m} = -\mu_k \cdot g$

so $v_{CM} = \int_{t=0}^{t} -\mu_k g \, dt = v_0 - \mu_k \cdot g \cdot t$

Rotat. N. 2nd law $\Sigma \vec{\tau} = I\vec{\alpha}$ (about the CM)

$\vec{\tau}_N = 0$; $\vec{\tau}_{mg} = 0$, so $f_k \cdot R = I\alpha$

$\alpha = \frac{f_k \cdot R}{I} = \frac{\mu_k mg \cdot R}{\frac{2}{5} mR^2} = \frac{5}{2} \cdot \frac{\mu_k \cdot g}{R}$, $\alpha = \frac{d\omega}{dt}$

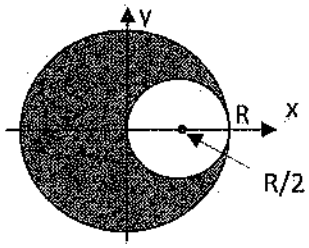
$\omega = \int_0^t \alpha \, dt = \omega_0^{\,0} + \alpha \cdot t = \left\| \frac{5\mu_k g}{2 \cdot R} \cdot t = \omega \right\|$ (slips w/out rolling)

Now, the condition for rolling w/out slipping is $v_{CM} = \omega \cdot R$

so $v_0 - \mu_k g \cdot t_1 = \frac{5\mu_k \cdot g}{2 \cdot R} t_1 \cdot R = \frac{5\mu_k g t_1}{2}$

$v_0 = \frac{7\mu_k g t_1}{2} \Rightarrow \left\| t_1 = \frac{2 \cdot v_0}{7\mu_k \cdot g} \right\|$

(A) A circular hole of radius R/2 is cut out in a uniform circular disc of surface density σ and radius R in such a way as to be internally tangent to the disc. Find the center of mass of the composite body.



Solution:

The coordinates of the center of mass of a system can be determined by subdividing the system, in an arbitrary manner, into more than one part, computing the mass center for each of the parts separately, and then in turn finding the center of mass of these centers. So, we can consider the body to be made up of two parts. The first one is a disc of radius R and a surface density σ and the second one is a disc of radius R/2 with a negative surface density (-σ). The masses of the two bodies are

$$M_\sigma = \sigma\pi R^2$$
$$M_{-\sigma} = -\sigma\pi(R/2)^2$$

The CM of the first body is at the point (x,y) = (0,0)
The CM of the second body is at the point (x_1,y_1) = (R/2,0)
Then, the CM of the composite body is

$$X = \frac{M_\sigma x + M_{-\sigma}x_1}{M_\sigma + M_{-\sigma}} = \frac{M_{-\sigma}x_1}{M_\sigma + M_{-\sigma}} = \frac{x_1}{1+\frac{M_\sigma}{M_{-\sigma}}} = \frac{R/2}{-3} = -\frac{R}{6}$$

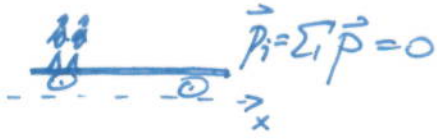
$$Y = 0 \text{ (due to symmetry)}$$

3.4 ** Two hobos, each of mass m_h, are standing at one end of a stationary railroad flatcar with frictionless wheels and mass m_{fc}. Either hobo can run to the other end of the flatcar and jump off with the same speed u (relative to the car). **(a)** Use conservation of momentum to find the speed of the recoiling car if the two men run and jump simultaneously. **(b)** What is it if the second man starts running only after the first has already jumped? Which procedure gives the greater speed to the car? [*Hint:* The speed u is the speed of either hobo, *relative to the car* just after he has jumped; it has the same value for either man and is the same in parts (a) and (b).]

• Let v be the speed of the flatcar immediately after jump (relative to the ground). Then $u-v$ is the speed of either man relative to the ground just after they jump.

a) • Conservation of momentum (M – mass of the flatcar)

INITIAL



$\vec{p}_i = \sum_i \vec{p} = 0$

FINAL



$\vec{p}_f = \sum_i \vec{p} = \vec{p}_1 + \vec{p}_2 + \vec{p}_{car} =$

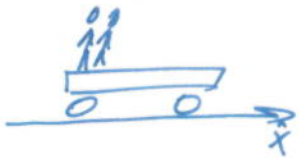
$= 2m(u-v)\hat{x} + Mv(-\hat{x})$

$\vec{p}_i = \vec{p}_f$, so

$0 = 2m(u-v) - M \cdot v \Rightarrow \boxed{v = \left(\frac{2m}{2m+M}\right) \cdot u}$

b) Let's devide this part into two: 1) the first guy jumps (the 2nd stays)
2) the 2nd jumps.

