Cohomology computations for Artin groups, Bestvina–Brady groups, and graph products

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Abstract

We compute:

- the cohomology with group ring coefficients of Artin groups (or actually, of their associated Salvetti complexes), Bestvina–Brady groups, and graph products of groups,
- the L^2-Betti numbers of Bestvina–Brady groups and of graph products of groups,
- the weighted L^2-Betti numbers of graph products of Coxeter groups.

In the case of arbitrary graph products there is an additional proviso: either all factors are infinite or all are finite. (However, for graph products of Coxeter groups this proviso is unnecessary.)

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Introduction

This paper concerns the calculation of the group cohomology, $H^*(G;N)$, for certain discrete groups G, where the G-module N is either $\mathbf{Z}G$ or a von Neumann algebra $\mathcal{N}(G)$. Here $\mathcal{N}(G)$ is a completion of the group algebra $\mathbf{R}G$ acting on $L^2(G)$, the Hilbert space of square summable functions on G. If G acts properly and cocompactly on a CW complex Y, then one has the reduced L^2-cohomology spaces, $L^2H^*(Y)$. These are Hilbert spaces with orthogonal G-actions. As such, each has a "von Neumann dimension" with respect to $\mathcal{N}(G)$. The dimension of $L^2H^n(Y)$ is the n^{th} *L^2-Betti number*, $L^2b^n(Y;G)$. When Y is acyclic, it is an invariant of G, denoted $L^2b^n(G)$. As shown in [28], these L^2-Betti numbers can be computed from the cohomology groups $H^*(G;\mathcal{N}(G))$ (i.e., the cohomology of BG with local coefficients in $\mathcal{N}(G)$).

For Coxeter groups, there is a refinement of the notion of L^2-Betti number. Suppose (W,S) is a Coxeter system and $\mathbf{p}$ is a multiparameter of positive real numbers (meaning that $\mathbf{p}$ is a function $S \to (0,\infty)$ which is constant on conjugacy classes). There is an associated "Hecke–von Neumann algebra," $\mathcal{N}_{\mathbf{p}}(W)$, which one can use to define the "weighted L^2-Betti numbers," $L^2_{\mathbf{p}}b^n(W)$, cf. [21], [15] or [11]. These numbers have been calculated provided $\mathbf{p}$ lies within a certain range, namely, when either $\mathbf{p}$ or $\mathbf{p}^{-1}$ lie within the region of convergence for the growth series of W (in [21], [15]).

Associated to (W,S), there is a finite CW complex X, called its *Salvetti complex*. It is homotopy equivalent to the quotient by W of the complement of the (possibly infinite) complex hyperplane arrangement associated to (W,S) (cf. [9]). The fundamental group of X is the *Artin group* A associated to (W,S). The $K(\pi,1)$ Conjecture for Artin groups asserts that X is a model for BA $(= K(A,1))$. This conjecture is known to hold in many cases, for example when W is either right-angled or finite.

Given a simplicial graph Γ with vertex set S and a family of groups $\{G_s\}_{s\in S}$, their *graph product*, $\prod_\Gamma G_s$, is the quotient of the free product of the G_s by the relations that elements of G_s and G_t commute whenever $\{s,t\}$ is an edge of Γ. Associated to Γ there is a flag complex L with 1-skeleton Γ. (L is defined by the requirement that a subset of S spans a simplex of L if and only if it is the vertex set of a complete subgraph of Γ.) A *right-angled Coxeter group* W_L (a RACG for short) is a graph product where each G_s is cyclic of order 2. Similarly, a *right-angled Artin group* (a RAAG for short) A_L is a graph product where each G_s is infinite cyclic. An arbitrary graph product of groups is an example of a right-angled building of type (W_L, S). In §5.2 we consider a family of arbitrary Coxeter systems $\{(V_s, T_s)\}_{s\in S}$. Their graph product, $V := \prod_\Gamma V_s$, gives a Coxeter system (V, T) (where T denotes the disjoint union of the T_s).

Given a right-angled Artin group A_L, let $\pi : A_L \to \mathbf{Z}$ be the homomorphism which sends each standard generator to 1. Put $BB_L := \operatorname{Ker}\pi$. In [4] Bestvina and Brady prove that BB_L is type FP if and only if the simplicial complex L is acyclic. If this is the case, BB_L is called a *Bestvina–Brady group*.

A subset of S (the set of generators of a Coxeter group W) is *spherical* if it generates a finite subgroup of W. Let $\mathcal{S}(W, S)$ denote the poset of spherical subsets of S and let K be the geometric realization of this poset. For each spherical subset J, let K_J (resp. ∂K_J) be the geometric realization of the subposet $\mathcal{S}(W, S)_{\geq J}$ (resp. $\mathcal{S}(W, S)_{>J}$). K_J is the cone on ∂K_J. (∂K is the barycentric subdivision of the nerve of (W, S); ∂K_J is the barycentric subdivision of the link of the simplex corresponding to J in the nerve.)

Many of the following computations are done by using a spectral sequence associated to a double complex. The E_∞ terms of such a spectral sequence compute the graded group, $\operatorname{Gr} H^*(\ )$, associated to the corresponding filtration of the cohomology group, $H^*(\ )$, in question.

In item (1) below, the Coxeter system is arbitrary while in (2), (3), (4) it is right-angled. (Within parentheses we refer either to the theorem in this paper where the computation appears or else we give a reference to the literature.) Here are the computations.

(1) Suppose A is the Artin group associated to a Coxeter system (W, S), X is the associated Salvetti complex and $\widetilde{X}$ is its universal cover. Then

  (a) (Theorem 4.1).

$$\operatorname{Gr} H^n(X; \mathbf{Z}A) = \bigoplus_{J\in\mathcal{S}(W,S)} H^{n-|J|}(K_J, \partial K_J) \otimes H^{|J|}(A_J; \mathbf{Z}A).$$

  (In the case of a RAAG this formula is the main result of [26].)

  (b) ([17, Cor. 2]).

$$L^2 b^n(\widetilde{X}; A) = b^n(K, \partial K).$$

  (Here $b^n(Y, Z)$ is the ordinary Betti number of a pair (Y, Z), i.e., $b^n(Y, Z) := \dim H^n(Y, Z; \mathbf{R})$.)

When the $K(\pi, 1)$ Conjecture holds for A, these formulas compute, $\operatorname{Gr} H^*(A; \mathbf{Z}A)$ and $L^2b^*(A)$, respectively.

(2) Suppose BB_L is the Bestvina–Brady subgroup of a RAAG, A_L.

(a) (Theorem 4.3). The cohomology of BB_L with group ring coefficients is isomorphic to that of A_L shifted up in degree by 1:

$$\operatorname{Gr} H^n(BB_L; \mathbf{Z}BB_L) = \bigoplus_{J \in \mathcal{S}(W,S)} H^{n-|J|+1}(K_J, \partial K_J) \otimes \mathbf{Z}(A_L/A_J).$$

(b) (Theorem 4.4). The L^2-Betti numbers of BB_L are given by

$$L^2b^n(BB_L) = \sum_{s \in S} b^n(K_s, \partial K_s),$$

(3) Suppose $G = \prod_\Gamma G_s$ is a graph product of groups, and let (W, S) be the RACS associated to the graph. For each spherical subset J, G_J denotes the direct product $\prod_{s \in J} G_s$.

(a) (Theorem 4.5). If each G_s is infinite, then

$$\operatorname{Gr} H^n(G; \mathbf{Z}G) = \bigoplus_{\substack{J \in \mathcal{S}(W,S) \\ i+j=n}} H^i(K_J, \partial K_J; H^j(G_J; \mathbf{Z}G)).$$

(b) (Theorem 4.6). If each G_s is infinite, then

$$L^2b^n(G) = \bigoplus_{\substack{J \in \mathcal{S}(W,S) \\ i+j=n}} b^i(K_J, \partial K_J) \cdot L^2b^j(G_J).$$

(c) ([13, Corollary 9.4]). If each G_s is finite, then

$$\operatorname{Gr} H^n(G; \mathbf{Z}G) = \bigoplus_{J \in \mathcal{S}(W,S)} H^n(K, K^{S-J}) \otimes \hat{A}^J,$$

Here K^{S-J} denotes the union of subcomplexes K_s, with $s \in S - J$, and $\hat{A}^J$ is a certain (free abelian) subgroup of $\mathbf{Z}(G/G_J)$.

(d) ([15, Theorem 13.8] and Corollary 6.5 below). Again, when each G_s is finite, G is a right-angled building of type (W, S) and its L^2-Betti numbers are determined by the weighted L^2-Betti numbers of (W, S) via the formula, $L^2b^n(G) = L^2_{\mathbf{p}}b^n(W)$, where the multiparameter $\mathbf{p} = (p_s)_{s\in S}$ is defined by $p_s = |G_s| - 1$.

(4) Suppose that V is a graph product of Coxeter groups $\{V_s\}_{s\in S}$, that (W, S) is the RACS associated to the graph and that $\mathbf{q}$ is multiparameter for (V, T). There are two results concerning the weighted L^2-Betti numbers of (V, T).

(a) (Theorem 7.2). Suppose $\mathbf{q}$ is "large" in the sense that it does not lie in the region of convergence for the growth series of any component group V_s. Then

$$L^2_{\mathbf{q}}b^n(V) = \sum_{\substack{i+j=n \\ J\in\mathcal{S}}} b^i(K_J, \partial K_J) \cdot L^2_{\mathbf{q}}b^j(V_J),$$

(b) (Theorem 7.7). For V as above and $\mathbf{q}$ "small" in the sense that it lies in the convergence region of each V_s, then

$$L^2_{\mathbf{q}}b^*(V) = L^2_{\mathbf{p}}b^*(W),$$

where the multiparameter $\mathbf{p}$ for W is defined by $p_s = V_s(\mathbf{q}) - 1$.

The calculations in (1), (2), (3a), (3b) and (4a) follow a similar line. They are based on the spectral sequence developed in §2. In all cases we are computing some type of cohomology of a CW complex Y, which is covered by a family of subcomplexes $\{Y_J\}_{J\in\mathcal{S}(W,S)}$, indexed by $\mathcal{S}(W, S)$. For a fixed j, the $E_1^{*,j}$ terms of the spectral sequence form a cochain complex for a nonconstant coefficient system on K, which associates to a simplex σ of K with minimum vertex J the group $H^j(Y_J)$. We first show that the spectral sequence decomposes at E_2 as a direct sum, with a component for each $J \in \mathcal{S}(W, S)$, and with the J-component consisting of a cochain complex of the form $C^*(K_J, \partial K_J)$ with some constant system of coefficients. We then show the spectral sequence collapses at E_2.

Since $\mathbf{Z}G$ is a G-bimodule, both sides of the formulas in (1a), (2a) and (3a) are right G-modules. One can ask if these formulas give isomorphisms of G-modules. The spectral sequence argument shows that this is indeed the case. Moreover, since the right hand side of each formula is finitely generated as a G-module, so is the left hand side (cf. [14]). In general, if we change the $\operatorname{Gr} H^n(-)$ on the left hand side to $H^n(-)$, then these formulas do not give isomorphisms of G-modules. For example, (3a) is not valid after this change in the case where G is the free product of two infinite cyclic groups (cf.[14, Example 5.2]). However, the possibility remains that after dropping the Gr's, both sides of these equations are still isomorphic as $\mathbf{Z}$-modules, which leads us to the following question.

Question. *On the left hand sides of the the formulas in (1a), (2a) and (3a), is* $\operatorname{Gr} H^n(-)$ *always isomorphic to* $H^*(-)$ *as an abelian group?*

The calculations in (3c), (3d) and (4b) come from a different direction based on [13] and [15]. In particular, the proof of (4b) uses an argument, similar to one in [15], relating the ordinary L^2-cohomology of a building to the weighted L^2-cohomology of its Coxeter system.

1 Basic definitions

1.1 Coxeter groups, Artin groups, buildings

Throughout this paper, we will be given as data a simplicial graph Γ with finite vertex set S and edge set, $\operatorname{Edge}(\Gamma)$, together with a labeling of the edges by integers ≥ 2. The label corresponding to $\{s,t\} \in \operatorname{Edge}(\Gamma)$ is denoted by $m(s,t)$. These data give a presentation of a *Coxeter group* W with generating set S and relations:

$$s^2 = 1, \ (st)^{m(s,t)} = 1, \quad \text{for all } s \in S \text{ and } \{s,t\} \in \operatorname{Edge}(\Gamma).$$

The pair (W,S) is a *Coxeter system*; Γ is its *presentation graph*. These same data determine a presentation for an *Artin group* A with generating set $\{a_s\}_{s\in S}$ and relations,

$$a_s a_t \cdots = a_t a_s \cdots, \qquad \text{for all } \{s,t\} \in \operatorname{Edge}(\Gamma),$$

where there are $m(s,t)$ terms on each side of the equation. Given a word $\mathbf{s} = (s_1, \ldots, s_n)$ in S, its *value* is the element $w(\mathbf{s})$ of W defined by

$$w(\mathbf{s}) := s_1 \cdots s_n.$$

The word $\mathbf{s} = (s_1, \dots, s_n)$ is a *reduced expression* if $\mathbf{s}$ is a word of minimum length for its value. The pair (W_J, J) is also a Coxeter system (cf. [11, Thm. 4.1.6]). For any $J \subseteq S$, the *special subgroup* W_J is the subgroup generated by J. The subset J is *spherical* if $|W_J| < \infty$. Let $\mathcal{S}(W, S)$ denote the poset of spherical subsets of S. The nonempty elements of $\mathcal{S}(W, S)$ form an abstract simplicial complex $L(W, S)$, called the *nerve* of the Coxeter system. ($\mathrm{Vert}(L(W, S)) = S$ and a nonempty subset $J \subseteq S$ spans a simplex if and only if it is spherical.)

Given a subset $I \subseteq S$, an element $w \in W$ is *I-reduced* if $l(sw) > l(w)$ for all $s \in I$. For $I \subseteq J \subseteq S$, let W_J^I be the subset of all I-reduced elements in the special subgroup W_J. (For example, $W_J^{\emptyset} = W_J$ and $W_J^J = \{1\}$.)

A *chamber system over a set* S is a set $\mathcal{C}$ (of "chambers") and a family of equivalence relations $\{\sim_s\}_{s \in S}$ on $\mathcal{C}$ indexed by S. (There is one equivalence relation for each $s \in S$.) An s-equivalence class is an *s-panel.* Distinct chambers $C, D \in \mathcal{C}$ are *s-adjacent* if they belong to the same s-panel. A *gallery* in $\mathcal{C}$ is a sequence $\mathbf{C} = (C_0, \dots, C_n)$ of adjacent chambers. Its *type* is the word $\mathbf{s} = (s_1, \dots, s_n)$ in S where the i^{th} letter of $\mathbf{s}$ is s_i if C_{i-1} be s_i-adjacent to C_i. A *building of type* (W, S) is a chamber system $\mathcal{C}$ over S equipped with a function $\delta : \mathcal{C} \times \mathcal{C} \to W$ (called a *Weyl distance*) such that

(1) Each panel contains at least two elements.

(2) Given a reduced expression $\mathbf{s}$ and chambers $C, D \in \mathcal{C}$, there is a gallery of type $\mathbf{s}$ from C to D if and only if $\delta(C, D) = w(\mathbf{s})$.

(The above definition of building is due to Ronan and Tits, a variant can be found in [1].) The building $\mathcal{C}$ is *locally finite* if each panel is finite.

Example. A Coxeter group W can be given the structure of a chamber system by declaring the s-panels to be the left cosets, wW_s, where W_s ($= W_{\{s\}}$) is the cyclic group of order two generated by s. Define $\delta_W : W \times W \to W$ by $\delta(w, w') = w^{-1}w'$. Then (W, δ_W) is a building, called the *standard thin building of type* (W, S).

