# Aspects of the Topology of Interactions on Loop Dynamics in One and Two Dimensions

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**Abstract.** This paper discusses aspects of topology as relevant for loop dynamics as they occur in physical modeling synthesis algorithms. Boundary and interaction point behavior is treated purely from a topological perspective for some dynamical systems in one and two dimensions.

## 1 Introduction

The purpose of this paper is to discuss aspects of topological ideas to interaction models of loop dynamics.

The theoretical approach is related but generally somewhat different to waveguide-like arguments (as reference see for example [1]). Specifically, we emphasize the topological structure of the problem over the details of the dynamics. This way arguments are valid for any dynamical situation for which the given topological structure holds. In this case the structure is one of closed and directional loops or orbits. Hence the only assumption made about the dynamics is that disturbances propagate, in some otherwise unspecified fashion, along such trajectories, either directly or in some useful approximation. This idea is in fact not new, but rather has been developed since Poincaré as a core aspect of the contemporary theory of dynamical systems. The goal here is to study simple one and two-dimensional situations that dynamically are very well-behaved yet occur frequently in physical modeling of musical instruments. That is, the periodic trajectories will be assumed to be structurally stable under perturbation and hence we are only concerned with aspects of the dynamics that is regular and integrable.

First I will develop the one-dimensional situation and introduce projection, lifting and covering arguments with respect to point interactions. Based on this discussion I will try to illustrate the benefit of postponing spectral arguments as essential features of interaction dynamics can be shown easily using the presented arguments. In addition the arguments are not confined to a specific dynamic situation and hence have a more general application that when a dynamical operator and domain shapes are starting points of the discussion.

Then I will extend this treatment to two-dimensional dynamics for which the dynamics can be represented to fall into families of tori in some related parameter-space. The action of point excitation will be extended to this situation.

A note on notation and treatment. I will use conventional notation and nomenclature as customary in the topology literature as far as possible, though in general, discussion and illustration of the core ideas by pictures will take precedence over detailed notation. The notation used is informed by texts by Hatcher [2, chap. 1] and Jänich [3, chap 9].

## 2 Remarks on the Relationship to Exact Wave-Equations

The relationship of orbit dynamics to the exact wave-equation is a very interesting and important one. However, it is also one that is difficult and in general yet unsolved. I will make no pretense that any contribution is made here to narrow this gap. The topological ideas are straight-forward for a class of billiard problems. For short wave-lengths the many properties of the wave-equation on the domain can be studied well using the billiard model. See for example [4, Section 2.7]. This was known to Keller and Rubinow already as they set up billiard style path constructions. The general case and specifically the behavior at long wave-lengths is far less understood but it is interesting that the discrepancy reported by Keller and Rubinow and repeated by Brack and Bhaduri [5] is small (less than 3%) and the error vanishes quite quickly [6]. In case of banded waveguides this error in the spectrum is usually avoided by tuning the model to exact frequencies which in turn corresponds to accepting spatial discrepancies [7].

## 3 Topology of Loop Dynamics in One Dimension

First I want to discuss the case of one dimensions. With dimensions, unless otherwise noted, I will mean the spatial dimensions of a related dynamical domain from which the topological construction is derived.

Let's start of with a one-dimensional domain of a line interval $I$. One could write $I \subset \mathbb{R}$ but strictly speaking we are not interested in the the metric properties that can be interpreted in $\mathbb{R}$, rather we are only interested in its connectivity. This also means that the line doesn't necessarily need to be straight. On the line $I$ we will assume a waveguide-like dynamic of left and right-traveling disturbances, which travel along the line $I$ staying on it through reflections at the domain limits $\partial I$.

By imposing a Euclidean distance metric and a constant wave speed on $I$ with a suitable discrete spatial representation one would then immediately recover a standard loss-less waveguide model. Instead we want to study the properties in the absence of defined metric (length of string), wave speed (string tension and density). The advantage we gain are insights that hold for other situations as well, which share the same topology though potentially vastly different metrics, geometric layout and wave dynamics.

