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Exercises

1. Check the distributive laws for $\cup$ and $\cap$, and DeMorgan's laws.

2. Determine which of the following statements are true for all sets A, B, C, and D. If a double implication fails, determine whether one or the other of the possible implications holds. If an equality fails, determine whether one or the other of the possible inclusions holds.
   (a) $A \supset C$ and $B \supset C \Longleftrightarrow (A \cup B) \supset C$.
   (b) $A \supset C$ or $B \supset C \Longleftrightarrow (A \cup B) \supset C$.
   (c) $A \supset C$ and $B \supset C \Longleftrightarrow (A \cap B) \supset C$.
   (d) $A \supset C$ or $B \supset C \Longleftrightarrow (A \cap B) \supset C$.
   (e) $A - (A - B) = B$.
   (f) $A - (B - A) = A - B$.
   (g) $A \cap (B - C) = (A \cap B) - (A \cap C)$.
   (h) $A \cup (B - C) = (A \cup B) - (A \cup C)$.
   (i) $(A \cap B) \cup (A - B) = A$.
   (j) $A \subset C$ and $B \subset D \Longrightarrow (A \times B) \subset (C \times D)$.
   (k) The converse of (j).
   (l) The converse of (j), assuming that A and B are nonempty.
   (m) $(A \times B) \cup (C \times D) = (A \cup C) \times (B \cup D)$.
   (n) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.
   (o) $A \times (B - C) = (A \times B) - (A \times C)$.
   (p) $(A - B) \times (C - D) = (A \times C - B \times C) - A \times D$.
   (q) $(A \times B) - (C \times D) = (A - C) \times (B - D)$.

3. (a) Write the contrapositive and converse of the following statement: "If $x < 0$, then $x^2 - x > 0$," and determine which (if any) of the three statements are true.
   (b) Do the same for the statement "If $x > 0$, then $x^2 - x > 0$."

4. Let A and B be sets of real numbers. Write the negation of each of the following statements:
   (a) For every $a \in A$, it is true that $a^2 \in B$.
   (b) For at least one $a \in A$, it is true that $a^2 \in B$.
   (c) For every $a \in A$, it is true that $a^2 \notin B$.
   (d) For at least one $a \notin A$, it is true that $a^2 \in B$.

5. Let $\mathcal{A}$ be a nonempty collection of sets. Determine the truth of each of the following statements, and of their converses:
   (a) $x \in \bigcup_{A \in \mathcal{A}} A \Rightarrow x \in A$ for at least one $A \in \mathcal{A}$.
   (b) $x \in \bigcup_{A \in \mathcal{A}} A \Rightarrow x \in A$ for every $A \in \mathcal{A}$.
   (c) $x \in \bigcap_{A \in \mathcal{A}} A \Rightarrow x \in A$ for at least one $A \in \mathcal{A}$.
   (d) $x \in \bigcap_{A \in \mathcal{A}} A \Rightarrow x \in A$ for every $A \in \mathcal{A}$.

6. Write the contrapositive of each of the statements of Exercise 5.

7. Given sets A, B, and C. Express each of the following sets in terms of A, B, and C, using the symbols $\cup$, $\cap$, and $-$.
$$D = \{x \mid x \in A \text{ and } (x \in B \text{ or } x \in C)\},$$
$$E = \{x \mid (x \in A \text{ and } x \in B) \text{ or } x \in C\},$$
$$F = \{x \mid x \in A \text{ and } (x \in B \Rightarrow x \in C)\}.$$

8. If a set A has two elements, show that $\mathcal{P}(A)$ has four elements. How many elements does $\mathcal{P}(A)$ have if A has one element? Three elements? No elements? Why is $\mathcal{P}(A)$ called the power set of A?