Given a building $(\mathcal{C}, \delta)$ of type (W, S) and a subset J of S, the *J-residue* containing a chamber C is the subset $R_J(C) \subset \mathcal{C}$ defined by

$$R_J(C) := \{D \in \mathcal{C} \mid \delta(C, D) \in W_J\}.$$

In other words, a J-residue is a "J-gallery connected component of $\mathcal{C}$." If $J = \emptyset$, then a J-residue is simply a chamber and if J has only one element, then a J-residue is a panel. In the standard thin building (W, δ_W), a J-residue is a left coset of W_J.

1.2 Posets, simplicial complexes, flag complexes

Suppose L is a simplicial complex with vertex set S and let $\mathcal{S}(L)$ denote the poset of (vertex sets of) simplices in L (including the empty simplex). If J is the vertex set of a simplex σ in L, then denote by $\mathrm{Lk}(J, L)$ (or simply $\mathrm{Lk}(J)$ when L is understood), the link of σ in L. The abstract simplicial complex $\mathrm{Lk}(J)$ has one simplex for each element of $\mathcal{S}(L)_{>J}$ $(= \{J' \in \mathcal{S}(L) \mid J' \supset J\})$.

A simplicial complex L is a *flag complex* if any finite, nonempty set of vertices, which are pairwise connected by edges, spans a simplex of L. A simplicial graph Γ determines a flag complex, $L(\Gamma)$: its simplices are the finite, nonempty sets of vertices which are pairwise connected by edges.

Suppose $\mathcal{P}$ is a poset. There is an abstract simplicial complex $\mathrm{Flag}(\mathcal{P})$ with vertex set $\mathcal{P}$ and with simplices the totally ordered, finite, nonempty subsets of $\mathcal{P}$. We note that $\mathrm{Flag}(\mathcal{P})$ is a flag complex. Given a simplex $\sigma \in \mathrm{Flag}(\mathcal{P})$, its least element is its *minimum vertex* and is denoted by $\min(\sigma)$. If L is a simplicial complex, then $\mathrm{Flag}(\mathcal{S}(L)_{>\emptyset})$ can be identified with the barycentric subdivision of L. Similarly, $\mathrm{Flag}(\mathcal{S}(L))$ is the cone on the barycentric subdivision of L. (The empty set provides the cone point.)

1.3 Davis complexes and Salvetti complexes.

Let M be a topological space. A *mirror structure* on M *over a set* S is a family of subspaces $\{M_s\}_{s\in S}$ indexed by S; M_s is the s*-mirror* of M. If M is CW complex and each M_s is a subcomplex, then M is *mirrored CW complex.* For each $x \in M$,

$$S(x) := \{s \in S \mid x \in M_s\}. \tag{1.1}$$

If M is a mirrored CW complex and c is a cell in M, then

$$S(c) := \{s \in S \mid c \subset M_s\}.$$

Given a building $\mathcal{C}$ of type (W, S) and mirrored CW complex M over S, define an equivalence relation $\sim$ on $\mathcal{C} \times M$ by $(C, x) \sim (C', x')$ if and only if $x = x'$ and C, C' belong to the same $S(x)$-residue. Give $\mathcal{C}$ the discrete topology, $\mathcal{C} \times M$ the product topology, and denote the quotient space by

$$\mathcal{B}(\mathcal{C}, M) := (\mathcal{C} \times M)/\sim \tag{1.2}$$

and call it the *M-realization* of $\mathcal{C}$. When $\mathcal{C}$ is the standard thin building W, put

$$\mathcal{U}(W, M) := \mathcal{B}(W, M) \tag{1.3}$$

and call it the *M-realization* of the Coxeter system (W, S).

Remark. In our previous work, e.g., in [11], we used the notation $\mathcal{U}(\ ,\ )$ to denote a topological realization of either a building or of a Coxeter system. However, in what follows we will study Coxeter systems (V, T) which will also have the structure of a RAB over an auxiliary RACS (W, S) and we will want to distinguish the two types of realizations of (V, T) (either as a Coxeter system or as a building).

The mirror structure is *W-finite* if $S(x)$ is spherical for each $x \in M$. (In this paper we shall always assume this.) When this is the case, $\mathcal{U}(W, M)$ is a locally finite complex and similarly, if $\mathcal{C}$ is locally finite building, then $\mathcal{B}(\mathcal{C}, M)$ is a locally finite complex.

We will use the following notation for unions and intersections of mirrors, for any $J \subseteq S$, put

$$M_J := \bigcap_{s \in J} M_s, \qquad M^J := \bigcup_{s \in J} M_s. \tag{1.4}$$

Also, we will write ∂M_J for the subset of M_J consisting of all points $x \in M$ such that $S(x)$ is a proper subset of J.

As in §1.2, $\mathcal{S}(W, S)$ (or simply $\mathcal{S}$) is the poset of spherical subsets of S (including the empty set). The geometric realization of this poset is the simplicial complex $\mathrm{Flag}(\mathcal{S}(W, S))$. We denote it by $K(W, S)$ (or simply K) and call it the *Davis chamber*. Most often we will want to take $M = K$. One gets a mirror structure on K by defining K_s to be the geometric realization of the subposet $\mathcal{S}_{\geq \{s\}}$. Then $\mathcal{U}(W, K)$ is the *Davis complex* and $\mathcal{B}(\mathcal{C}, K)$ is the *standard realization* of $\mathcal{B}$. By construction, the W-action on $\mathcal{U}(W, K)$ is proper (i.e., each isotropy subgroup is finite) and the quotient space K is a finite complex, hence, compact. Moreover, $\mathcal{U}(W, K)$ is contractible (by [11, Thm. 8.2.13]). Note that for any $J \in \mathcal{S}$, the subcomplex ∂K_J is the barycentric subdivision $\mathrm{Lk}(J)$ (the link in L of the simplex with vertex set J). Also, K_J is the cone on ∂K_J (i.e., $K_J \cong \mathrm{Cone}(\mathrm{Lk}(J))$).

The Salvetti complex. Let A be the Artin group associated to (W, S) and let $q : A \to W$ denote the natural homomorphism sending a_s to s. There is a set-theoretic section for q, denoted by $w \mapsto a_w$, defined as follows: if $s_1 \cdots s_n$ is any reduced expression for w, then $a_w := a_{s_1} \cdots a_{s_n}$. As explained in [8, p. 602], it follows from Tits' solution to the word problem for Coxeter groups that $w \mapsto a_w$ is well-defined.

Define a partial order on $W \times \mathcal{S}$ by $(w, I) < (v, J)$ if and only if $I < J$ and $v^{-1}w \in W_J^I$ (where W_J^I was defined in §1.1). It is proved in [9] that $W \times \mathcal{S}$ is the poset of cells of a cell complex X' on which W acts freely so that each cell of X' is a Coxeter cell. (Flag($W \times \mathcal{S}$) is the barycentric subdivision of X'.) The quotient space $X := X'/W$ is the *Salvetti complex* of (W, S). It is known that $\pi_1(X) = A$. The universal cover of X is denoted by $\widetilde{X}$. For each $(w, J) \in W \times \mathcal{S}$, the flag complex on $(W \times \mathcal{S})_{\leq (w,J)}$ can be identified with the barycentric subdivision of a Coxeter cell of type (W_J). (A *Coxeter cell* of type W_J means the convex hull of a generic orbit in the canonical representation of W_J; see [11, §7.3].) So, X' (or X)) can be given the structure of a CW complex where the cells are Coxeter cells. In particular, each vertex of X' corresponds to an element of $W \times \mathcal{S}$ of the form $(w, \emptyset)$ and each 1-cell of X' corresponds to an element of the form $(w, \{s\})$. Orient this edge by declaring $(w, \emptyset)$ to be its initial vertex and $(ws, \emptyset)$ its terminal vertex. Since the W-action preserves edge orientations, these orientations pass to the edges of X. Call a vertex x of a cell C of X a *top vertex* of C if each edge of C which contains x points away from x (cf. [17, §7]). One can then explicitly describe CW structure on X as follows. For each $J \in \mathcal{S}$, take a Coxeter cell C_J of type W_J. Now for each $I < J$ and each $u \in W_J^I$ glue together C_I and uC_I via the homeomorphism induced by u. The result is denoted X_J. (It is the Salvetti complex for A_J and therefore, a $K(A_J, 1)$.) To construct X, start with the disjoint union of the X_J, for $J \in \mathcal{S}$, and then use the natural maps to identify X_I with a subcomplex of X_J whenever $I < J$. This description exhibits X as a "poset of spaces over $\mathcal{S}$" (as defined in §2). On the level of fundamental groups we know that the inclusion $X_J \to X$ induces the natural injection $A_J \to A$ and that the associated "simple complex of groups" is the one discussed in [8, §3]. Similarly, for each $J \in \mathcal{S}$, let $\widetilde{X}_J$ denote the inverse image of X_J in $\widetilde{X}$. ($\widetilde{X}_J$ is a disjoint union of copies of the universal cover X_J, one copy for each coset of A_J in A.) Thus, $\widetilde{X}$ also has the structure of a poset of spaces over $\mathcal{S}$.

In the right-angled case, the Salvetti complex has a simple description as a subcomplex of a torus (and we will denote it by T_L instead of X). Let T^S

denote the S-fold product of copies of S^1. Define

$$T_L := \bigcup_{J \in \mathcal{S}(L)} T^J. \tag{1.5}$$

(This is a special case of the "polyhedral product" construction discussed in the next subsection and in [2, 20].) According to [9], its universal cover $\widetilde{T}_L$ is a CAT(0)-cubical complex and hence, is contractible.

1.4 Graph products of groups and spaces

As in the Introduction, Γ is a simplicial graph and suppose each edge is labeled 2. Also, $S = \mathrm{Vert}(\Gamma)$, L is the flag complex determined by Γ and (W_L, S) is the associated RACS. (We shall generally reserve the notation W_L for the case when the Coxeter group is right-angled with nerve L and similarly, A_L for the RAAG associated to L.) Suppose $\{G_s\}_{s\in S}$ is a family of groups indexed by S. The *graph product* of the G_s, denoted $\prod_\Gamma G_s$, is the quotient of the free product of the G_s, $s \in S$, by the normal subgroup generated by all commutators of the form, $[g_s, g_t]$, where $\{s,t\} \in \mathrm{Edge}(\Gamma)$, $g_s \in G_s$ and $g_t \in G_t$. For example, if $\mathrm{Edge}(\Gamma) = \emptyset$, then $\prod_\Gamma G_s$ is the free product, while if Γ is the complete graph on S, then $\prod_\Gamma G_s$ is the direct sum.

Suppose $G = \prod_\Gamma G_s$. We want to see that G has the structure of a RAB of type (W_L, S). The group G can be given the structure of a chamber system as follows: the s-panels are the left cosets gG_s, with $g \in G$. Write G_s^* for $G_s - \{1\}$. The projections $G_s^* \mapsto s$ induce a map (*not a homomorphism*) $\pi : G \to W_L$, as follows: any element $g \in G$ can be written as $g_{s_1} \cdots g_{s_n}$, with $g_{s_i} \in G_{s_i}^*$ so that $s_1 \cdots s_n$ is a reduced expression for an element $w \in W$. Moreover, w depends only on g. The map π sends g to w. The Weyl distance $\delta : G \times G \to W_L$ is defined by $\delta(g,h) = \pi(g^{-1}h)$. The following lemma is easily checked.

Lemma 1.1 ([11, Ex. 18.1.10])**.** *(G, δ) is a building of type (W_L, S).*

Polyhedral products. Suppose, for the moment, that S is the vertex set of an arbitrary simplicial complex L. Let $\{(Z_s, A_s)\}_{s\in S}$ be a family of pairs of spaces indexed by S. For each $J \in \mathcal{S}(L)$, let Z'_J be the subspace of the product $\prod_{s\in S} Z_s$, consisting of all S-tuples $(x_s)_{s\in S}$ such that

$$x_s \in \begin{cases} Z_s, & \text{if } s \in J, \\ A_s & \text{if } s \notin J. \end{cases}$$

The *polyhedral product* of this family, denoted $\mathcal{Z}_L(Z_s, A_s)$, is defined to be the following subspace of $\prod_{s\in S} Z_s$:

$$\mathcal{Z}_L(Z_s, A_s) := \bigcup_{J\in\mathcal{S}(L)} Z'_J. \tag{1.6}$$

(This terminology comes from [2]. In [20] it is called a "generalized moment angle complex.")

Example 1.2. Suppose each $(Z_s, A_s) = ([0,1], 0)$. Then $\mathcal{Z}_L([0,1], 0)$ can be identified with $\mathrm{Flag}(\mathcal{S}(L))$ in such a fashion that a standard subdivision of each cube in the polyhedral product is a subcomplex of $\mathrm{Flag}(\mathcal{S}(L))$. In particular, if L is the nerve of a RACS, then $\mathcal{Z}_L([0,1], 0) = K$, the Davis chamber from §1.3.

Graph products of spaces. We return to the assumption that L is the flag complex determined by Γ.

Example 1.3. Suppose each $(Z_s, A_s) = (\mathrm{Cone}(G_s), G_s)$ for a family of discrete groups $\{G_s\}_{s\in S}$. The group G_s acts on $Z_s = \mathrm{Cone}(G_s)$ and $A_s = G_s$ is an invariant subspace. Let $G' := \prod_{s\in S} G_s$ denote the direct product. Then G' acts on $\prod_{s\in S} Z_s$ and $Z' := \mathcal{Z}_L(\mathrm{Cone}(G_s), G_s)$ is an invariant subspace. The quotient space Z'/G' can be identified with the chamber $K = \mathcal{Z}_L([0,1], 0)$ of the previous example. It is proved in [12] that the universal cover of Z' is the standard realization of a RAB. In particular, the group G of all lifts of elements in G' is the graph product, $G = \prod_\Gamma G_s$. An explanation for this, which is different from that in [12], is given in the following lemma.

Lemma 1.4. *With notation as in Example 1.3, the fundamental group of* $Z' = \mathcal{Z}_L(\mathrm{Cone}(G_s), G_s)$ *can be identified with the kernel of the natural surjection* $G = \prod_\Gamma G_s \to G' = \prod_{s\in S} G_s$*. Moreover, if* $Z \to Z'$ *is the corresponding covering space, then the* G'*-action on* Z' *lifts to a* G*-action on* Z*.*

Proof. First consider the special case where S consists of two elements s and t and Γ has no edges. Then $\mathrm{Cone}(G_s) \times \mathrm{Cone}(G_t)$ is a 2-complex and the polyhedral product Z' is the union of 1-cells which do not contain the product of the cone points. Such a 1-cell either has the form $\mathrm{Cone}(g_s) \times g_t$ or $g_s \times \mathrm{Cone}(g_t)$ for some $(g_s, g_t) \in G_s \times G_t$. These two 1-cells fit together to give a single edge $e(g_s, g_t) := (\mathrm{Cone}(g_s)\times g_t)\cup(g_s\times \mathrm{Cone}(g_t))$ connecting g_t to g_s. In this way we see that Z' is identified with (the barycentric subdivision of) the join of G_s and G_t, which we denote $G_s \odot G_t$. The group $G_s \times G_t$ acts on $G_s \odot G_t$ and the vertex stabilizers are either G_s or G_t. The universal cover of $G_s \odot G_t$ is a tree T. The group of all lifts of the $(G_s \times G_t)$-action is transitive on edges, and the quotient space is a single edge (with distinct vertices). Hence, the group of lifts is the free product $G_s * G_t$ and T is the corresponding Bass–Serre tree.