In the above description, the dynamics is uneventful except for the boundary points at which additional treatment has to be imposed. This special treatment can be generalized by seeing the one-dimensional line as a projection $\pi$ of a two-dimensional loop, for example the circle $\mathcal{S}^1$ into the line $I$. This projection can be visualized as a circle lying flat and its two branches overlapping in ones view. We will call the operation of finding a smooth connected path in a higher-dimensional space *lifting*. The circle as a closed path is everywhere smooth, though we will keep track of the former position of the reflection by keeping singular markers at reflection points. While in the $I$ an orbit apparently changes direction at the boundary $\partial I$ this is not true on the lifted path $S^1$, where an orbit never changes direction. If the reflection points do nothing but keep disturbance within $I$, this constitutes already the first purely topological representation of the dynamics, meaning that all dynamical changes have been converted into smooth connectivity properties. We will call the cover *trivial* if reflections do not add dynamical behavior to a lifted space. An example of a waveguide model of this type would be the Karplus-Strong model without loss-filter.

In order to study interaction behavior on the lifted topology, we will first introduce another scenario. Instead of assuming a simple lifted space, we will assume that the lift consists of two layers which we will call *sheets*. The number of sheets are sometimes called the *degree* of a cover and if disconnected are labeled by sets of integers for each path-connected component [8, p. 464]. To illustrate a situation that leads to non-trivial covers and hence more than one sheet, consider the case of a loss-less string or a loss-less waveguide structure with Dirichlet boundary conditions. In this case, disturbances reflect with a sign inversion.

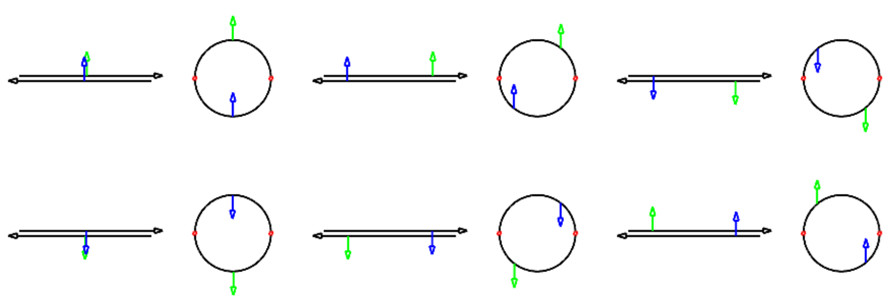

**Fig. 1.** Sequence of disturbance states on the line domain and its lifted circle domain.

Now let us place a disturbance of positive sign into the solution traveling in one direction and the same into the opposite direction. We trace the sign of the disturbance over time. At boundaries the impulses invert. We observe that the impulse traveling in one direction will always return with its sign inversed. Hence the two positive impulses trace two separate covers that are distinct and the total behavior can be seen as a projection of these distinct states onto the same covering space. See figure 2.

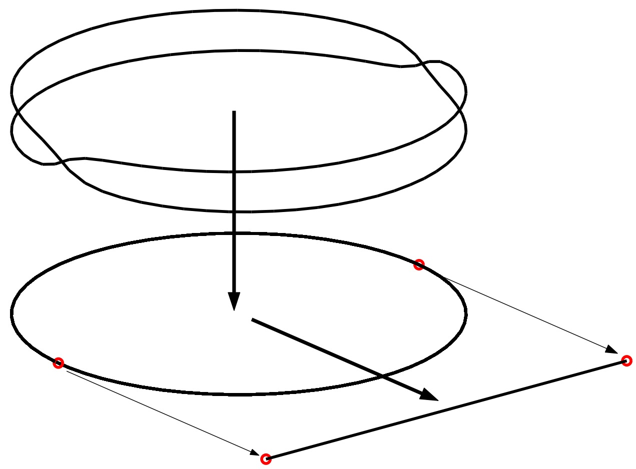

**Fig. 2.** The topology of two inverting reflections on a string.

To compare this with other boundary condition situations, we take Neumann conditions (no sign change at the boundary). Observe that we still have a non-trivial cover as positive and negative disturbance never share the same space. The cover maintains this separation everywhere as is depicted in Figure 3.