9. Let R denote the set of real numbers. For each of the following subsets of $R \times R$, determine whether it is equal to the cartesian product of two subsets of R.
   (a) $\{(x, y) \mid x \text{ is an integer}\}$.
   (b) $\{(x, y) \mid 0 < y \leq 1\}$.
   (c) $\{(x, y) \mid y > x\}$.
   (d) $\{(x, y) \mid x \text{ is not an integer and } y \text{ is an integer}\}$.
   (e) $\{(x, y) \mid x^2 + y^2 < 1\}$.

Exercises 2

1. Let $f: A \longrightarrow B$. Let $A_0 \subset A$ and $B_0 \subset B$.
   (a) Show that $f^{-1}(f(A_0)) \supset A_0$ and that equality holds if f is injective.
   (b) Show that $f(f^{-1}(B_0)) \subset B_0$ and that equality holds if f is surjective.

2. Let $f: A \longrightarrow B$ and let $A_i \subset A$ and $B_i \subset B$ for $i = 0$ and $i = 1$. Show that f^{-1} preserves inclusions, unions, intersections, and differences of sets:
   (a) $B_0 \subset B_1 \Rightarrow f^{-1}(B_0) \subset f^{-1}(B_1)$.
   (b) $f^{-1}(B_0 \cup B_1) = f^{-1}(B_0) \cup f^{-1}(B_1)$.
   (c) $f^{-1}(B_0 \cap B_1) = f^{-1}(B_0) \cap f^{-1}(B_1)$.
   (d) $f^{-1}(B_0 - B_1) = f^{-1}(B_0) - f^{-1}(B_1)$.
   Show that f preserves inclusions and unions only:
   (e) $A_0 \subset A_1 \Rightarrow f(A_0) \subset f(A_1)$.
   (f) $f(A_0 \cup A_1) = f(A_0) \cup f(A_1)$.
   (g) $f(A_0 \cap A_1) \subset f(A_0) \cap f(A_1)$; give an example where equality fails.
   (h) $f(A_0 - A_1) \supset f(A_0) - f(A_1)$; give an example where equality fails.

3. Show that (b), (c), (f), and (g) of Exercise 2 hold for arbitrary unions and intersections.

4. Let $f: A \longrightarrow B$ and $g: B \longrightarrow C$.
   (a) If $C_0 \subset C$, show that $(g \circ f)^{-1}(C_0) = f^{-1}(g^{-1}(C_0))$.
   (b) If f and g are injective, show that $g \circ f$ is injective.
   (c) If $g \circ f$ is injective, what can you say about injectivity of f and g?
   (d) If f and g are surjective, show that $g \circ f$ is surjective.
   (e) If $g \circ f$ is surjective, what can you say about surjectivity of f and g?
   (f) Summarize your answers to (b)–(e) in the form of a theorem.

5. In general, let us denote the **identity function** for a set C by i_C. That is, define $i_C : C \longrightarrow C$ to be the function given by the rule $i_C(x) = x$ for all $x \in C$. Given $f: A \longrightarrow B$, we say that a function $g: B \longrightarrow A$ is a **left inverse** for f if $g \circ f = i_A$; and we say that $h: B \longrightarrow A$ is a **right inverse** for f if $f \circ h = i_B$.
   (a) Show that if f has a left inverse, f is injective; and if f has a right inverse, f is surjective.
   (b) Give an example of a function that has a left inverse but no right inverse.
   (c) Give an example of a function that has a right inverse but no left inverse.
   (d) Can a function have more than one left inverse? More than one right inverse?
   (e) Show that if f has both a left inverse g and a right inverse h, then f is bijective and $g = h = f^{-1}$.

6. Let $f: R \longrightarrow R$ be the function $f(x) = x^3 - x$. By restricting the domain and range of f appropriately, obtain from f a bijective function g. Draw the graphs of g and g^{-1}. (There are several possible choices for g.)

3. Let A be a set; let X be the two-element set $\{0, 1\}$. Show that there is a bijective correspondence between the set $\mathcal{P}(A)$ of all subsets of A and the cartesian product X^A.