1) INITIAL



$\sum \vec{p} = 0$

FINAL



$\sum_i \vec{p} = (M+m) \cdot v_1 (-\hat{x}) + m(u-v_1) \cdot \hat{x}$

Conserv. of momentum:

$0 = m(u-v_1) - (M+m)v_1 \Rightarrow \boxed{v_1 = \frac{m}{M+2m} \cdot u}$ similar to part a.

2) INITIAL



$\sum_i \vec{p} = (M+m) \cdot v_1 (-\hat{x})$

FINAL



$\sum_i \vec{p} = (-\hat{x}) M \cdot v_2 + \hat{x} \cdot m(u-v_2)$

Conserv. of momentum:

$-(M+m)v_1 = m(u-v_2) - M\cdot v_2$

$v_2 = \frac{1}{m+M}\cdot\left[mu + (M+m)v_1\right] = \left\| \text{use } v_1 \text{ from the 1st part: } v_1 = \frac{m}{M+2m}u \right\|$

$= \left[\frac{m(2M+3m)}{(m+M)(M+2m)}\right]\cdot u$

To compare with the result from part a, let's separate v from v_2:

$v_2 = \left[\frac{2M+3m}{2(m+M)}\right]\cdot\left(\frac{2mu}{2m+M}\right) = \left(\frac{2M+3m}{2(m+M)}\right)\cdot v$, so $v_2 > v$

The second procedure gives the larger final velocity.

3.11

3.11 ** **(a)** Consider a rocket traveling in a straight line subject to an external force F^{ext} acting alc the same line. Show that the equation of motion is

$$m\dot{v} = -\dot{m}v_{\text{ex}} + F^{\text{ext}}. \qquad (3.$$

[Review the derivation of Equation (3.6) but keep the external force term.] **(b)** Specialize to the case a rocket taking off vertically (from rest) in a gravitational field g, so the equation of motion becom

$$m\dot{v} = -\dot{m}v_{\text{ex}} - mg. \qquad (3.$$

Assume that the rocket ejects mass at a constant rate, $\dot{m} = -k$ (where k is a positive constant), that $m = m_0 - kt$. Solve equation (3.30) for v as a function of t, using separation of variables (t is, rewriting the equation so that all terms involving v are on the left and all terms involving t on right). **(c)** Using the rough data from Problem 3.7, find the space shuttle's speed two minutes into flig assuming (what is nearly true) that it travels vertically up during this period and that g doesn't char appreciably. Compare with the corresponding result if there were no gravity. **(d)** Describe what wo

(a) The change in momentum (rocket + fuel) in time is given by (3.4)

$dP = m\,dv + dm\,v_{ex}$

Also we remember, that the rate of change of the system's total momentum is determined by the external force

(3.1) $\dot{P} = F_{ext} \Rightarrow dP = F_{ext}\,dt$

Equating these two expressions for dP, we'll get

$m\,dv + dm\,v_{ex} = F_{ext}\,dt$ or

$$\boxed{m\dot{v} = -\dot{m}v_{ex} + F_{ext}.}$$

(b) since $m = m_0 - kt \Rightarrow \dot{m} = -k$; $\bar{F}_{ext} = -mg$

so the eq-n from part (a) is

$$m\dot{v} = k v_{ex} - mg \Rightarrow \frac{dv}{dt} = \frac{k}{m} v_{ex} - g$$

$$dv = \left(\frac{k}{m_0 - kt} v_{ex} - g\right) \cdot dt$$

$$\int_{v_0=0}^{v} dv = \int_0^t \left[\frac{k v_{ex}}{m_0 - kt} - g\right] dt = k v_{ex} \left(-\frac{1}{k}\right) \int_0^t \frac{d(m_0 - kt)}{m_0 - kt} - g \cdot t$$

$$v = -v_{ex} \ln(m_0 - kt)\Big|_0^t - gt = -v_{ex} \ln \frac{(m_0 - kt)}{m_0} - gt$$

$$v(t) = v_{ex} \ln\left(\frac{m_0}{m_0 - kt}\right) - g \cdot t = \left\| m_0 - kt = m \right\| = v_{ex} \cdot \ln\left(\frac{m_0}{m}\right) - gt \quad \therefore$$