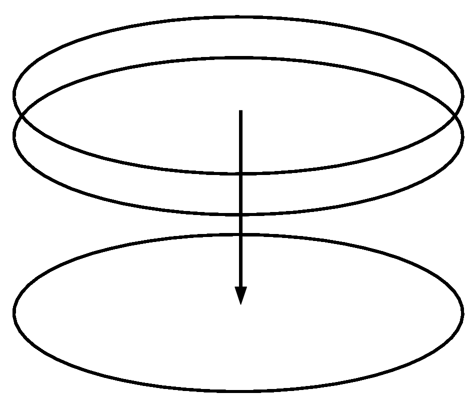

**Fig. 3.** The topology of two non-inverting reflections on a string or of a radial vibration of a cylinder.

This case is interesting because it can be used to visually prove the uniform velocity of a velocity excited string. The initial velocity distribution of some overall sign will be preserved in the lifted spaces and will be integrated over once per traversal through the covering loop, hence string will travel in the direction of the initial velocity distribution sign with the velocity defined by the traversal duration [9].

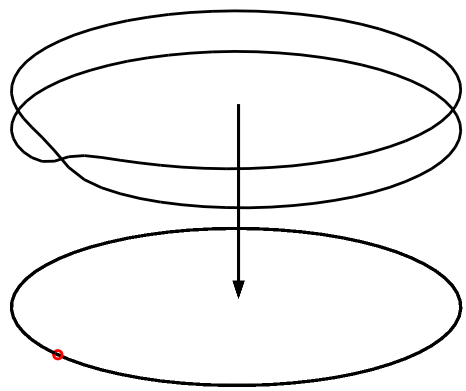

**Fig. 4.** The topology of one inverting reflections on a string.

A particularly interesting case is given by one Dirichlet boundary on one end combined with a Neumann condition on the other. In this case a disturbance will traverse the covering space twice to return to its original state. Hence, as opposed to the previous two cases the lifted space is thoroughly path-connected. See Figure 4. This implies the well-known fact that given the same metric and propagation behavior, this configuration has half the fundamental frequency. However we immediately also see that there are no two distinct sets of excitation configurations as opposed to the previous case. As a note (see [3, p. 57]), this topology is just the Möbius band which in turn can be described by the interval $I = [-1, 1]$ and the ends of the interval identified with a sign inversion $\alpha(i) = -i$, $i \in [-1, 1]$.

The degree of the cover the matched boundary conditions of either Neumann or Dirichlet type is the same, namely $\{1, 1\}$. The degree of the cover with mixed boundary conditions is $\{2\}$.

This already gives the topological result for the open pipe compared to the closed one. A disturbance needs to travel the loop twice to return to its origin in the same configuration and hence the wave-length is doubled for the pipe with one open end [10, p. 51].

Additionally the difference in degree between implies that all disturbance configurations can be reached by considering only one point in the lifted space of the mixed situation, whereas in the matched cases there are unreachable configurations. In practice one has however additional constraints on the excitations of the loops.

To describe the treatment we will assume that a notion of length is defined. This length describes how far disturbances travel over time. Additionally we will confine the current discussion to the situation that is familiar for the wave equation: Disturbances will travel to the left and right with equal contribution and with sign dependent on the particular variable chosen [1].