The general case follows in the same manner by considering the universal cover of the 2-skeleton of Z'. □

Next suppose that for each $s \in S$ we are given a path connected G_s-space Z_s and a basepoint $b_s \in Z_s$ lying in some free orbit. We can find a G_s-equivariant map of pairs $f_s : (Z_s, G_s b_s) \to (\mathrm{Cone}(G_s), G_s)$. We want to define a space $\prod_\Gamma(Z_s, G_s b_s)$, together with a G-action on it (where $G := \prod_\Gamma G_s$). It will be called the *graph product* of the $(Z_s, G_s b_s)$. The f_s induce a map, well-defined up to G'-equivariant homotopy, $f : \boldsymbol{\pi}_L(Z_s, G_s b_s) \to \boldsymbol{\pi}_L(\mathrm{Cone}(G_s), G_s)$. It is easy to see that f induces a surjection on fundamental groups. Pulling back the universal cover of $\boldsymbol{\pi}_L(\mathrm{Cone}(G_s), G_s)$, we get a covering space, $Z \to \boldsymbol{\pi}_L(Z_s, G_s b_s)$. We use the notation $\prod_\Gamma(Z_s, G_s b_s) := Z$ for this covering space. Notice that the G'-action on $\boldsymbol{\pi}_L(Z_s, G_s b_s)$ lifts to a G-action on Z. Also, notice that if each Z_s is simply connected, then Z is just the universal cover of the polyhedral product.

Example 1.5. Suppose $\{(B_s, p_s)\}_{s\in S}$ is a collection of path connected spaces with base points and that $B = \boldsymbol{\pi}_L(B_s, p_s)$ is the polyhedral product. For each $s \in S$, let $\pi_s : Z_s \to B_s$ be the universal cover and let $G_s = \pi_1(B_s, p_s)$. Let Z' denote the polyhedral product $\boldsymbol{\pi}_L(Z_s, \pi_s^{-1}(p_s))$. Then $Z' \to B$ is a regular covering space with group of deck transformations G' (the product of the G_s). It follows that the universal cover of Z' can be identified with the graph product of the $(Z_s, \pi_s^{-1}(p_s))$. Hence, $\pi_1(B)$ is the graph product of the G_s.

Example 1.6. Suppose $Z_s = EG_s$, the universal cover of the classifying space BG_s. A simple argument using induction on the number of elements of S (cf. [8, Remark p. 619]) shows that the polyhedral product of the (BG_s, p_s) is aspherical; hence, it is a model for BG and its universal cover $\prod_\Gamma(EG_s, G_s b_s)$ is EG.

Lemma 1.7. *(i) If each G_s acts properly on Z_s, then G acts properly on the graph product $\prod_\Gamma(Z_s, G_s b_s)$.*

(ii) If each Z_s is acyclic, then so is $\prod_\Gamma(Z_s, G_s b_s)$.

Proof. The proof of (i) is trivial. To prove (ii), consider the cover of Z ($:= \prod_\Gamma(Z_s, G_s b_s)$) by components of the inverse images of the $\{Z_J\}_{J\in\mathcal{S}(L)}$. By the Künneth Formula, each Z_J is acyclic and the same is true for each component of its inverse image (since such a component projects homeomorphically). There is a similar cover of $\prod_\Gamma(\mathrm{Cone}(G_s), G_s)$ with the same nerve. So, Z and $\prod_\Gamma(\mathrm{Cone}(G_s), G_s)$ have the same homology. Since $\prod_\Gamma(\mathrm{Cone}(G_s), G_s)$ is the standard realization of a building, it is contractible; hence, acyclic. Statement (ii) follows. $\square$

Remark. Probably the correct level of generality at which to define the graph product of a family of pairs of spaces is the following. Suppose we are given a family of pairs $\{(Z_s, A_s)\}_{s\in S}$, where each Z_s is path connected and where A_s is a not necessarily connected, closed nonempty subspace. Then $\prod_\Gamma(Z_s, A_s)$ can be defined in the following manner. First notice that Example 1.3 works when the discrete groups G_s are replaced by discrete spaces D_s. Let H denote the fundamental group of $\mathcal{Z}_L(\mathrm{Cone}(D_s), D_s)$. If $D_s = \pi_0(A_s)$ denotes the set of components of A_s, then, as before, we have maps $f_s : (Z_s, A_s) \to (\mathrm{Cone}(D_s), D_s)$. The f_s induce a map $f : \mathcal{Z}_L(Z_s, A_s) \to \mathcal{Z}_L(\mathrm{Cone}(D_s), D_s)$. Moreover, the induced map of fundamental groups $f_* : \pi_1(\mathcal{Z}_L(Z_s, A_s)) \to H$ is surjective. The corresponding covering space Z is called the *graph product* of the (Z_s, A_s) and is denoted by $\prod_\Gamma(Z_s, A_s)$. (Notice that if each A_s is connected, then the graph product is the polyhedral product, $\mathcal{Z}_L(Z_s, A_s)$.) In particular, this allows us to deal with the case where each A_s is a G_s-orbit (not necessarily a free G_s-orbit). So, suppose $A_s = G_s/H_s$. Then the group of lifts of the G'-action to the universal cover of $\mathcal{Z}_L(\mathrm{Cone}(G_s/H_s), G_s/H_s)$ is the "graph product of pairs," $\prod_\Gamma(G_s, H_s)$, defined previously in [25].

2 A spectral sequence

A *poset of coefficients* is a contravariant functor $\mathcal{A}$ from a poset $\mathcal{P}$ to the category of abelian groups (i.e., it is a collection $\{\mathcal{A}_a\}_{a\in\mathcal{P}}$ of abelian groups together with homomorphisms $\varphi_{ba} : \mathcal{A}_a \to \mathcal{A}_b$, defined whenever $a > b$, such that $\varphi_{ca} = \varphi_{cb}\varphi_{ba}$, whenever $a > b > c$). The functor $\mathcal{A}$ also gives us a system of coefficients on the cell complex Flag($\mathcal{P}$): it associates to the simplex σ the abelian group $\mathcal{A}_{\min(\sigma)}$. Hence, we get a cochain complex

$$C^j(\mathrm{Flag}(\mathcal{P}); \mathcal{A}) := \bigoplus_{\sigma \in \mathrm{Flag}(\mathcal{P})^{(j)}} \mathcal{A}_{\min(\sigma)}, \tag{2.1}$$

where $\mathrm{Flag}(\mathcal{P})^{(j)}$ means the set of j-simplices in Flag($\mathcal{P}$).

Given a CW complex Y, a *poset of spaces* in Y over $\mathcal{P}$ is a cover $\mathcal{V} = \{Y_a\}_{a \in \mathcal{P}}$ of Y by subcomplexes indexed by $\mathcal{P}$ so that if $N(\mathcal{V})$ denotes the nerve of the cover, then

(i) $a < b \implies Y_a \subset Y_b$, and

(ii) the vertex set Vert(σ) of each simplex of $N(\mathcal{V})$ has the greatest lower bound $\wedge\sigma$ in $\mathcal{P}$, and

(iii) $\mathcal{V}$ is closed under taking finite nonempty intersections, i.e., for any simplex σ of $N(\mathcal{V})$,
$$\bigcap_{a \in \sigma} Y_a = Y_{\wedge\sigma}.$$

Remark. Any cover leads to a poset of spaces by taking all nonempty intersections as elements of new cover, and removing duplicates. The resulting poset is the *set* of all nonempty intersections, ordered by inclusion.

Lemma 2.1 (cf. [23])**.** *Suppose $\mathcal{V} = \{Y_a\}_{a \in \mathcal{P}}$ is a poset of spaces for Y over $\mathcal{P}$. There is a Mayer–Vietoris type spectral sequence converging to $H^*(Y)$ with E_1-term:*
$$E_1^{i,j} = C^i(\mathrm{Flag}(\mathcal{P}); \mathcal{H}^j(\mathcal{V})),$$
and E_2-term:
$$E_2^{i,j} = H^i(\mathrm{Flag}(\mathcal{P}); \mathcal{H}^j(\mathcal{V})),$$
where the coefficient system $\mathcal{H}^j(\mathcal{V})$ is given by $\mathcal{H}^j(\mathcal{V})(\sigma) = H^j(Y_{\min(\sigma)})$.

Proof. We follow the line laid down in [6, Ch. VII, §3,4]. Consider the following double complex:

$$E_0^{i,j} = \bigoplus_{\substack{\sigma \in \mathrm{Flag}(\mathcal{P}) \\ \dim \sigma = i}} C^j(Y_{\min(\sigma)}), \tag{2.2}$$

where the differentials are defined as follows. The vertical differentials are direct sums of the differentials $d : C^j(Y_{\min(\sigma)}) \to C^{j+1}(Y_{\min(\sigma)})$. Similarly, the horizontal differentials are direct sums of homomorphisms $\delta : C^j(Y_{\min(\sigma)}) \to C^j(Y_{\min(\tau)})$ where the matrix entry corresponding to $\sigma\tau$ is $[\tau : \sigma]i_{\tau\sigma}$, where $[\tau : \sigma]$ is the incidence number and $i_{\tau\sigma} : C^j(Y_{\min(\sigma)}) \to C^j(Y_{\min(\tau)})$ is the restriction map if τ is a coface of σ and 0 otherwise. As in [6, p.165], there are two spectral sequences associated to the double complex. The first begins by taking vertical cohomology to get E_1 and then takes horizontal cohomology to get E_2. The differential on the E_r sheet has bidegree $(r, -r+1)$. The second spectral sequence begins with the horizontal differential so that the differential on the E_r sheet has bidegree $(-r+1, r)$

The usual inclusion-exclusion argument using properties (i)–(iii) of a poset of spaces shows that the rows of the double complex are exact, except when $i = 0$, where we get $C^j(Y)$ as the $E_1^{0,j}$-term of the second spectral sequence. This implies that the second spectral sequence collapses at E_2 and that the cohomology of the double complex is $H^*(Y)$ (cf. the exercise in [6, p.165]).

We can rewrite (2.2) as $E_0^{i,j} = C^i(\mathrm{Flag}(\mathcal{P}); \mathcal{C}^j(\mathcal{V}))$, where the coefficient systems are defined by $\mathcal{C}^j(\mathcal{V})(\sigma) = C^j(Y_{\min(\sigma)})$. So, the first spectral sequence is the one claimed in the lemma. □

For $a \in \mathcal{P}$, let $Y_{<a} := \bigcup_{b<a} Y_b$. For any maximal element $a \in \mathcal{P}$, put $Y_{\neq a} := \bigcup_{b \neq a} Y_b$. Consider the following two conditions on the poset of spaces.

- (Z′) For any $a, b \in \mathcal{P}$ with $b < a$, the induced homomorphism $H^*(Y_a) \to H^*(Y_b)$ is the zero map.
- (Z) For any $a \in \mathcal{P}$, the induced homomorphism $H^*(Y_a) \to H^*(Y_{<a})$ is the zero map.

Note that (Z) implies (Z′) since the map $H^*(Y_a) \to H^*(Y_b)$ factors through $H^*(Y_{<a})$.

Lemma 2.2. *Suppose $\mathcal{V} = \{Y_a\}_{a \in \mathcal{P}}$ is a poset of spaces for Y over $\mathcal{P}$.*

(i) *If (Z′) holds, then*

$$E_2^{i,j} = \bigoplus_{a \in \mathcal{P}} H^i(\mathrm{Flag}(\mathcal{P}_{\geq a}), \mathrm{Flag}(\mathcal{P}_{>a}); H^j(Y_a)).$$

(ii) *If (Z) holds, then the spectral sequence degenerates at* E_2 *and*

$$\operatorname{Gr} H^*(Y) = \bigoplus_{a \in \mathcal{P}} H^i(\operatorname{Flag}(\mathcal{P}_{\geq a}), \operatorname{Flag}(\mathcal{P}_{>a}); H^j(Y_a)).$$

Proof. We use the double complex from the proof of Lemma 2.1. The cochain group decomposes as a direct sum:

$$C^i(\operatorname{Flag}(\mathcal{P}); \mathcal{C}^j(\mathcal{V})) = \bigoplus_{a \in \mathcal{P}} C^i(\operatorname{Flag}(\mathcal{P}_{\geq a}), \operatorname{Flag}(\mathcal{P}_{>a}); C^j(Y_a)).$$

The vertical differentials at E_0 respect this decomposition, so at E_1 the spectral sequence always decomposes as a direct sum:

$$E_1^{i,j} = \bigoplus_{a \in \mathcal{P}} C^i(\operatorname{Flag}(\mathcal{P}_{\geq a}), \operatorname{Flag}(\mathcal{P}_{>a}); H^j(Y_a)).$$

In general, the differentials at E_1 do not respect this decomposition; however, condition (Z′) implies that they do, and therefore, the spectral sequence also decomposes at E_2:

$$E_2^{i,j} = \bigoplus_{a \in \mathcal{P}} H^i(\operatorname{Flag}(\mathcal{P}_{\geq a}), \operatorname{Flag}(\mathcal{P}_{>a}); H^j(Y_a)). \tag{2.3}$$

This proves (i).

Now suppose (Z) holds. By induction, we can assume that (ii) is true for all posets with fewer elements. If $z \in E_0^{i,j}$ is a vertical cocycle, then its higher differential is given by $d_r(z) = \delta(x_r)$, where $(x_0 = z, x_1, \ldots, x_r)$ is any sequence of elements satisfying $x_k \in E_0^{i+k,j-k}$ and $\delta(x_k) = d(x_{k+1})$. Since the columns of double complex split as direct sums over σ, the vertical cocycles split as a direct sum, and it suffices to show that higher differentials vanish for each summand. So, let σ be a simplex in $\operatorname{Flag}(\mathcal{P})$, and consider the term $C^j(Y_{\min(\sigma)})$. There are two cases.

1) σ is a face of a simplex τ with $\min(\tau) = \min(\sigma)$. Then $i_{\tau\sigma}$ is the identity map and this forces higher differentials to be trivial on this term. Indeed, if $z \in C^j(Y_{\min(\sigma)})$, then $d(x_1) = \delta(z)$, and therefore $d(x_{1\tau}) = \pm z$, where $x_{1\tau}$ denotes the τ component of x_1. Thus we can choose $x_1' = \pm\delta(i_{\tau\sigma}^{-1}(x_{1\tau}))$ and all higher $x_k = 0$.

2) σ is a "maximal" face, i.e., all its cofaces have strictly smaller minimum vertices. Then $a = \max(\sigma)$ is a maximal element of $\mathcal{P}$. The cover of $Y_{\neq a}$ by

$\{Y_b \mid b \neq a\}$ is a poset of spaces over $\mathcal{P}_{\neq a}$. Let $E_{0,a}$ be the subcomplex of E_0 corresponding to the pair $(Y, Y_{\neq a})$:

$$E_{0,a}^{i,j} = \bigoplus_{\substack{\max(\tau)=a \\ \dim \tau = i}} C^j(Y_{\min(\tau)}).$$

Note that $C^j(Y_\sigma)$ is contained in this subcomplex, so it suffices to show that the higher differentials vanish for $E_{0,a}$. The pair $(Y, Y_{\neq a})$ excises to the pair $(Y_a, Y_{<a})$. So we have a short exact sequence:

$$0 \to E_{0,a} \to E_{0,\leq a} \to E_{0,<a} \to 0\,,$$

where

$$E_{0,\leq a}^{i,j} = \bigoplus_{\substack{\tau \in \mathrm{Flag}(\mathcal{P}_{\leq a}) \\ \dim \tau = i}} C^j(Y_{\min(\tau)}),$$

and

$$E_{0,<a}^{i,j} = \bigoplus_{\substack{\tau \in \mathrm{Flag}(\mathcal{P}_{<a}) \\ \dim \tau = i}} C^j(Y_{\min(\tau)}).$$

are double complexes corresponding to the covers (posets of spaces) of Y_a by $\{Y_b \mid b \leq a\}$ and of $Y_{<a}$ by $\{Y_b \mid b < a\}$.