(c) using $v_{ex} = 3000\ m/s$, $m_0/m = 2$, $t = 120\ sec$

$$v(t=120s) = 3000\ m/s \ln 2 - 9.8\ m/s^2 \cdot 120\ sec \approx 2079 - 1200 \approx 880\ m/s$$

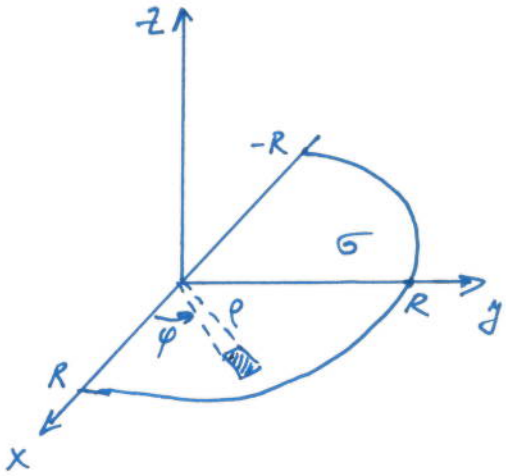
if there is no gravity: $g = 0$

$$v(t=120s) = 2079\ m/s$$

So the gravity reduces the speed acquired in the first 2min to a little less than half its weight-free value.

(d) If the thrust in v_{ex} is less than the weight mg, the rocket will just sit on the ground until it has shed enough mass that the thrust can overcome the weight. Not a good design.

3.21 ** A uniform thin sheet of metal is cut in the shape of a semicircle of radius R and lies in the xy plane with its center at the origin and diameter lying along the x axis. Find the position of the CM using polar coordinates. [In this case the sum (3.9) that defines the CM position becomes a two-dimensional integral of the form $\int \mathbf{r}\,\sigma\,dA$ where σ denotes the surface mass density (mass/area) of the sheet and dA is the element of area $dA = r\,dr\,d\phi$.]



The CM position is

$$\vec{R}_{CM} = \frac{1}{M}\int \vec{r}\,dm = \frac{1}{M}\int \vec{r}\,\sigma\,dA$$

$$dA = d\rho\cdot\rho\cdot d\varphi\,, \quad \vec{R} = (X_{CM}, Y_{CM}, Z_{CM})$$

By symmetry $X_{CM} = Z_{CM} = 0$.

$$M = \int \sigma\,dA = \frac{\pi R^2}{2}\sigma$$

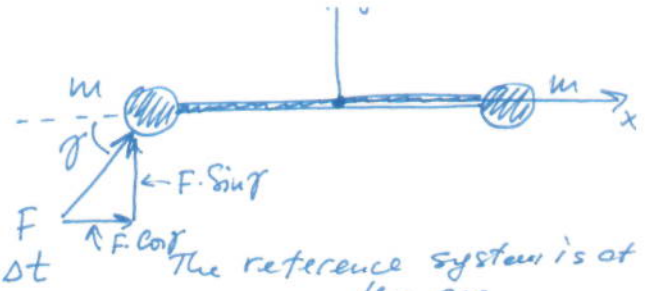
$$Y_{CM} = \frac{1}{M}\int y\,\sigma\,dA = \frac{1}{M}\int_0^R\int_0^{\pi} d\varphi\cdot y\cdot\sigma\cdot\rho\,d\rho$$

$$y = \rho\cdot\sin\varphi$$

$$Y_{CM} = \frac{\sigma}{M}\cdot\int_0^R \rho^2 d\rho\int_0^{\pi}\sin\varphi\,d\varphi = \frac{2}{3}\frac{\sigma R^3}{M} = \left\| M = \frac{\pi R^2}{2}\sigma \right| = \boxed{\frac{4\cdot R}{3\cdot\pi} = Y_{CM}}$$

$$\vec{R}_{CM} = \left(0;\ \frac{4R}{3\cdot\pi};\ 0\right)$$

3.36



The reference system is at the CM.

The impulse does two things:
- it provides momentum to the system
- it provides a torque since it's applied off-center.