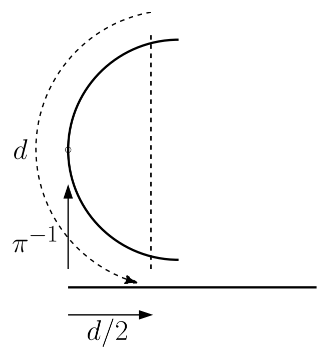


**Fig. 5.** Distance of coincidences to reflection points.

This is a situation where the projection of disturbance contribution in the lifted space *coincide*. Using this notion of distance we can then study when such coincidences will occur as disturbances propagate on the loops. Generally it is easy to see (for example in Figure 5) that disturbances coincide in the lifted space if they have the same distance from lifted domain boundaries. This situation can also be seen in Figure 1.

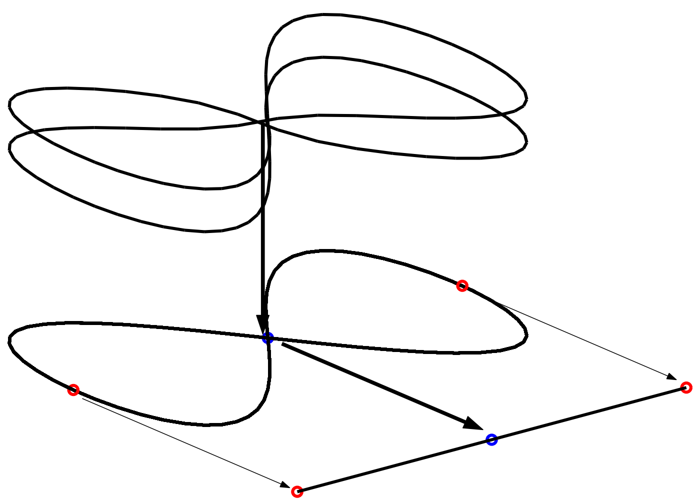

**Fig. 6.** Topology of disturbance coincidences at equal distance to both boundaries.

It also shows how an initial coincidence situation leads to another one after a half-rotation in the lifted space, reaching again a situation where the distance is symmetric and $d/2$ with respect to the second domain boundary. If the original interaction point was in the middle, i.e. $d/2$ both to the left and the right boundary. This is in fact a special situation. In Figure 6 we see the topology of the disturbance coincidence loop. It is a symmetric eight-shaped loop where the intersection point describes the lifted point of coincidence. In general the situation is somewhat more complicated.

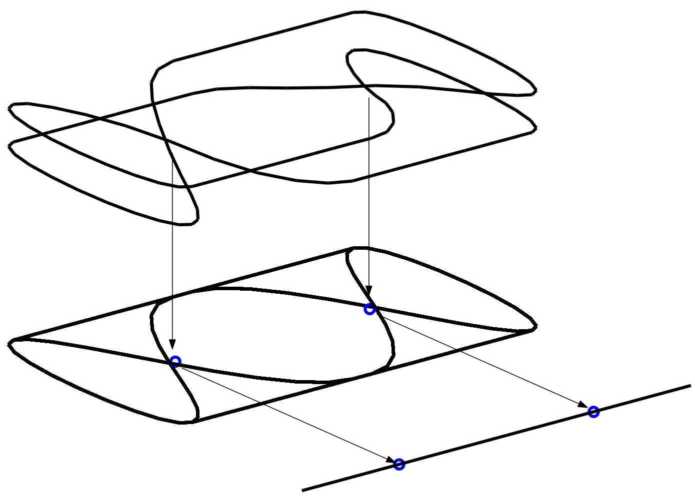

**Fig. 7.** Topology of disturbance coincidences at arbitrary distance to a boundary.

The complete situation can be seen in Figure 7. The double-sheet eight with one double-intersection splits into a double-sheet of two path-connected loops that cross at two separate coincidence points. The crossing[1] has the form depicted in Figure 11 (a1). A single path-connected loop is depicted in Figure 8.

As is to be expected, the lifted loop does not self-intersect, as a single traveling disturbance in only one direction would never find an intersection point with itself. The loop is changing elevation at coincidence point and hence unfolds an apparent region of coincidence in the plane cover.