The E_2 terms of the spectral sequences $E_{\leq a}$ and $E_{<a}$ are

$$E_{2,\leq a}^{i,j} = \bigoplus_{b \leq a} H^i(\mathrm{Flag}(\mathcal{P}[b,a]), \mathrm{Flag}(\mathcal{P}(b,a]); H^j(Y_b)),$$

and

$$E_{2,<a}^{i,j} = \bigoplus_{b < a} H^i(\mathrm{Flag}(\mathcal{P}[b,a)), \mathrm{Flag}(\mathcal{P}(b,a)); H^j(Y_b)).$$

For $b < a$, $\mathrm{Flag}([b,a])$ is a cone on $\mathrm{Flag}((b,a])$; therefore, the only nonzero terms in $E_{2,\leq a}$ come from $b = a$ and $i = 0$, i.e., $E_{2,\leq a}$ has $H^j(Y)$ in the 0-row and 0 everywhere else. In particular, it collapses at E_2. By inductive assumption $E_{2,<a}$ also collapses at E_2. Since, by hypothesis, $H^*(Y_a) \to H^*(Y_{<a})$ is the 0-map, the long exact sequence of the pair $(Y_a, Y_{<a})$ splits into short exact sequences, and similarly for the E_2 terms, it follows that the spectral sequence E_a also collapses at E_2 term. Thus, the higher differentials in E are 0. □

3 Some previous cohomology computations

Suppose G is a discrete group and Y is a G-CW complex. Let N be a left G-module. The *G-equivariant cochain complex* is defined by,

$$C^i_G(Y;N) := \mathrm{Hom}_G(C_i(Y);N),$$

where $C_*(Y)$ denotes the usual cellular chain complex. Its cohomology is denoted $H^*_G(Y;N)$. If G acts freely on Y, then $C^*_G(Y;N)$ can be identified with $C^*(Y/G;N)$, the cellular cochains on the quotient space with local coefficients in N. There is a similar result even when the action is not free; however, the coefficients will no longer be locally constant. Rather, the coefficients will be in a contravariant functor $\mathcal{I}(N)$ from the poset of cells in Y/G to the category of abelian groups: $\mathcal{I}(N)$ assigns to a cell c the fixed submodule N^{G_c}, where G_c denotes the stabilizer of some lift of c and where

$$C^i_G(Y) = C^i(Y/G;\mathcal{I}(N)). \tag{3.1}$$

For $Y = EG$, the universal cover of the classifying space BG, define the *cochains and cohomology of G with coefficients in N* by

$$\begin{aligned} C^*(G;N) &:= C^*_G(EG;N) = C^*(BG;N), \\ H^*(G;N) &:= H^*_G(EG;N) = H^*(BG;N). \end{aligned}$$

The spectral sequence arguments from §2 work with equivariant cochains (in particular with cochains with local coefficients) as long as the cover $\mathcal{V}$ is G-equivariant.

In what follows we will be interested principally in two cases: $N = \mathbf{Z}G$, the group ring, and $N = \mathcal{N}(G)$, the group von Neumann algebra. We recall the definitions and some results which have been proved previously.

Group ring coefficients. In the case of group ring coefficients, if G acts properly and cocompactly on Y, there is the following formula (cf. [6, Ex. 4, p. 209]),

$$H^*_G(Y;\mathbf{Z}G) = H^*_c(Y),$$

where $H^*_c(Y)$ means cohomology with compact supports. Thus, Lemma 1.7 implies the following.

Corollary 3.1. *For each $s \in S$, suppose G_s is a discrete group and that $(Z_s, G_s b_s)$ a G_s-CW complex together with a free orbit. Also suppose each G_s-action is proper and cocompact and that Z_s is acyclic. Then for $G = \prod_\Gamma G_s$ and $Z = \prod_\Gamma (Z_s, G_s b_s)$, we have*

$$H^*(G; \mathbf{Z}G) = H_c^*(Z).$$

The cohomology of Coxeter groups are computed as follows.

Theorem 3.2 ([10] as well as [14])**.**

$$H^n(W; \mathbf{Z}W) = H_c^n(\mathcal{U}(W, K)) = \bigoplus_{J \in \mathcal{S}(W,S)} H^n(K, K^{S-J}) \otimes \hat{A}(W)^J,$$

where $\hat{A}(W)^J$ is the free abelian group on the set of $w \in W$ which have reduced expressions ending with letters in J.

Using this, Jensen and Meier proved the following. (A different proof of this will be given in §4.1.)

Theorem 3.3 (Jensen–Meier [26])**.** *Suppose (W, S) is a RACS and A_L is the associated RAAG. Then*

$$H^n(A_L; \mathbf{Z}A_L) = \bigoplus_{J \in \mathcal{S}(W,S)} H^{n-|J|}(K_J, \partial K_J) \otimes \mathbf{Z}(A_L/A_J).$$

Theorem 3.4. ([13, Corollary 8.2]). *Suppose $\mathcal{C}$ is a locally finite building of type (W, S). Then*

$$H_c^n(\mathcal{B}(W, K)) = \bigoplus_{J \in \mathcal{S}(W,S)} H^n(K, K^{S-J}) \otimes \hat{A}(\mathcal{C})^J,$$

where $\hat{A}(\mathcal{C})^J$ is a certain subgroup of the free abelian group on $\mathcal{C}$.

In particular, by Lemma 1.1, this theorem gives the following computation of [14] of the compactly supported cohomology of a locally finite RAB and hence, of the cohomology with group ring coefficients of the graph product of a collection of finite groups.

Theorem 3.5 ([14, Theorem 6.6])**.** *Suppose $\{G_s\}_{s \in S}$ is a collection of finite groups and that $G = \prod_\Gamma G_s$. Then*

$$H^n(G; \mathbf{Z}G) = \bigoplus_{J \in \mathcal{S}(L)} H^n(K, K^{S-J}) \otimes \hat{A}(J),$$

where $\hat{A}(J)$ is a certain (free abelian) subgroup of $\mathbf{Z}(G/G_J)$.

L^2-cohomology and L^2-Betti numbers. The real *group algebra*, $\mathbf{R}G$, of G consists of all finitely supported functions $G \to \mathbf{R}$. Its *standard basis* is $\{e_g\}_{g\in G}$, where e_g denotes the indicator function of $\{g\}$. The *standard inner product* on $\mathbf{R}G$ is defined by $e_g \cdot e_h = \delta_{gh}$, where δ_{gh} is the Kronecker delta. The Hilbert space completion of $\mathbf{R}G$, denoted $L^2(G)$, consists of all square summable functions $G \to \mathbf{R}$. The group G acts orthogonally on $L^2(G)$ by either left or right translation. To fix ideas, let us say that it is the right action defined by left translation. The *von Neumann algebra* of G, denoted by $\mathcal{N}(G)$, is the commutant of the G-action. It acts on $L^2(G)$ from the left. For $\varphi \in \mathcal{N}(G)$, define

$$\mathrm{tr}_{\mathcal{N}(G)}(\varphi) := (\varphi e_1) \cdot (e_1).$$

If V is a closed G-stable subspace of a finite direct sum of copies of $L^2(G)$, then its *von Neumann dimension* is defined by

$$\dim_{\mathcal{N}(G)} V := \mathrm{tr}_{\mathcal{N}(G)}(p_V),$$

where $p_V : \oplus L^2(G) \to \oplus L^2(G)$ is orthogonal projection onto V.

Suppose the G-CW complex Y is proper and cocompact. Define $L^2C^*(Y)$ to be the cochain complex of real-valued, square summable cochains on Y. Denote its reduced cohomology group by $L^2H^*(Y)$. (Here "reduced" means $\mathrm{Ker}\,\delta/\overline{\mathrm{Im}\,\delta}$, where $\delta : L^2C^i(Y) \to L^2C^{i+1}(Y)$ is the coboundary operator. It is necessary to take the closure of $\mathrm{Im}\,\delta$ for the quotient to be a Hilbert space.) Define the i^{th} L^2-Betti number by

$$L^2b^i(Y;G) := \dim_{\mathcal{N}(G)} L^2H^i(Y).$$

If Y is acyclic, then $L^2H^i(Y)$ depends only on G and is denoted by $L^2H^i(G)$ and similarly, $L^2b^i(G) := L^2b^i(Y;G)$. Thus, Lemma 1.7 implies the following.

Corollary 3.6. *For each $s \in S$, suppose G_s is a discrete group and that (Z_s, G_sb_s) a G_s-CW complex together with a free orbit. Also suppose each G_s-action is proper and cocompact and that Z_s is acyclic. Then for $G = \prod_\Gamma G_s$ and $Z = \prod_\Gamma(Z_s, G_sb_s)$, we have*

$$L^2b^*(G) = L^2b^i(Z;G).$$

The L^2-Betti numbers of Coxeter groups have proved to be difficult to compute. Some partial results and conjectures can be found in [19]. For locally finite buildings of very large thickness there is a complete calculation due to Dymara–Januszkiewicz [22]. The requirement on the thickness is reduced in [15] (cf. Theorems 6.3 and 6.4 in §6.2).

In the case of Artin groups, we have the following easy computation of [17]. (A proof of this will be given in the next section.)

Theorem 3.7 ([17]). *$L^2b^n(\widetilde{X}; A) = b^n(K, \partial K) = \overline{b}^{n-1}(L)$, where, as usual, $b^n(K, \partial K) = \dim(H^n(K, \partial K; \mathbf{R}))$ and $\overline{b}^{n-1}(L) = \dim(\overline{H}^{n-1}(L; \mathbf{R}))$.*

In [28], Lück shows that there is an equivalence of categories between the category of finitely generated $\mathcal{N}(G)$-modules and the category of orthogonal representations of G on Hilbert spaces which are G-isomorphic to closed, G-stable subspaces of a finite direct sum of copies of $L^2(G)$. Given a finitely generated $\mathcal{N}(G)$-module E, define $\dim_{\mathcal{N}(G)} E$ to be the von Neumann dimension of the corresponding Hilbert space. Then

$$L^2b^i(Y; G) = \dim_{\mathcal{N}(G)} H^i_G(Y; \mathcal{N}(G)).$$

Just as in (3.1), we have that

$$H^*_G(Y; \mathcal{N}(G)) = H^*\big(Y/G; \mathcal{I}(\mathcal{N}(G))\big). \tag{3.2}$$

4 Computations

4.1 Artin groups

As in §1.3, A is the Artin group associated to a Coxeter system (W, S) and X is its Salvetti complex. As usual, $L = L(W, S)$, $\mathcal{S} = \mathcal{S}(W, S)$ and $K := \mathrm{Flag}(\mathcal{S})$. We wish to compute $H^*(X; \mathbf{Z}A)$. Given a spherical subset $J \in \mathcal{S}$, A_J is the corresponding Artin group and X_J is its Salvetti complex. We know that X_J is the classifying space for A_J. By [29] (see also [3]), for each spherical subset J, A_J is a duality group of dimension $|J|$. This means that $H^*(A_J; \mathbf{Z}A_J)$ is zero for $* \neq |J|$ and that $F_J := H^{|J|}(A_J; \mathbf{Z}A_J)$ is free abelian.

As explained in §1.3, the cover $\mathcal{V} = \{X_J\}_{J \in \mathcal{S}}$ is a poset of spaces for X. In the case of group ring coefficients, we have a spectral sequence of the type considered in §2 converging to $H^*(X; \mathbf{Z}A)$. It has E_2 term: $E_2^{i,j} = H^i(K; \mathcal{H}^j(\mathcal{V}))$, where $\mathcal{H}^j(\mathcal{V})$ is the coefficient system, $\sigma \mapsto H^i(X_{\min \sigma}; \mathbf{Z}A)$,. By Lemma 2.2, once we establish condition (Z) of §2 we will get the following calculation.

Theorem 4.1.
$$\operatorname{Gr} H^n(X;\mathbf{Z}A) = \bigoplus_{J\in\mathcal{S}(W,S)} H^{n-|J|}(K_J,\partial K_J)\otimes H^{|J|}(A_J;\mathbf{Z}A).$$

A similar argument can be used recover the calculation of the L^2-Betti numbers of X in [17]. (This computation was stated earlier as Theorem 3.7.) The spectral sequence has $E_2^{i,j} = H^i(K;\mathcal{H}^j(\mathcal{V}))$, where $\mathcal{H}^j(\mathcal{V})$ is the coefficient system $\sigma\mapsto H^i(X_{\min\sigma};\mathcal{N}(A))$. The key observation in [17] for proving Theorem 3.7 was that for $J\neq\emptyset$, all L^2-Betti numbers of A_J vanish. In particular, condition (Z) of §2 holds (since all cohomology groups vanish except when $J=\emptyset$). So, $E_1^{i,j}$ is 0 for $j\neq 0$ while
$$E_1^{i,0} = C^i(K,\partial K;\mathcal{N}(A)),$$
where the coefficients are now constant. It follows that $H^n(X;\mathcal{N}(A))\cong H^n(K,\partial K)\otimes\mathcal{N}(A)$; whence, Theorem 3.7.

In the case of Theorem 4.1, Condition (Z) is basically the following lemma.

Lemma 4.2. *For any $J\in\mathcal{S}$, $H^*(X_J;\mathbf{Z}A)$ is concentrated in degree $|J|$, where it is equal to the free abelian group $F_J\otimes_{A_J}\mathbf{Z}A$. Hence, $H^*(X_J;\mathbf{Z}A)\to H^*(X_{<J};\mathbf{Z}A)$ is the zero map.*

Proof. The first sentence is from [29]. The second sentence follows since X_J is a $|J|$-dimensional CW complex, $H^*(X_J;\mathbf{Z}A_J)$ is concentrated in the top dimension and $X_{<J}$ is a subcomplex of one less dimension. □

Theorem 4.1 follows immediately from Lemma 2.2. We note that if the $K(\pi,1)$ Conjecture holds for A (i.e, if $X=BA$), then the formula in Theorem 4.1 is a calculation of $H^*(A;\mathbf{Z}A)$ and Theorem 3.7 gives a formula for $L^2b^n(A)$. In particular, since the $K(\pi,1)$ Conjecture holds for RAAG's, Theorem 4.1 gives as a corollary, a different proof the Jensen–Meier calculation in [26] (stated previously as Theorem 3.3).

4.2 Bestvina–Brady groups

In this subsection A_L is a RAAG, T_L is its Salvetti complex defined in (1.5) and $\pi: A_L\to\mathbf{Z}$ is the standard homomorphism. We have a π-equivariant map $p:\widetilde{T}_L\to\mathbf{R}$ and $BB_L=\operatorname{Ker}\pi$. Put $Z_L=p^{-1}(t)$ for some $t\in\mathbf{R}-\mathbf{Z}$ (say for $t=\frac{1}{2}$). It is proved in [4] that if L is acyclic, then so is Z_L. If this is the case, BB_L is called a *Bestvina–Brady group*. We can compute equivariant cohomology of Z_L by the method used in the proof of Theorem. 4.1.

Theorem 4.3. *The compactly supported cohomology of Z_L is isomorphic to that of $\widetilde{T}_L$ shifted up in degree by 1, i.e.,*

$$\begin{aligned} H_c^n(Z_L) &= H^n_{BB_L}(Z_L; \mathbf{Z}BB_L) \\ &= \bigoplus_{J \in \mathcal{S}(L)_{>\emptyset}} H^{n-|J|+1}(K_J, \partial K_J) \otimes \mathbf{Z}(BB_L/(BB_L \cap A_J)). \end{aligned}$$

When L is acyclic, this is a calculation $H^j(BB_L; \mathbf{Z}BB_L)$.