- momentum: (as we did in class) $\frac{d\vec{P}}{dt} = \vec{F}$

$$\int_{p(t=0)=0}^{\vec{p}} d\vec{p} = \int_{t=0}^{\Delta t} \vec{F}\,dt \Rightarrow \vec{p} = \vec{F}\Delta t$$ - mom. transfered to the system

This momentum will cause a translational motion which can be described by the CM motion (Eq. 3.12)

$$\vec{P} = 2m\cdot\dot{\vec{R}}_{CM}, \text{ so}$$

$$2m\dot{\vec{R}}_{CM} = \vec{F}\Delta t \Rightarrow \dot{\vec{R}}_{CM} = \vec{V}_{CM} = \frac{\vec{F}\Delta t}{2m}$$ - it moves with constant linear velocity

So, still, it moves in the $\vec{F}$ direction.

$$\vec{V}_{CM} = \left(\frac{F\cos\gamma\,\Delta t}{2m}, \frac{F\sin\gamma\,\Delta t}{2m}\right)$$ now it moves in x and y direction

- $\vec{F}$ also provides a torque which causes a change in angular momentum

$$\dot{\vec{L}} = \vec{\tau}_{ext} = \vec{r}\times\vec{F} = b(-\hat{x})\times\left(F_x(+\hat{x}) + F_y(\hat{y})\right) = \left\| \begin{matrix} F_x = F\cos\gamma \\ F_y = F\cdot\sin\gamma \end{matrix} \right\| =$$

$$= -bF_y\cdot\hat{z} = -b\cdot F\cdot\sin\gamma\cdot\hat{z}$$

$$\int_{\vec{L}(t=0)=0}^{\vec{L}} d\vec{L} = \int_{t=0}^{\Delta t} -bF\sin\gamma\,\hat{z}$$

$$\vec{L} = -bF\sin\gamma\,\hat{z}\,\Delta t = I\cdot\vec{\omega} = (mb^2+mb^2)\vec{\omega} = 2mb^2\cdot\vec{\omega}$$

$$\vec{\omega} = -\frac{bF\sin\gamma\,\Delta t}{2mb^2}\hat{z}$$

$$\vec{\omega} = (-\hat{z})\cdot\left[\frac{F\sin\gamma\,\Delta t}{2mb}\right]$$ - it rotates with constant angular velocity in clockwise direction (right-hand rule)

- So, the system motion can be represented by two motions: translational motion of the CM and rotational motion around the CM
- Velocities of the bodies:

$$\vec{V}_{left} = \vec{V}_{CM} + \underbrace{(\vec{\omega}\times\vec{r})}_{\text{linear velocity due to rotation}} = \vec{V}_{CM} + \omega(-\hat{z})\times(-\hat{x})b =$$

$$= \vec{V}_{CM} + \omega b\,\hat{y}$$

$$(V_{left})_y = \frac{F\cdot \sin\gamma\,\Delta t}{2m} + \frac{F\cdot \sin\gamma\,\Delta t}{2mb}\cdot b = \frac{F\cdot \sin\gamma\,\Delta t}{m}$$

$$(V_{left})_x = V_{CM_x} = \frac{F\cdot \cos\gamma\,\Delta t}{2m}$$

$$\vec{V}_{Right} = \vec{V}_{CM} + (\vec{\omega}\times\vec{r}) = \vec{V}_{CM} + \omega(-\hat{z})\times(\hat{x})\,b = \vec{V}_{CM} + \omega b(-\hat{y})$$

$$(V_{Right})_y = \frac{F\cdot \sin\gamma\,\Delta t}{2m} - \frac{F\cdot \sin\gamma\,\Delta t}{2mb}\cdot b = 0$$ (right after the impact the right body does not move in y direction)

$$(V_{Right})_x = \frac{F\cdot \cos\gamma\,\Delta t}{2m} - 0 = \frac{F\cdot \cos\gamma\cdot\Delta t}{2m}$$

- If $\gamma = 90^\circ$ (the case of Example 3.4)
  $\vec{V}_{CM} = \left(0, \frac{F\cdot\Delta t}{2m}\right)$; $\vec{\omega} = (-\hat{z})\frac{F\Delta t}{2mb}$; $\vec{V}_{left} = \left(0, \frac{F\Delta t}{2m}\right)$; $\vec{V}_{Right} = (0,0)$
  we got exactly the same answers.

- If $\gamma = 0$, the system was tapped in x direction.
  $\vec{V}_{CM} = \left(\frac{F\Delta t}{2m}; 0\right)$; $\vec{\omega} = (-\hat{z})\cdot 0 = 0$
  so the dumbbell moves in x direction without rotation.