Coincidence arguments are of interest as they allow to make "spectral-like" arguments that are short-lived. At points of coincidence a disturbance can be annihilated by matching it with an equal but sign-inverted disturbance. The conditions of coincidence is defined by the topology described above plus the loop speed between coincidence points, which here is a global property. In the

[1] We note that this crossing has the form of the Vassiliev knot invariant [8], but knot-theoretic treatment will be work of the future.

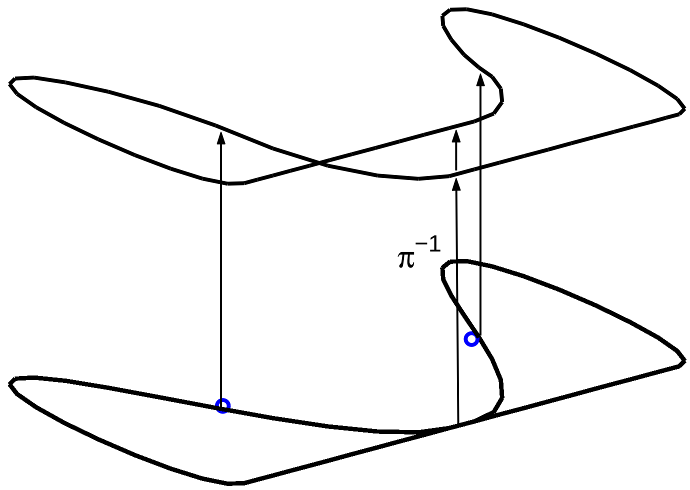


**Fig. 8.** One loop of the generic disturbance coincidence topology.

case of the center excitation such annihilation can happen every half-rotation at the same point, whereas otherwise it requires a full rotation for the configuration to return to the same spot. Similar arguments have been used to derive non-propagating excitations locally [11].

## 4 Topology of Loop Dynamics in Two Dimension

The situation naturally extends to two and more dimensions. Here we will only discuss this extension to two dimension because of its applicability to vibrating flat structures. It also already indicates how in general the extension behaves for higher dimensions.

In the one-dimensional case we lifted a line-domain into loops in two dimensions. In two dimensions we will principally be concerned with plane domains which will be lifted into loops in three dimensions. More precisely the concern is with a toroidal topology embedded in three dimensions. This is the flat, potentially layered 2-torus, meaning that the metric of the related dynamic is "flat" Euclidean space and that it is a 2-dimensional torus embedded in 3 dimensional space. The layers correspond to the sheets of circle covers discussed in the previous section.

The details how toroidal structures relate to dynamical situations has been known and has also been previously discussed in relation to physical models for sound synthesis using banded waveguides [12].

To quickly give an intuition of this relation, see Figure 9. A family of parallel rays will become straight through reflections if the plane is extended via mirror images. As the direction of the rays repeat after two reflections, these edges can be identified and form a tube, which we here depict to be have an extended