Proof. We intersect the cover $\{\widetilde{T}_J\}_{J \in \mathcal{S}(L)}$ of $\widetilde{T}_L$ with Z_L. Put $Z_J := Z_L \cap \widetilde{T}_J$. Since t is not an integer, Z_L does not contain any vertices of the cubical complex $\widetilde{T}_L$, i.e., $Z_L \cap \widetilde{T}^{\emptyset} = \emptyset$. On the other hand, when J is nonempty, the intersection of Z_L with any component of $\widetilde{T}_J$ is a Euclidean subspace of codimension one and the collection of such intersections is in one-to-one correspondence with the cosets of $BB_L \cap A_J$ in BB_L. So, $\{Z_J\}_{J \in \mathcal{S}(L)_{>\emptyset}}$ is a poset of spaces on Z_L. The simplicial complex $\operatorname{Flag}(\mathcal{S}(L)_{>\emptyset})$ is equal to ∂K (i.e., the barycentric subdivision of L). The $E_1^{i,j}$-term of the spectral sequence is $C^i(K; \mathcal{H}^j(\mathcal{V}))$, where the coefficient system takes σ to $H^j_{BB_L}(Z_{\min \sigma}; \mathbf{Z}BB_L)$. Since each component of Z_J is Euclidean space of dimension $|J|-1$, the coefficients, $H^j_{BB_L}(Z_J; \mathbf{Z}BB_L)$ are 0 whenever $j \neq |J|-1$. Moreover, for $j = |J|-1$,

$$H^j_{BB_L}(Z_J; \mathbf{Z}BB_L) = H^j_c(\mathbf{R}^j) \otimes_{BB_L \cap A_J} \mathbf{Z}BB_L = \mathbf{Z}(BB_L/(BB_L \cap A_J)).$$

It follows that conditions (Z$'$) and (Z) of §2 hold (because $Z_{<J}$ is a subcomplex of dimension $|J|-2$). Hence, by Lemma 2.2, the spectral sequence degenerates at E_2 and $E_2^{i,j} = \bigoplus_J E^{i,j}_{2,J}$, where $E^{i,j}_{2,J}$ is nonzero only for $j = |J|-1$, in which case,

$$E^{i,|J|-1}_{2,J} = H^i(K_J, \partial K_J) \otimes \mathbf{Z}(BB_L/(BB_L \cap A_J)).$$

The theorem follows. (We also note that for $J \neq \emptyset$, $\pi : A_J = \mathbf{Z}^J \to \mathbf{Z}$ is onto, so that $BB_L/(BB_L \cap A_J) \cong A_L/A_J$.) $\square$

Similarly, we compute the L^2-Betti numbers of Z_L as follows.

Theorem 4.4. *Suppose Z_L is a generic level set of the function $p : \widetilde{T}_L \to \mathbf{R}$. Then*

$$L^2 b^n(Z_L; BB_L) = \sum_{s \in S} b^n(K_s, \partial K_s) = \sum_{s \in S} \overline{b}^{n-1}(\operatorname{Lk}(s)).$$

In particular, when L is acyclic,

$$L^2 b^n(BB_L) = \sum_{s \in S} b^n(K_s, \partial K_s).$$

Proof. As before, the $E_1^{i,j}$-term of the spectral sequence is $C^i(K; \mathcal{H}^j(\mathcal{V}))$, where the coefficient system takes σ to $H^j(Z_{\min \sigma}; \mathcal{N}(BB_L))$. Since each component of Z_J is Euclidean space of dimension $|J|-1$, the coefficients, $H^*(Z_J; \mathcal{N}(BB_L))$ are 0 whenever $|J| \neq 1$. Hence, by Lemma 2.2, the spectral sequence degenerates at E_2 and only $E_2^{i,0}$ can be nonzero, where

$$E_2^{i,0} = \bigoplus_{s \in S} H^i(K_s, \partial K_s; \mathcal{N}(BB_L)).$$

The theorem follows. $\square$

Remark. Here is a different proof of Theorem 4.3 when L is acyclic. Put $Y_+ := p^{-1}([\frac{1}{2}, \infty))$ and $Y_- := p^{-1}((\infty, \frac{1}{2}])$. We first claim that the compactly supported cohomology of $Y_\pm$ vanishes in all degrees. It suffices to consider Y_+, the argument for Y_- being similar. The arguments of [4] show that when L is acyclic the inclusion of any level set $p^{-1}(t)$ into a sublevel set $p^{-1}([\frac{1}{2}, t])$ induces an isomorphism on homology. The same argument shows that it induces an isomorphism on compactly supported cohomology, $H_c^*(p^{-1}([\frac{1}{2}, t])) \to H_c^*(p^{-1}(t))$. Hence, $H_c^*(p^{-1}([\frac{1}{2}, t]), p^{-1}(t)) = 0$. Since there is an excision, $H_c^*\big(Y_+, p^{-1}([t, \infty))\big) \cong H_c^*(p^{-1}([\frac{1}{2}, t]), p^{-1}(t))$, the left hand side also vanishes. For any compact subset $C \subset Y_+$ we have that $Y_+ - C \supset p^{-1}([t, \infty])$ for large enough t; so,

$$H_c^*(Y_+) = \lim_{t \to \infty} H_c^*\big(Y_+, p^{-1}([t, \infty))\big),$$

and by the previous discussion the right hand side vanishes. Hence, so does $H_c^*(Y_+)$. We have $Y_+ \cup Y_- = \widetilde{T}_L$ and $Y_+ \cap Y_- = Z_L$ and a Mayer–Vietoris sequence:

$$0 = H_c^*(Y_+) \oplus H_c^*(Y_-) \to H^*(Z_L) \to H_c^{*+1}(\widetilde{T}_L) \to 0.$$

The theorem follows from the computation of $H_c^{*+1}(\widetilde{T}_L)$ in Theorem 4.1 (or in Theorem 3.3).

Remark. Theorem 3.3 provides a calculation of $H^*(A_L; \mathbf{Z}A_L)$ as a sum of terms involving the $H^{*-|J|}(K_J, \partial K_J)$, where $\partial K_J \cong \mathrm{Lk}(J)$. In the calculation of L^2-cohomology in Theorem 3.7 only the term with $J = \emptyset$ enters. Hence, under the canonical map, $H^*(A_L; \mathbf{Z}A_L) \to L^2H^*(A_L)$, all the terms with $J \neq \emptyset$ go to 0. Similarly, in Theorem 4.3 we calculated $H^*(BB_L; \mathbf{Z}BB_L)$ as a sum of terms involving $H^{*-|J|+1}(K_J, \partial K_J)$. (Since L is acyclic, the term with $J = \emptyset$ does not appear.) On the other hand, in Theorem 4.4 for $L^2H^*(BB_L)$ only the terms with $|J| = 1$ occur. So, the canonical map $H^*(BB_L; \mathbf{Z}BB_L) \to L^2H^*(BB_L)$ takes all the terms with $|J| > 1$ to 0.

Remark. The cohomology of BB_L with trivial coefficients was computed by Leary and Saadetoğlu in [27].

4.3 Graph products of infinite groups

As in the Introduction, $\{G_s\}_{s\in S}$ is a family of groups and $G = \prod_\Gamma G_s$ is the graph product with respect to the simplicial graph Γ. The associated flag complex is L. For each J in $\mathcal{S}(L)$, G_J denotes the direct product of the G_s with $s \in J$. In this subsection we shall also suppose that *each G_s is infinite.* Put $Y = EG$. As in Example 1.6, EG is the graph product of the $(EG_s, G_s b_s)$. The cover $\mathcal{V} = \{Y_J\}_{J\in\mathcal{S}(L)}$, where $Y_J = G \times_{G_J} EG_J$, is a poset of spaces structure for EG. Let N stand for $\mathbf{Z}G$ or $\mathcal{N}_{\mathbf{q}}(G)$. The spectral sequence of §2 converges to $H^*(G; N)$ and has E_2 term:

$$E_2^{j.k} = H^i(K; \mathcal{H}^j(\mathcal{V})),$$

where the coefficient system is given by $\mathcal{H}^j(\mathcal{V})(\sigma) = H^j(G_{\min(\sigma)}; N)$. Once we verify that Condition (Z) holds, Lemma 2.2, will provide the following calculations.

Theorem 4.5. *Let G be a graph product of groups G_s, each of which is infinite. Then*

$$\mathrm{Gr}\, H^n(G; \mathbf{Z}G) = \bigoplus_{\substack{J\in\mathcal{S}(L)\\ i+j=n}} H^i(K_J, \partial K_J; H^j(G_J; \mathbf{Z}G)).$$

(Note that $H^j(G_J; \mathbf{Z}G) = H^j(G_J; \mathbf{Z}G_J) \otimes_{G_J} \mathbf{Z}G$.)

Theorem 4.6. *Let G be a graph product of groups G_s, each of which is infinite. Then*

$$L^2b^n(G) = \sum_{\substack{J\in\mathcal{S}(L)\\ i+j=n}} b^i(K_J, \partial K_J) \cdot L^2b^j(G_J).$$

To establish Theorem 4.5 we need to verify conditions (Z′) and (Z) (or in fact just condition (Z)) which precedes Lemma 2.2. These conditions follow from statements (i) and (ii), respectively, in the next lemma.

Lemma 4.7. *Suppose $G_J = \prod_{s\in J} G_s$ is the direct product of a collection of infinite groups indexed by a finite set J. Let $\{Y_s\}_{s\in J}$ be a collection of connected CW complexes with proper G_s-actions such that each Y_s contains a free orbit, $G_s b_s$, and put $Y_J := \prod_{s\in J} Y_s$. As in §1.4, for each $I \subset J$, define*

$$Y'_I := \prod_{s\in I} Y_s \times \prod_{s\in J-I} G_s b_s.$$

Let N stand for either $\mathbf{Z}G_J$ or $\mathcal{N}(G_J)$. Then

(i) The map induced by inclusion, $H^*_{G_J}(Y_J; N) \to H^*_{G_J}(Y'_I; N)$, is the zero map.

(ii) More generally, if

$$Y_{<J} := \bigcup_{s\in J} Y'_{J-s},$$

then the map induced by inclusion, $H^*_{G_J}(Y_J; N) \to H^*_{G_J}(Y_{<J}; N)$, is the zero map.

Proof. We shall prove this only in the case $N = \mathbf{Z}G_J$, the case $N = \mathcal{N}(G_J)$ being entirely similar. The relative version of the Künneth Formula states that for pairs of spaces (A, B) and (A', B'),

$$H^n((A, B) \times (A', B')) = \bigoplus_{i+j=n} H^i(A, B; H^j(A', B')),$$

(where $(A, B)\times(A', B') = (A\times A', (A\times B')\cup(B\times A'))$). Similarly, if (A, B) is a pair of H-spaces and (A', B') a pair of H'-spaces, then for left H- and H'-modules M and M',

$$H^n_{H\times H'}((A, B) \times (A', B'); M \otimes M') = \bigoplus_{i+j=n} H^i_H(A, B; M \otimes H^j_{H'}(A', B'; M')),$$

By the exact sequence of the pair, showing $H^*_{G_J}(Y_J; \mathbf{Z}G_J) \to H^*_{G_J}(Y'_I; \mathbf{Z}G_J)$ is zero is equivalent to showing that

$$H^*_{G_J}(Y_I \times (Y_{J-I}, G_{J-I}b); M \otimes M') \to H^*_{G_J}(Y_I \times Y_{J-I}; M \otimes M') \qquad (4.1)$$

is onto, where $M = \mathbf{Z}G_I$, $M' = \mathbf{Z}G_{J-I}$ and $b \in Y_{J-I}$ is a basepoint. If $J - I \neq \emptyset$, then since G_{J-I} is infinite and acts properly, Y_{J-I} is noncompact; hence, $H^0_{G_{J-I}}(Y_{J-I}; M') = 0$ and so

$$H^j_{G_{J-I}}(Y_{J-I}, G_{J-I}b; M') \to H^j_{G_{J-I}}(Y_{J-I}; M')$$

is onto. Hence,

$$H^i_{G_I}(Y_I; M \otimes H^j_{G_{J-I}}(Y_{J-I}, G_{J-I}b; M')) \to H^i_{G_I}(Y_I; M \otimes H^j_{G_{J-I}}(Y_{J-I}; M'))$$

is onto. It follows from the relative Künneth Formula that the map in (4.1) is onto.

The proof of the second statement is similar using induction on the cardinality of J. Choose $s \in J$. Then $Y_{<J} = (Y_{<(J-s)} \times Y_s) \cup (Y_{J-s} \times G_s b_s)$. Hence,

$$(Y_J, Y_{<J}) = (Y_{J-s}, Y_{<(J-s)}) \times (Y_s, G_s b_s).$$

Let $M = \mathbf{Z}G_{J-s}$ and $M' = \mathbf{Z}G_s$. Then

$$H^j_{G_s}(Y_s, G_s b_s; M') \to H^j_{G_s}(Y_s; M')$$

is onto by the argument in the previous paragraph, and

$$H^i_{G_{J-s}}(Y_{J-s}, Y_{<(J-s)}; M) \to H^i_{G_{J-s}}(Y_{J-s}; M)$$

is onto by inductive hypothesis. Combining these two surjections, we see that

$$H^i_{G_{J-s}}(Y_{J-s}, Y_{<(J-s)}; M \otimes H^j_{G_s}(Y_s, G_s b_s; M')) \to H^i_{G_{J-s}}(Y_{J-s}; M \otimes H^j_{G_s}(Y_s; M'))$$

is onto. So, by the relative Künneth Formula,

$$H^n_{G_{J-s} \times G_s}((Y_{J-s}, Y_{<(J-s)}) \times (Y_s, G_s b_s); \mathbf{Z}G_J) \to H^n_{G_{J-s} \times G_s}(Y_{J-s} \times Y_s; \mathbf{Z}G_J)$$

is onto, which completes the proof. □

Other coefficients. As before, $G = \prod_\Gamma G_s$ is the graph product and $G' = \prod_{s\in S} G_s$ is the direct product. Let $p : G \to G'$ be the natural projection. We say that a group H *lies between* G *and* G' if $H = G/N$ for some normal subgroup $N \subset G$ with $\operatorname{Ker} p \subseteq N$. If this is the case, then p factors as

$$G \xrightarrow{f} H \longrightarrow G'$$

where f is the natural epimorphism. In this way $\mathbf{Z}H$ becomes a G-module. There is the following generalization of Theorem 4.5.

Theorem 4.8. *Suppose each G_s is infinite and H lies between G and G'. Then*

$$\operatorname{Gr} H^n(G;\mathbf{Z}H) = \bigoplus_{\substack{J\in\mathcal{S}(L)\\ i+j=n}} H^i(K_J,\partial K_J; H^j(G_J;\mathbf{Z}H))$$

where as before, $H^j(G_J;\mathbf{Z}H) = H^j(G_J;\mathbf{Z}G_J)\otimes_{G_J}\mathbf{Z}H$. Similarly,

$$L^2b^n((EG)/N;H) = \sum_{\substack{J\in\mathcal{S}(L)\\ i+j=n}} b^i(K_J,\partial K_J)\cdot L^2b^j(G_J).$$

We have $H^n(G;\mathbf{Z}H) = H^n_H((EG)/N)$ and $(EG)/N$ is covered by complexes of the form $H\times_{G_J} EG_J$. The proof then goes through in the same manner as that of Theorem 4.5.

5 Graph products of Coxeter groups

5.1 Polyhedral joins

As in §1.2 and §1.4, let L be a simplicial complex with vertex set S. For each $s\in S$, suppose given a simplicial complex $\mathcal{L}(s)$ with vertex set T_s. For each $J\in\mathcal{S}(L)$, define $\mathcal{L}(J)$ to be the join,

$$\mathcal{L}(J) := \mathop{\ast}_{s\in J} \mathcal{L}(s), \tag{5.1}$$

and then define the *polyhedral join* of the $\mathcal{L}(s)$ with respect to L by

$$\ast_L\ \mathcal{L}(s) := \bigcup_{J\in\mathcal{S}(L)} \mathcal{L}(J). \tag{5.2}$$

To simplify notation put $\mathcal{L} := \ast_L \ \mathcal{L}(s)$. Here is an equivalent definition. Let T denote the disjoint union

$$T := \bigcup_{s \in S} T_s.$$

and $\pi : T \to S$ the natural projection. Any subset I of T can be decomposed as

$$I = \bigcup_{s \in \pi(I)} I_s,$$

where $I_s \subseteq T_s$. Then I is the vertex set of a simplex in $\mathcal{L}$ if and only if $\pi(I) \in \mathcal{S}(L)$, and $I_s \in \mathcal{S}(\mathcal{L}(s))$ for each $s \in S$. In other words, a simplex I of $\mathcal{L}$ is determined by a simplex $J \in \mathcal{S}(L)$ and a collection of simplices $\{I_s\}_{s \in J}$, where each $I_s \in \mathcal{S}(\mathcal{L}(s))$.