4. (a) A real number x is said to be **algebraic** (over the rationals) if it satisfies some polynomial equation of positive degree

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$$

with rational coefficients a_i. Assuming that each polynomial equation has only finitely many roots, show that the set of algebraic numbers is countable.

(b) A real number is said to be **transcendental** if it is not algebraic. Assuming the reals are uncountable, show that the transcendental numbers are uncountable. (It is a somewhat surprising fact that only two transcendental numbers are familiar to us: e and π. Even proving these two numbers transcendental is highly nontrivial.)

5. Determine, for each of the following sets, whether or not it is countable. Justify your answers.
(a) The set A of all functions $f: \{0, 1\} \longrightarrow Z_+$.
(b) The set B_n of all functions $f: \{1, \ldots, n\} \longrightarrow Z_+$.
(c) The set $C = \bigcup_{n \in Z_+} B_n$.
(d) The set D of all functions $f: Z_+ \longrightarrow Z_+$.
(e) The set E of all functions $f: Z_+ \longrightarrow \{0, 1\}$.
(f) The set F of all functions $f: Z_+ \longrightarrow \{0, 1\}$ that are "eventually zero." [We say that f is **eventually zero** if there is a positive integer N such that $f(n) = 0$ for all $n \geq N$.]
(g) The set G of all functions $f: Z_+ \longrightarrow Z_+$ that are eventually 1.
(h) The set H of all functions $f: Z_+ \longrightarrow Z_+$ that are eventually constant.
(i) The set I of all two-element subsets of Z_+.
(j) The set J of all finite subsets of Z_+.

6. We say that two sets A and B **have the same cardinality** if there is a bijection of A with B.
(a) Show that if $B \subset A$ and if there is an injection

$$f: A \longrightarrow B,$$

then A and B have the same cardinality. [*Hint:* Define $A_1 = A$, $B_1 = B$, and for $n > 1$, $A_n = f(A_{n-1})$ and $B_n = f(B_{n-1})$. (Recursive definition again!) Note that $A_1 \supset B_1 \supset A_2 \supset B_2 \supset A_3 \supset \cdots$. Define $h: A \longrightarrow B$ by the rule

$$h(x) = \begin{cases} f(x) & \text{if } x \in A_n - B_n \text{ for some } n, \\ x & \text{otherwise.]} \end{cases}$$

(b) *Theorem (Schroeder–Bernstein theorem). If there are injections $f: A \longrightarrow C$ and $g: C \longrightarrow A$, then A and C have the same cardinality.*

7. Show that the sets D and E of Exercise 5 have the same cardinality.

8. Let X denote the two-element set $\{0, 1\}$; let $\mathcal{B}$ be the set of all *countable* subsets of X^ω. Show that X^ω and $\mathcal{B}$ have the same cardinality.

9. (a) The recursion formula

$$(*) \qquad \begin{aligned} h(1) &= 1, \\ h(2) &= 2, \\ h(n) &= [h(n+1)]^2 - [h(n-1)]^2 \quad \text{for } n \geq 2 \end{aligned}$$

is not one to which the principle of recursive definition applies. Show that nevertheless there does exist a function $h: Z_+ \longrightarrow R$ satisfying this formula. [*Hint:* Reformulate $(*)$ so that the principle will apply and require h to be positive.]

(b) Show that the formula $(*)$ of part (a) does not determine h uniquely. [*Hint:* If h is a positive function satisfying $(*)$, let $f(i) = h(i)$ for $i \neq 3$, and let $f(3) = -h(3)$.]

(c) Show that there is no function $h: Z_+ \longrightarrow R$ satisfying the recursion formula

$$\begin{aligned} h(1) &= 1, \\ h(2) &= 2, \\ h(n) &= [h(n+1)]^2 + [h(n-1)]^2 \quad \text{for } n \geq 2. \end{aligned}$$