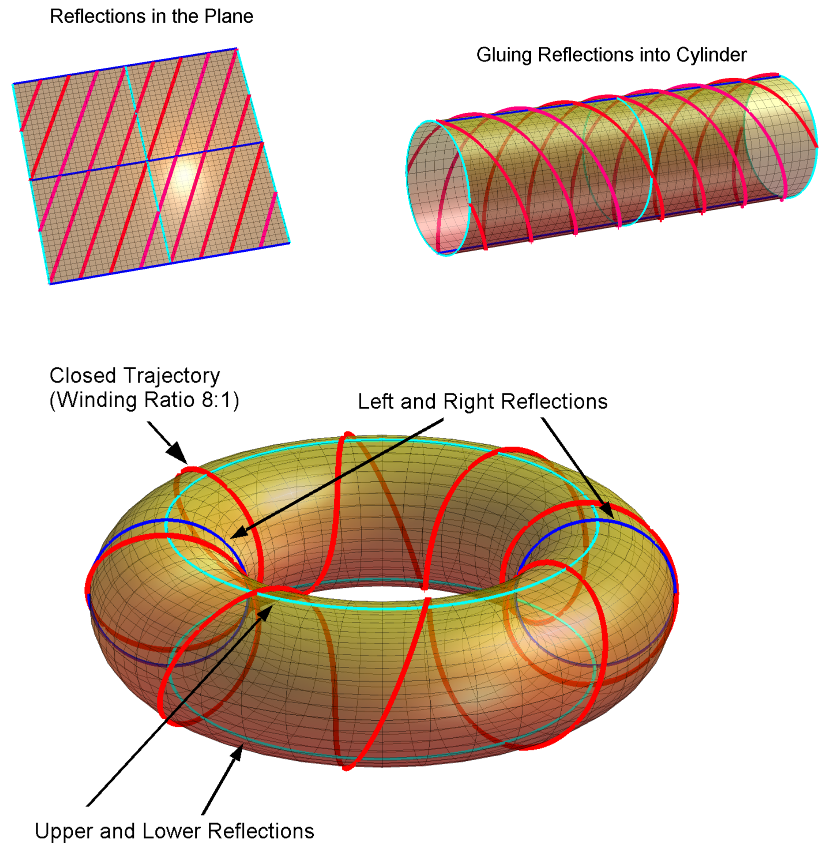


**Fig. 9.** Folding of path-connected families from a plane sheet into 2-torus by identifying edges representing reflections.

volume, though the related dynamics is still flat. With the same argument the ends of the tube can be identified after two reflections and then form a torus. If the number of windings in both dimensions of the torus are integer, ray paths will close and form loops.

For our purpose here, we will not consider the varied connections of such loops to dynamical systems and refer to [6, 12, 7] for further details.

The torus has a cover of the square as follows from the construction above. When projecting the torus back to the cover to undo the construction we see that the four sides of the torus occupy the same space. In the case of mapping the loops onto the line we usually encountered two loop branches to coincide on the line. Reflections on the other hand are only concerned with one point on the loop, namely where the loop intersects with the reflection circle. Similarly in one dimensions the loop intersected with the two reflection points.

Hence we can extend the treatment of reflection points simply to the two-dimensional case by noting that a reflection constitutes a change in sheet of the torus. Neumann boundary in two dimensions corresponds to a single-sheet, unlayered torus. If the total number of crossings with reflection circles in both toroidal dimensions are even then we get two separate crossing sheets whereas if it is odd, we get a single path-connected 2-sheet loop, all in analogy with the one-dimensional situation.