Remark. Similarly, given any family of spaces $\{X(s)\}_{s \in S}$, for each $J \in \mathcal{S}(L)$, define $X(J)$ to be the join of the $X(s)$ and the *polyhedral join*, $\ast_L X(s)$, to be the union of the $X(J)$ as in (5.2).

Recall that the notion of "polyhedral product" was defined by (1.6). The proof of the next lemma is straightforward.

Lemma 5.1. *The operation of applying K to simplicial complexes intertwines the polyhedral join with the polyhedral product (defined in §1.4), i.e.,*

$$K(\ast_L \ \mathcal{L}(s)) = \prod_L \big(K(\mathcal{L}(s)), \partial K(\mathcal{L}(s))\big).$$

The following lemma is a well-known consequence of the Künneth Formula.

Lemma 5.2. *Given two spaces A and B,*

$$\overline{H}^n(A \ast B) = \bigoplus_{i+j=n-1} \overline{H}^i(A; \overline{H}^j(B)).$$

Also, $\overline{H}^*(A \ast B) \to \overline{H}^*(A)$ is the zero map whenever B is nonempty. (Here $\overline{H}^*(\ )$ means reduced cohomology. Also, we follow the convention that the reduced cohomology of the empty set is $\mathbf{Z}$ in degree -1.)

Remark. If we take coefficients in a field $\mathbb{F}$, the formula in lemma 5.2 reads

$$\overline{H}^n(A * B; \mathbb{F}) = \bigoplus_{i+j=n-1} \overline{H}^i(A; \mathbb{F}) \otimes \overline{H}^j(B; \mathbb{F})$$

Similarly, for the J-fold join, $X(J)$, of $\{X(s)\}_{s\in J}$,

$$\overline{H}^n(X(J); \mathbb{F}) = \bigoplus_{\sum i_s = n-|J|-1} \bigotimes_{s\in J} \overline{H}^{i_s}(X(s); \mathbb{F}).$$

Proof of Lemma 5.2. Write CA and CB for the cones on A and B, respectively. Then $(CA, A) \times (CB, B) = (CA \times CB, A * B)$. Since $CA \times CB$ is contractible, $\overline{H}^n(A * B) = H^{n+1}((CA, A) \times (CB, B))$. Hence,

$$\begin{aligned} \overline{H}^n(A * B) &= H^{n+1}((CA, A) \times (CB, B)) \\ &= \bigoplus_{i+j+2=n+1} H^{i+1}(CA, A; H^{j+1}(CB, B)) \\ &= \bigoplus_{i+j=n-1} \overline{H}^i(A; \overline{H}^j(B)), \end{aligned}$$

where the second equation is the relative Künneth Formula. This proves the first sentence. To prove the second, note that the subspace $A * \emptyset \subset A * B$ is homotopy equivalent to $A \times CB \subset A * B$. So, by the exact sequence of the pair, we need only show $H^*(A * B, A \times CB) \to \overline{H}^*(A * B)$ is onto. We have

$$\begin{aligned} H^n(A * B, A \times CB) = H^n(CA \times B, A \times B) &= H^n((CA, A) \times B) \\ &= \bigoplus_{i+j=n} H^i(CA, A; H^j(B)). \end{aligned}$$

Since the connecting homomorphism $\overline{H}^j(B) \to H^{j+1}(CB, B)$ is an isomorphism, it follows that $H^n(A * B, A) \to \overline{H}^n(A * B)$ is onto. $\square$

For any $J \in \mathcal{S}(L)$, put

$$\mathcal{L}(< J) := \bigcup_{s\in J} \mathcal{L}(J - s),$$

where $\mathcal{L}(J - s)$ is defined by (5.1).

Lemma 5.3 (cf. Lemma 4.7)**.** *The map $\overline{H}^*(\mathcal{L}(J)) \to \overline{H}^*(\mathcal{L}(< J))$, induced by the inclusion, is the zero homomorphism.*

Proof. The proof is by induction on the cardinality of J. It is trivially true for $|J| = 1$. So assume $|J| > 1$. We first claim that for each $s \in J$, $H^*(\mathcal{L}(J), \mathcal{L}(J-s)) \to H^*(\mathcal{L}(< J), \mathcal{L}(J-s))$ is the zero map. We have $(\mathcal{L}(J), \mathcal{L}(J-s)) = \mathcal{L}(J-s) * (\mathcal{L}(s), \emptyset)$. Hence, as in Lemma 5.2,

$$H^n(\mathcal{L}(J), \mathcal{L}(J-s)) = \bigoplus_{i+j=n-1} \overline{H}^i\big(\mathcal{L}(J-s); H^j(\mathcal{L}(s))\big). \tag{5.3}$$

Similarly, $\mathcal{L}(< J) = (\mathcal{L}(< (J-s)) * \mathcal{L}(s)) \cup (\mathcal{L}(J-s) * \emptyset)$; so,

$$\begin{aligned} H^n(\mathcal{L}(< J), \mathcal{L}(J-s)) &= H^n(\mathcal{L}(< (J-s)) * (\mathcal{L}(s), \emptyset)) \\ &= \bigoplus_{i+j=n-1} \overline{H}^i\big(\mathcal{L}(< (J-s)); H^j(\mathcal{L}(s))\big). \end{aligned} \tag{5.4}$$

By inductive hypothesis, $\overline{H}^i(\mathcal{L}(J-s)) \to \overline{H}^i\big(\mathcal{L}(< (J-s))\big)$ is zero. Comparing (5.3) and (5.4), we see that $H^*(\mathcal{L}(J), \mathcal{L}(J-s)) \to H^*(\mathcal{L}(< J), \mathcal{L}(J-s))$ is the zero map, which proves the claim. By the exact sequence of the triple, this is equivalent to the statement that $H^*(\mathcal{L}(J), \mathcal{L}(< J)) \to H^*(\mathcal{L}(J), \mathcal{L}(J-s))$ is onto. By Lemma 5.2, $H^*(\mathcal{L}(J), \mathcal{L}(J-s)) \to \overline{H}^*(\mathcal{L}(J))$ is also onto; hence, so is their composition, $H^*(\mathcal{L}(J), \mathcal{L}(< J)) \to \overline{H}^*(\mathcal{L}(J))$. But this is equivalent to the statement that $\overline{H}^*(\mathcal{L}(J)) \to \overline{H}^*(\mathcal{L}(< J))$ is zero, which is what we wanted to prove. □

Put $\mathcal{K} := K(\mathcal{L})$. We want to use the spectral sequence of §2 to compute the cohomology of $(\mathcal{K}, \mathcal{K}^{T-I})$ for any $I \in \mathcal{S}(\mathcal{L})$. To warm up, let us do first the case $I = \emptyset$. We note that $\mathcal{K}^T$ is (the barycentric subdivision of) $\mathcal{L}$ and $\mathcal{K}$ is the cone on $\mathcal{L}$. Since

$$\mathcal{K} = \prod_L (K(\mathcal{L}(s)), \mathcal{L}(s)) \quad \text{and} \quad \mathcal{L} = \mathop{\ast}_L \mathcal{L}(s),$$

$(\mathcal{K}, \mathcal{L})$ is a pair of posets of spaces over $\mathcal{S}(L)$ (cf. (5.1) and (5.2)). Lemma 5.3 says that condition (Z) of §2 holds; so Lemma 2.2 gives the following,

$$\operatorname{Gr} H^n(\mathcal{K}, \mathcal{K}^T) = \bigoplus_{\substack{J \in \mathcal{S}(L) \\ i+j=n}} H^i\big(K_J, \partial K_J; H^j(\operatorname{Cone}(\mathcal{L}(J)), \mathcal{L}(J))\big). \tag{5.5}$$

Note that the term $H^j(\operatorname{Cone}(\mathcal{L}(J)), \mathcal{L}(J))$ can be replaced by $\overline{H}^{j-1}(\mathcal{L}(J))$. Let $F := \{s \in \mathcal{S}(L) \mid \mathcal{L}(s) \text{ is the simplex on } T_s\}$. Note that if $J \cap F \neq \emptyset$, then the join $\mathcal{L}(J)$ is contractible (since one of its factors is a simplex). So, in this case the coefficients in (5.5), $H^j(\mathrm{Cone}(\mathcal{L}(J)), \mathcal{L}(J))$ vanish for all j.

Next, fix $I \in \mathcal{S}(\mathcal{L})$. For each $s \in S$, $I_s = I \cap T_s$. Define a subset $G(I)$ of S by

$$G(I) := \{s \in \pi(I) \cap F \mid I_s = T_s\}. \tag{5.6}$$

Then $G(I)$ is the vertex set of a simplex σ of L (since $G(I) \subseteq \pi(I)$). Let ${}^I L = L = \sigma$ be the full subcomplex of L spanned by $S - G(I)$, and let ${}^I K := K({}^I L)$ be the Davis chamber. It is not hard to see that ${}^I L$ is homotopy equivalent to $K^{S-G(I)}$ (see [11, Lemma A.5.5, p. 416]), where σ denotes the simplex corresponding to $G(I)$. For each $s \in S - G(I)$, let $\mathcal{L}^I(s)$ denote the full subcomplex of $\mathcal{L}(s)$ spanned by $T_s - I_s$ and for each $J \in \mathcal{S}({}^I L)$, put

$$\mathcal{L}^I(J) := \mathop{\ast}_{s \in J} \mathcal{L}^I(s). \tag{5.7}$$

The usual spectral sequence argument proves the following.

Theorem 5.4. *With notation as above,*

$$\mathrm{Gr}\, H^n(\mathcal{K}, \mathcal{K}^{T-I}) = \bigoplus_{\substack{J \in \mathcal{S}({}^I L) \\ J \cap F = \emptyset \\ i+j=n-1}} H^i\big({}^I K_J, \partial\, {}^I K_J; \overline{H}^j(\mathcal{L}^I(J))\big).$$

There are two extreme cases of Theorem 5.4.

Corollary 5.5. *Suppose each $\mathcal{L}(s)$ is a simplex. Then*

$$H^n(\mathcal{K}, \mathcal{K}^{T-I}) = H^n({}^I K, \partial\, {}^I K)$$

($= \overline{H}^{n-1}(K^{S-G(I)}) = \overline{H}^{n-1}({}^I L)$).

Proof. If each $\mathcal{L}(s)$ is a simplex, then in Theorem 5.4, $J \cap F = J$ is nonempty unless $J = \emptyset$. When $J = \emptyset$, $\overline{H}^j(\mathcal{L}^I(J))$ is nonzero only for $j = -1$ and we get the formula in the corollary. □

Corollary 5.6. *Suppose no $\mathcal{L}(s)$ is a simplex. Then*

$$\mathrm{Gr}\, H^n(\mathcal{K}, \mathcal{K}^{T-I}) = \bigoplus_{\substack{J \in \mathcal{S}(L) \\ i+j=n-1}} H^i\big(K_J, \partial K_J; \overline{H}^j(\mathcal{L}^I(J))\big).$$

Proof. The hypothesis implies $G(I) = \emptyset$; hence, ${}^I L = L$ and ${}^I K = K$. □

5.2 Cohomology of graph products of Coxeter groups with group ring coefficients

We continue from §1.4. We are given a family of Coxeter systems $\{(V_s, T_s)\}_{s\in S}$ and we form the graph product with respect to Γ. Let T denote the disjoint union of the T_s and put $V = \prod_\Gamma V_s$. Then (V, T) is (obviously) a Coxeter system. The projection $\pi : V \to W_L$ restricts to the natural projection $T \to S$ which sends T_s to s. For each $s \in S$ define $\mathcal{L}(s)$ to be $L(V_s, T_s)$. Clearly,

$$K(V,T) = \pi_{L(W,S)} K(V_s, T_s) := \mathcal{K}$$

$$L(V,T) = *_{L(W,S)} \mathcal{L}(s) := \mathcal{L}. \tag{5.8}$$

Henceforth, we write L and K for $L(W, S)$ and $K(W, S)$, respectively.

Notation. For each $J \in \mathcal{S}(W, S)$, put $T(J) = \pi^{-1}(J)$ and

$$V_J := V_{\pi^{-1}(J)} = \prod_{s\in J} V_s.$$

We can combine Theorem 5.4 with Theorem 3.2 to get the following calculation of $H^*(V; \mathbf{Z}V)$.

Theorem 5.7.

$$\operatorname{Gr} H^n(V; \mathbf{Z}V) = \bigoplus_{I\in\mathcal{S}(V,T)} \bigoplus_{\substack{J\in\mathcal{S}({}^IL)\\ J\cap F=\emptyset\\ i+j=n}} H^i\big({}^IK_J, \partial\, {}^IK_J; \overline{H}^{\,j-1}(\mathcal{L}^I(J))\big) \otimes \hat{A}(V)^I.$$

On the other hand, in Theorem 4.5 we calculated the cohomology of an arbitrary graph product of infinite groups and in Theorem 3.5 for an arbitrary graph product of finite groups. We would like to see that these answers agree with the above in the case of Coxeter groups.

For any $J \in \mathcal{S}(L)$, put

$$\mathcal{I}(J) := \{I \in \mathcal{S}(\mathcal{L}) \mid G(I) = J\}.$$

When each V_s is finite,

$$\begin{aligned}
\operatorname{Gr} H^n(V;\mathbf{Z}V) &= \bigoplus_{I\in\mathcal{S}(\mathcal{L})} H^n(\mathcal{K},\mathcal{K}^{T-I})\otimes \hat{A}(V)^I && \text{(by Theorem 3.2)}\\
&= \bigoplus_{J\in\mathcal{S}(L)}\ \bigoplus_{I\in\mathcal{I}(J)} \overline{H}^{\,n-1}(K^{S-J})\otimes \hat{A}(V)^I && \text{(by Corollary 5.5)}\\
&= \bigoplus_{J\in\mathcal{S}(L)} H^n(K,K^{S-J})\otimes \bigoplus_{I\in\mathcal{I}(J)} \hat{A}(V)^I, && (5.9)
\end{aligned}$$

where (5.9) agrees with Theorem 3.5 with

$$\hat{A}(J) := \bigoplus_{I\in\mathcal{I}(J)} \hat{A}(V)^I.$$

Next consider the situation where all V_s are infinite. First we consider the special case where the base complex L is a simplex.