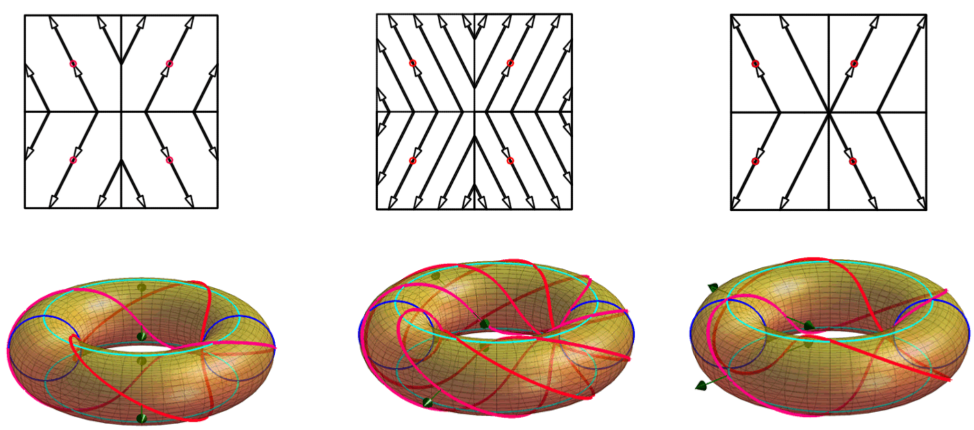

**Fig. 10.** Excitation point (center, left, far-left) and their loops. Top: cover. Bottom: torus.

Properties of placement of disturbances on the covering space can by seen in the depiction of Figure 10. In the case of center excitation and one off-center excitation we see that for the given winding numbers, the disturbances trace only two disjoint loops, whereas in the case of the other depicted off-center excitation one gets four disjoint loops. Clearly, these are the only two cases possible for integer winding numbers. On close inspection we see that in fact the center case and the off-center case with two disjoint loops differ in the way the excitation paths coincide. In the center case the disturbance contributions starting on the outside of the torus share the same loops, whereas in the off-center case the front outside shares with the back inside and the front inside disturbance contribution shares with the back outside. The respective conditions are captured in the following equations (1) and (2) with $n$ being the overall sum of winding numbers of the loop on the torus, $d_{1,2}$ is the distance across reflection and $n_{1,2}$ are winding counts along independent torus dimensions:

$$d_1 = n_1, \qquad d_2 = n_2, \qquad n_1, n_2 \in \mathbb{N}, \qquad n_1 + n_2 < n \tag{1}$$
$$d_1 = n_1, \qquad d_2 = (2n_2 - 1)/2, \qquad n_1, n_2 \in \mathbb{N}, \qquad n_1 + n_2 < n \tag{2}$$

The coincidence loops are somewhat more complicated to depict than the ones in one dimensions. Like in the one-dimensional case we expect a crossing between paths from all possible directions to occur at a point of excitation. Given that four directions (up-left, up-right, down-left, down-right) are possible on the torus, this also defines the number of directed lines forming the crossing at the coincidence point as is depicted in Figure 11 (b1).

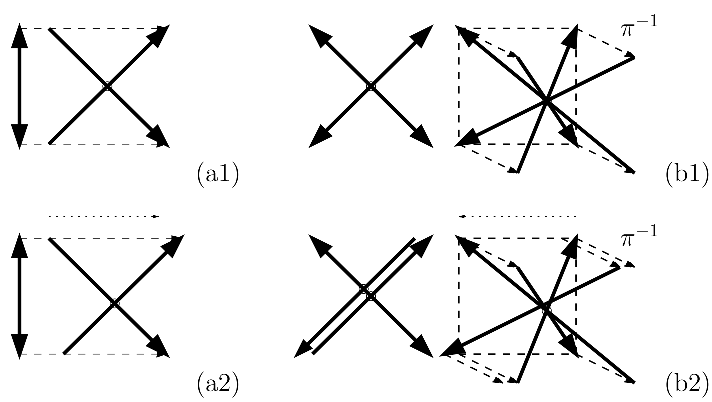


**Fig. 11.** The coincidence crossing in one and two dimensions. (a) shows the one dimensional two-path crossing and (b) shows the flat and lifted version of the four-path crossing of the two-dimensional case.

Depending on whether or not one of the conditions (1) and (2) hold, the coincidence point will intersect two or four otherwise disjoint loops away from it in both directions. The positions of coincidences cannot be as simply connected to reflections as in the one-dimensional case, but are easily observed on the torus topology. If we ascribe an overall loop-length, at least one such coincidence point will persist at the original location. Under what conditions more coincidence points exist is an open question. The length measures of all four path components need to match and coincide in the projected plane. Alternatively one can say that the knot doesn't persist under perturbation. This is illustrated in Figure 11 (b2) where one path is perturbed in the lifted space and then projected back. Notice that the coincidence does not persist. While in the one-dimensional case, equal length points from the reflection will always be forced by the projection into a point (see Figure 11 (a1-a2)), this is not the case in the plane. Hence it is a rather special condition as intersection points of trajectories constitute a sparse discrete set.

## 5 Conclusions

We discussed aspects of the topology of loop dynamics in one and two dimensions as they relate to effects of boundaries, excitation paths and disturbance coincidences of transportation type physical models for sound synthesis. Certain properties can be read immediately from such topologies without reference to the details of the dynamics involved.

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