Lemma 5.8. *Suppose V_J is the J-fold product of $\{V_s\}_{s\in J}$, where each V_s is infinite. For each $I\in\mathcal{S}(V_J,T(J))$, $\mathcal{L}^I(J)$ denotes the J-fold join defined by* (5.7)*. Then*

$$H^n(V_J;\mathbf{Z}V_J) = \bigoplus_{I\in\mathcal{S}(V_J,T(J))} \overline{H}^{\,n-1}(\mathcal{L}^I(J))\otimes \hat{A}(V_J)^I.$$

Proof. Let $\mathcal{K}(J)$ denote the Davis chamber for $(V_J,T(J))$. By Theorem 3.2,

$$H^n(V_J;\mathbf{Z}V_J) = \bigoplus_{I\in\mathcal{S}(V_J,T(J))} H^n(\mathcal{K}(J),\mathcal{K}^{T(J)-I}(J))\otimes \hat{A}(V_J)^I.$$

Moreover, $\mathcal{K}^{T(J)-I}(J)$ is homotopy equivalent to $\mathcal{L}^I(J)$. The formula in the lemma follows. □

Finally, consider the general case when each V_s is infinite:

$$\begin{aligned}
\operatorname{Gr} H^n(V;\mathbf{Z}V) &= \bigoplus_{I\in\mathcal{S}(V,T)} H^n(\mathcal{K},\mathcal{K}^{T-I})\otimes \hat{A}(V)^I \\
&= \bigoplus_{I\in\mathcal{S}(V,T)} \bigoplus_{\substack{J\in\mathcal{S}(W,S)\\ i+j=n}} H^i(K_J,\partial K_J;\overline{H}^{j-1}\big(\mathcal{L}^I(J))\big)\otimes \hat{A}(V)^I \\
&= \bigoplus_{\substack{J\in\mathcal{S}(W,S)\\ i+j=n}} H^i(K_J,\partial K_J;\bigoplus_{I\in\mathcal{S}(V,T)} \overline{H}^{j-1}(\mathcal{L}^I(J))\otimes \hat{A}(V)^I) \\
&= \bigoplus_{\substack{J\in\mathcal{S}(W,S)\\ i+j=n}} H^i(K_J,\partial K_J;H^j(V;\mathbf{Z}V)), \qquad (5.10)
\end{aligned}$$

where the first equation follows from Theorem 3.2, the second from Corollary 5.6, the third from the fact that $\hat{A}(V)^I$ is free abelian, and the last from Lemma 5.8. Moreover, (5.10) agrees with Theorem 4.5.

6 Weighted L^2-cohomology of buildings and Coxeter groups

6.1 Hecke–von Neumann algebras

Suppose given a Coxeter system (W,S) and a function $i: S\to I$ to an index set I such that $i(s)=i(t)$ whenever s and t are conjugate in W. A *multiparameter* for (W,S) is an I-tuple $\mathbf{t}=(t_i)_{i\in I}$ of indeterminates (or of numbers). Write t_s instead of $t_{i(s)}$. If $s_1\cdots s_n$ is a reduced expression for an element $w\in W$, then the monomial

$$\mathbf{t}_w := t_{s_1}\cdots t_{s_n}$$

depends only on w and not on the choice of reduced expression for it. (This follows from Tits' solution to the Word Problem for Coxeter groups, cf. [11, p. 315].) The *growth series* of W is power series in $\mathbf{t}$ defined by,

$$W(\mathbf{t}) := \sum_{w\in W} \mathbf{t}_w.$$

This power series has a region of convergence $\mathcal{R}(W)$ (a subset of $\mathbf{C}^I$). If W is finite, then $W(\mathbf{t})$ is a polynomial. For any Coxeter group W, it can be shown that $W(\mathbf{t})$ is a rational function of $\mathbf{t}$ (cf. [11, Cor. 17.1.6]).

For any set X, $\mathbf{R}^X$ denotes the vector space of finitely supported real-valued functions on X. For each $x \in X$, e_x denotes the indicator function of $\{x\}$ so that $\{e_x\}_{x\in X}$ is the *standard basis* for $\mathbf{R}^X$. For a multiparameter $\mathbf{q}$ of positive real numbers, define a inner product $\langle\ ,\ \rangle_{\mathbf{q}}$ on $\mathbf{R}^W$ by

$$\langle e_w, e_{w'}\rangle_{\mathbf{q}} = \begin{cases} \mathbf{q}_w & \text{if } w = w', \\ 0 & \text{otherwise.} \end{cases}$$

Let $L^2_{\mathbf{q}}(W)$ denote the Hilbert space completion of $\mathbf{R}^W$ with respect to $\langle\ ,\ \rangle_{\mathbf{q}}$.

Using $\mathbf{q}$, one can also give $\mathbf{R}^W$ the structure of a *Hecke algebra*, determined by the formula

$$e_s e_w = \begin{cases} e_{sw} & \text{if } l(sw) > l(w), \\ q_s e_{sw} + (q_s - 1)e_w & \text{if } l(sw) < l(w). \end{cases}$$

When $\mathbf{q} = \mathbf{1}$ (the multiparameter which is identically 1), $\mathbf{R}_{\mathbf{q}}(W)$ is the group algebra of W.

Define an anti-involution $*$ on $\mathbf{R}_{\mathbf{q}}W$, by $(\sum x_w e_w)^* := \sum x_{w^{-1}} e_w$. The inner product $\langle\ ,\ \rangle_{\mathbf{q}}$ and the anti-involution $*$ give $\mathbf{R}_{\mathbf{q}}W$ the structure of a Hilbert algebra (see [21, Prop. 2.1]). (In other words, x^*, the image of x under the anti-involution of algebras, is equal to the adjoint of x with respect to $\langle\ ,\ \rangle_{\mathbf{q}}$.) This implies that there is an associated von Neumann algebra $\mathcal{N}_{\mathbf{q}}(W)$ (called the *Hecke–von Neumann algebra*) acting from the right on $L^2_{\mathbf{q}}(W)$. One definition of $\mathcal{N}_{\mathbf{q}}(W)$ is that it is the algebra of all bounded linear endomorphisms of $L^2_{\mathbf{q}}(W)$ which commute with the left $\mathbf{R}_{\mathbf{q}}(W)$-action. An equivalent definition is that it is the weak closure of the elements $\mathbf{R}_{\mathbf{q}}(W)$ which act from the right on $L^2_{\mathbf{q}}(W)$ as bounded linear operators.

Define the *von Neumann trace* of $\varphi \in \mathcal{N}_{\mathbf{q}}(W)$ by $\operatorname{tr}_{\mathcal{N}_{\mathbf{q}}}(\varphi) := \langle e_1\varphi, e_1\rangle_{\mathbf{q}}$ and similarly, for any $(n\times n)$-matrix with coefficients in $\mathcal{N}_{\mathbf{q}}(W)$. This allows us to define the *von Neumann dimension* of any closed subspace of an n-fold orthogonal direct sum of copies of $L^2_{\mathbf{q}}(W)$ which is stable under the diagonal $\mathbf{R}_{\mathbf{q}}(W)$-action: if $V \subset (L^2_{\mathbf{q}}(W))^n$ is such a subspace and $p_V : (L^2_{\mathbf{q}}(W))^n \to (L^2_{\mathbf{q}}(W))^n$ is orthogonal projection onto V, then $p_v \in \mathcal{N}_{\mathbf{q}}(W)$, so define

$$\dim_{\mathcal{N}_{\mathbf{q}}} V := \operatorname{tr}_{\mathcal{N}_{\mathbf{q}}}(p_V). \tag{6.1}$$

For any $J \subset S$ and $\mathbf{q} \in \mathcal{R}(W_J)$, there is a self-adjoint idempotent $a_J \in \mathcal{N}_{\mathbf{q}}(W)$ defined by

$$a_J := \frac{1}{W_J(\mathbf{q})} \sum_{w \in W_J} e_w$$

(cf. [11, Lemma 19.2.5]). For $s \in S$, write a_s instead of $a_{\{s\}}$. For $s \in S$ and $J \subset S$, define subspaces of $L^2_{\mathbf{q}}(W)$ by

$$A^s := L^2_{\mathbf{q}}(W) a_s \ , \qquad A^J := \bigcap_{s \in J} A^s.$$

These subspaces are stable under the action of $\mathbf{R}_{\mathbf{q}} W$ from the left. Moreover, A^J is the image of the idempotent a_J if $\mathbf{q} \in \mathcal{R}(W_J)$, and $A^J = 0$ whenever $\mathbf{q} \notin \mathcal{R}(W_J)$ (cf. [11, §19.2]).

Let $A^{>J}$ denote the subspace $\sum_{I>J} A^I$ of A^J and put

$$D^J := A^J \cap (A^{>J})^{\perp}$$

The following is one of the main results of [15] (or see [11, Thm 20.6.1]).

Theorem 6.1 (The Decomposition Theorem of [15])**.** *If $\mathbf{q} \in \overline{\mathcal{R}} \cup \overline{\mathcal{R}^{-1}}$, then*

$$\sum_{I \supseteq J} D^I$$

is a direct sum and a dense subspace of A^J. In particular, taking $J = \emptyset$,

$$L^2_{\mathbf{q}} = \overline{\sum D^I}.$$

If $\mathbf{q} \in \overline{\mathcal{R}}$, the only nonzero terms in this sum are those with I cospherical (i.e., with $S - I \in \mathcal{S}$), and if $\mathbf{q}^{-1} \in \overline{\mathcal{R}}$, the only nonzero terms are those with I spherical. Moreover, for $\mathbf{q}^{-1} \in \overline{\mathcal{R}}$,

$$\dim_{\mathcal{N}_{\mathbf{q}}} D^J = \sum_{I \in \mathcal{S}_{\geq J}} \frac{(-1)^{|I-J|}}{W_I(\mathbf{q})} \ .$$

6.2 Weighted L^2-Betti numbers

Suppose M is a mirrored CW complex over S. Let $\mathbf{p}$ be a multiparameter of positive real numbers for a Coxeter system (W, S). Define a measure $\mu_{\mathbf{p}}$ on the set of cells of $\mathcal{U}(W, M)$ by $\mu_{\mathbf{p}}(c) := \mathbf{p}_w$, where $w \in W$ is the element of shortest length which moves c into the base chamber M (= the image of $1 \times M$ in $\mathcal{U}(W, M)$). As in [15] or [11], we can use $\mu_{\mathbf{p}}$ to define the *weighted L^2-cochains*, $L^2_{\mathbf{p}}C^*(\mathcal{U}(W, M))$. The corresponding reduced cohomology groups are denoted $L^2_{\mathbf{p}}H^*(\mathcal{U}(W, M))$. The weighted cochains on $\mathcal{U}(W, M)$ can also be regarded as cochains on M with respect to a certain coefficient system $\mathcal{I}(L^2_{\mathbf{p}})$. This coefficient system associates to a cell c in M, the left $\mathcal{N}_{\mathbf{p}}$-module, $L^2_{\mathbf{p}}(W)a_{S(c)}$, i.e.,

$$\mathcal{I}(L^2_{\mathbf{p}})(c) := L^2_{\mathbf{p}}(W)a_{S(c)}.$$

The corresponding cochain complex is denoted $C^*(M; \mathcal{I}(L^2_{\mathbf{p}}))$ and its reduced cohomology by $H^*(M; \mathcal{I}(L^2_{\mathbf{p}}))$. We have natural identifications,

$$C^*(M; \mathcal{I}(L^2_{\mathbf{p}})) \cong L^2_{\mathbf{p}}C^*(\mathcal{U}(W, M)),$$
$$H^*(M; \mathcal{I}(L^2_{\mathbf{p}})) \cong L^2_{\mathbf{p}}H^*(\mathcal{U}(W, M)).$$

(This is completely analogous to (3.1) and (3.2) of §3. See [14, 15].) The *j^{th}-weighted L^2-Betti number* of $\mathcal{U}(W, M)$ is defined by

$$L^2_{\mathbf{p}}b^j(\mathcal{U}(W, M)) := \dim_{\mathcal{N}_{\mathbf{p}}} L^2_{\mathbf{p}}H^j(\mathcal{U}(W, M)) = \dim_{\mathcal{N}_{\mathbf{p}}} H^j(M; \mathcal{I}(L^2_{\mathbf{p}})),$$

where $\dim_{\mathcal{N}_{\mathbf{p}}}$ is defined by (6.1). Also, put

$$L^2_{\mathbf{p}}b^j(W) := L^2_{\mathbf{p}}b^j(\mathcal{U}(W, K)).$$

We recall some of the main results of [15] and [21].

Theorem 6.2 (Dymara [21])**.** *Suppose $\mathbf{p} \in \overline{\mathcal{R}}$. Then $L^2_{\mathbf{p}}b^j(W) = 0$ for $j > 0$, while*

$$b^0(W) = \dim_{\mathcal{N}_{\mathbf{p}}} A^S = \frac{1}{W(\mathbf{p})}.$$

Theorem 6.3 ([15, Thm. 10.3])**.** *Suppose $\mathbf{p}^{-1} \in \overline{\mathcal{R}}$. Then*

$$L^2_{\mathbf{p}}b^j(\mathcal{U}(W, M)) = \sum_{J \in \mathcal{S}(W,S)} b^j(M, M^{S-J}) \dim_{\mathcal{N}_{\mathbf{p}}} D^J,$$

where the formula for $\dim_{\mathcal{N}_{\mathbf{p}}} D^J$ is given in Theorem 6.1.

Theorem 6.4 ([15, Thm. 13.8])**.** *Suppose $(\mathcal{C}, \delta)$ is a locally finite building of type (W, S) with a chamber transitive automorphism group G and that the thickness of $\mathcal{C}$ is given by a multiparameter $\mathbf{p}$ of integers. (In other words, each s-panel contains $p_s + 1$ chambers.) Let M be a mirrored CW complex over S (with a W-finite mirror structure). Then*

$$L^2 b^j(\mathcal{B}(\mathcal{C}, M), G) = L^2_{\mathbf{p}} b^j(\mathcal{U}(W, M)).$$

(The group G need not be discrete. The von Neumann dimensions with respect to G are defined using Haar measure on G, normalized so that the stabilizer of a chamber has measure 1.)

Corollary 6.5. *Let G be a graph product of finite groups $\{G_s\}_{s\in S}$. Let $\mathbf{p}$ be the multiparameter defined by $p_s := |G_s| - 1$. Then*

$$L^2 b^j(G) = \sum_{J\in\mathcal{S}(W,S)} L^2_{\mathbf{p}} b^j(K, K^{S-J}).$$

As in the last paragraph of §3, there is a different method which can be used to define weighted L^2-Betti numbers by using ideas of Lück [28]. As in [28] there is an equivalence of categories between the category of Hilbert $\mathcal{N}_{\mathbf{p}}(W)$-modules (i.e. $\mathcal{N}_{\mathbf{p}}(W)$-stable closed subspaces of $L^2_{\mathbf{p}}(W)^n$) and the category of ordinary projective modules for $\mathcal{N}_{\mathbf{p}}(W)$. This allows us to define a "dimension," $\dim_{\mathcal{N}_{\mathbf{p}}} M$, of a finitely generated, projective $\mathcal{N}_{\mathbf{p}}(W)$-module which agrees with the dimension of the corresponding Hilbert $\mathcal{N}_{\mathbf{p}}(W)$-module. The $\mathcal{N}_{\mathbf{p}}(W)$-dimension of an arbitrary $\mathcal{N}_{\mathbf{p}}(W)$-module is then defined to be the dimension of its projective part.

There is a coefficient system $\mathcal{I}(\mathcal{N}_{\mathbf{p}})$ on the mirrored CW complex M defined by

$$\mathcal{I}(\mathcal{N}_{\mathbf{p}})(c) := \mathcal{N}_{\mathbf{p}}(W) a_{S(c)}.$$

The corresponding cohomology groups are denoted $H^*(M; \mathcal{I}(\mathcal{N}_{\mathbf{p}}))$. The dimension of $H^j(M; \mathcal{I}(\mathcal{N}_{\mathbf{p}}))$ is equal to that of $H^j(M; \mathcal{I}(L^2_{\mathbf{p}}))$ (and they are both equal to j^{th}-weighted $L^2_{\mathbf{p}}$-Betti number of $\mathcal{U}(W, M)$). (The advantage of using the coefficient system $\mathcal{I}(\mathcal{N}_{\mathbf{p}})$ instead of $\mathcal{I}(L^2_{\mathbf{p}})$ is that it is not necessary to use reduced cohomology and then have to keep taking closures of images.)

7 Weighted L^2-Betti numbers of graph products of Coxeter groups

As in §5.2, (W_L, S) is the RACS associated to a graph Γ, $\{(V_s, T_s)\}_{s\in S}$ is a family of Coxeter systems and (V, T) is the corresponding graph product of Coxeter systems. Let $\mathbf{q}$ be a multiparameter for (V, T). It restricts to a multiparameter for each V_s, which we will denote by the same letter. By Lemma 1.1, V is a RAB of type (W_L, S).

Let $\mathbf{p}$ be the multiparameter for (W_L, S) given by $p_s = V_s(\mathbf{q}) - 1$. The following lemma shows that the growth series of (V, T) and (W_L, S) are related by a change of variables $\mathbf{q} \to \mathbf{p}$.

Lemma 7.1. *For $w \in W_L$,*

$$\sum_{v\in\pi^{-1}(w)} \mathbf{q}_v = \mathbf{p}_w, \tag{7.1}$$

and, therefore,

$$V(\mathbf{q}) = W(\mathbf{p}).$$

Proof. Let $s_1 \cdots s_n$ be a reduced expression for $w \in W$ and let $v \in \pi^{-1}(w)$. Then v factors as a product $v_{s_1} \cdots v_{s_n}$, with $v_{s_i} \in V^*_{s_i}$, and this factorization gives one-to-one correspondence between $\pi^{-1}(w)$ and $V^*_{s_1} \times \cdots \times V^*_{s_n}$; moreover, $\mathbf{q}_v = \mathbf{q}_{v_{s_1}} \cdots \mathbf{q}_{v_{s_n}}$. (Recall from §1.4 that $V^*_s = V_s - \{1\}$.) Hence, the growth series of $\pi^{-1}(w)$ is the product of the growth series of the $V^*_{s_i}$, and the result follows. □

In the first subsection we compute the weighted L^2-Betti numbers of V in the case where $\mathbf{q} \notin \mathcal{R}(V_s)$ for each $s \in S$. Notice that this necessarily entails that each V_s is infinite. The proof uses the spectral sequence of §2 in the same way as in §4.3. In the second subsection we consider the opposite situation where $\mathbf{q} \in \overline{\mathcal{R}(V_s)}$ for each $s \in S$. For example, this holds for all $\mathbf{q}$ when each V_s is finite. In this case the proofs are based on arguments from [15].

7.1 Large weights

In this subsection *we assume* $\mathbf{q} \notin \mathcal{R}(V_s)$ *for each* $s \in S$.

Theorem 7.2.

$$L^2_{\mathbf{q}} b^n(V) = \sum_{\substack{i+j=n \\ J \in \mathcal{S}(W,S)}} b^i(K_J, \partial K_J) \cdot L^2_{\mathbf{q}} b^j(V_J),$$

where

$$L^2_{\mathbf{q}} b^j(V_J) = \prod_{\substack{\sum k(s)=k \\ s \in J}} L^2_{\mathbf{q}} b^{k(s)}(V_s).$$

Proof of Theorem 7.2. The proof is almost the same as the proof of Theorem 4.5. Since (V, T) is a Coxeter system, we prefer to use its natural action on its Davis complex rather than on EV. Let $Y := \mathcal{U}(V, \mathcal{K})$ be the Davis complex and for each $J \in \mathcal{S}(W, S)$, put

$$Y'_J := \mathcal{U}(V_J, \mathcal{K}(J)), \quad Y_J := V \times_{V_J} Y'_J.$$

As before,

$$Y'_J = \prod_{s \in J} Y'_s, \quad \text{where} \quad Y'_s := \mathcal{U}(V_s, \mathcal{K}(s)).$$

Then $\mathcal{V} = \{Y_J\}_{J \in \mathcal{S}(W,S)}$ is a poset of spaces on Y. The spectral sequence of §2 has $E_1^{i,j} = C^j(K; \mathcal{H}^j(\mathcal{V}))$, where the coefficient system is defined by $\sigma \mapsto H^j_V(Y_{\min \sigma}; \mathcal{N}_{\mathbf{q}}(V))$. It converges to $H^*_V(Y; \mathcal{N}_{\mathbf{q}}(V))$ and the $\mathcal{N}_{\mathbf{q}}$-dimensions of these cohomology spaces are the $L^2_{\mathbf{q}}$-Betti numbers. Since $\mathbf{q} \notin \mathcal{R}(V_s)$ for each $s \in S$, $H^0_{V_s}(Y'_s; \mathcal{N}_{\mathbf{q}}(V_s)) = 0$ by [21].

and the relative Künneth Formula gives that

$$H^*_{V_J}(Y'_J; \mathcal{N}_{\mathbf{q}}(V_J)) \to H^*_{V_J}(Y'_{<J}; \mathcal{N}_{\mathbf{q}}(V_J))$$

is the zero map. By Lemma 2.2,

$$E_2^{i,j} = \bigoplus_{J \in \mathcal{S}(W,S)} H^i(K_J, \partial K_J) \otimes \mathcal{N}_{\mathbf{q}}(V)$$

and the spectral sequence degenerates at E_2. Taking von Neumann dimensions, we get the formula for weighted L^2-Betti numbers. The last formula also follows from the Künneth Formula. □

We also have a weighted version of Theorem 4.8. Let V' denote the direct sum $\prod_{s \in S} V_s$. A Coxeter system (V'', T) *lies between* V *and* V' if its presentation is given by changing certain entries of its Coxeter matrix from ∞ to an even integers ≥ 2 , more specifically, for any (t_1, t_2) with $t_i \in T_{s_i}$ and $\{s_1, s_2\} \notin \text{Edge}(\Gamma)$, we are allowed change $m(t_1, t_2)$ from ∞ to an even integer. Note that a multiparameter $\mathbf{q}$ for (V, T) is also a multiparameter for (V', T) and for (V'', T). The proof of Theorem 7.2 also gives the following.

Theorem 7.3 (cf. Theorem 4.8)**.**

$$L^2_{\mathbf{q}} b^n(\mathcal{U}(V'', \mathcal{K})) = \sum_{\substack{i+j=n \\ J \in \mathcal{S}}} b^i(K_J, \partial K_J) \cdot L^2_{\mathbf{q}} b^j(V_J).$$

7.2 Small Weights

Throughout this subsection we suppose that the multiparameter $\mathbf{q}$ is "small" in the sense that $\mathbf{q} \in \mathcal{R}(V_s)$ for each $s \in S$. Let M be a mirrored CW complex over S and let $\mathcal{B}(V, M)$ be the M-realization of V defined by (1.2) of §1.3.

As before, define a measure $\mu_{\mathbf{q}}$ on the set of cells of $\mathcal{B}(V, M)$ by putting $\mu_{\mathbf{q}}(c) := \mathbf{q}_v$, where $v \in V$ is the shortest element such that vc lies in the base chamber M. Again, we get a cochain complex, $L^2_{\mathbf{q}} C^*(\mathcal{B}(V, M))$. (If Y is a mirrored CW complex over T, then associated to the Coxeter system (V, T) there is a different cochain complex, $L^2_{\mathbf{q}} C^*(\mathcal{U}(V, Y))$.) There is a coefficient system $\mathcal{I}(\mathcal{N}_{\mathbf{q}})$ on M defined by

$$\mathcal{I}(\mathcal{N}_{\mathbf{q}})(c) := \mathcal{N}_{\mathbf{q}}(V) a_{\pi^{-1}(S(c))}$$

and the $\mathcal{N}_{\mathbf{q}}(V)$-dimension of $H^j(M; \mathcal{I}(\mathcal{N}_{\mathbf{q}}))$ is $L^2_{\mathbf{q}} b^j(\mathcal{B}(V, M))$.

Let K and $\mathcal{K}$ denote the geometric realizations of $\mathcal{S}(W, S)$ and $\mathcal{S}(V, T)$, respectively.

Theorem 7.4. $L^2_{\mathbf{q}} b^*(\mathcal{U}(V, \mathcal{K})) = L^2_{\mathbf{q}} b^*(\mathcal{B}(V, K))$.

Proof. We use the same spectral sequence as in the proof of Theorem 7.2. It converges to $H^*_V(\mathcal{U}(V, \mathcal{K}); \mathcal{N}_{\mathbf{q}}(V))$ and has E_1-term:

$$E_1^{i,j} = C^j(K; \mathcal{H}^j(\mathcal{V})).$$

where the coefficient system is defined by $\sigma \mapsto H^j_V(X_{\min \sigma}; \mathcal{N}_{\mathbf{q}}(V))$. Since $\mathbf{q} \in \mathcal{R}(V_s)$, for each $s \in S$, by Dymara's result, Theorem 6.2, the coefficients are nonzero only for $j = 0$. For $j = 0$ the coefficient system is associated to the poset of coefficients $J \mapsto \mathcal{N}_{\mathbf{q}} a_{\pi^{-1}(J)}$. In §6.2 we denoted this coefficient system by $\mathcal{I}(\mathcal{N}_{\mathbf{q}})$. So, $E_1^{i,0}$ is the cochain complex $C^j(K;\mathcal{I}(\mathcal{N}_{\mathbf{q}}))$, in other words, it is the cochain complex whose cohomology gives the $b^*_{\mathbf{q}}(\mathcal{B}(V,K))$. Thus,

$$b^*_{\mathbf{q}}(\mathcal{U}(V,\mathcal{K})) = b^*_{\mathbf{q}}(\mathcal{B}(V,K)). \qquad \square$$

Remark. Suppose each V_s is finite. Then each link $\mathcal{L}(s)$ is a simplex, and it follows that the natural map $\mathcal{L} \to L$, induced by π, has contractible fibers. It follows that $\mathcal{K}$ deformation retracts to K, respecting the mirror structure. This deformation retraction induces a V-equivariant, proper homotopy equivalence $\mathcal{U}(V,\mathcal{K}) \to \mathcal{B}(V,K)$. So, when each V_s is finite, Theorem 7.4 is the expected result.

Theorem 7.5. $L^2_{\mathbf{q}}b^*(\mathcal{B}(V,M)) = L^2_{\mathbf{p}}b^*(\mathcal{U}(W,M))$.

Remark. If each V_s is finite and $\mathbf{q} = \mathbf{1}$, then V is a locally finite building of type (W,S) of thickness $\mathbf{p}_s = |V_s| - 1$; so this reduces to Theorem 6.4 (i.e., [15, Thm. 13.8] or [11, Thm. 20.8.4]). The key point of the proof, which goes back to [21], is that the folding map π pulls back p-weighted harmonic cochains to q-weighted harmonic cochains. The proof of Theorem 7.5 is a minor generalization of the proof of [15, Thm. 13.8] to locally infinite buildings. It occupies the end of this subsection.

Example 7.6. Figure 1 depicts the folding map $\pi : \mathcal{B}(V,M) \to \mathcal{U}(W,M)$ in the case of the free product of two infinite dihedral groups. Here the graph Γ is two disjoint points s and t, so W is $\mathbf{D}_\infty$ generated by s and t. The vertex groups are also $\mathbf{D}_\infty$, generated by $\{s^+, s^-\}$ and $\{t^+, t^-\}$, respectively. So, $V = \mathbf{D}_\infty * \mathbf{D}_\infty$, and we let M be a segment K.

Combining the two previous theorems we get the following.

Theorem 7.7. $L^2_{\mathbf{q}}b^*(V) = L^2_{\mathbf{p}}b^*(W)$.

Proof.

$$\begin{aligned} L^2_{\mathbf{q}}b^*(V) &:= L^2_{\mathbf{q}}b^*(\mathcal{U}(V,\mathcal{K})) = L^2_{\mathbf{q}}b^*(\mathcal{B}(V,K)) && \text{(by Theorem 7.4)}\\ &= L^2_{\mathbf{p}}b^*(\mathcal{U}(W,K)) && \text{(by Theorem 7.5)}\\ &:= L^2_{\mathbf{p}}b^*(W). && \square \end{aligned}$$

Remark 7.8. (*Graph products of spherical buildings*). Suppose $\{\mathcal{C}_s\}_{s\in S}$ is a family of buildings where $\mathcal{C}_s$ is type (V_s, T_s). In [12, Ex. 3.1 (3)] the first

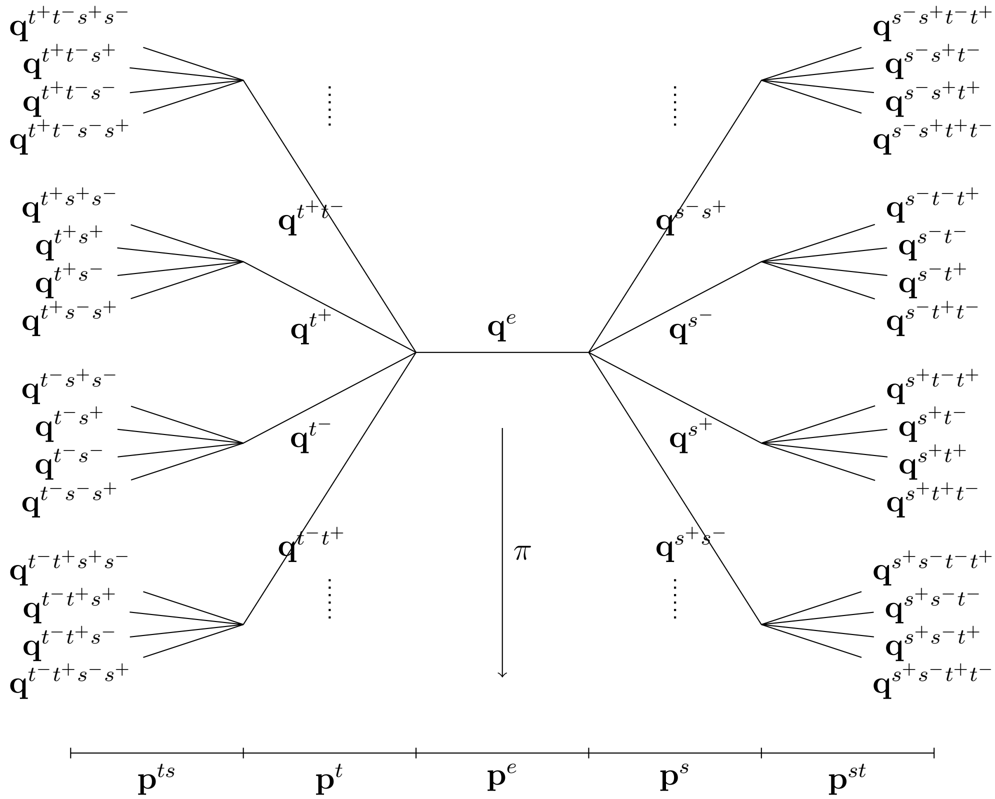

Figure 1: $\mathbf{D}_\infty * \mathbf{D}_\infty$ as a locally infinite building over $\mathbf{D}_\infty$.

author defined the notion of a "graph product of buildings," $\prod_\Gamma \mathcal{C}_s$. It is a building of type (V, T). Suppose each $\mathcal{C}_s$ is spherical of thickness $\mathbf{q}_s$ with a chamber-transitive automorphism group $G_s := \mathrm{Aut}(\mathcal{C}_s)$ (i.e., each V_s is a finite Coxeter group and the number of chambers of $\mathcal{C}_s$ in a panel of type t_s is $q_{t_s}+1$). By Theorem 6.4, the ordinary L^2-Betti numbers of $\mathcal{C} := \prod_\Gamma \mathcal{C}_s$ with respect to $G := \prod_\Gamma G_s$ are given by

$$L^2b^j(\mathcal{B}(\mathcal{C},\mathcal{K})) = L^2_{\mathbf{q}}b^j(V) = L^2_{\mathbf{p}}b^j(W_L),$$

where $p_s = V_s(\mathbf{q}) - 1 = |\mathcal{C}_s| - 1$. In other words, the L^2-Betti numbers of a graph product of spherical buildings depend only on the thickness of the buildings and the weighted L^2-Betti numbers of the associated RACS, (W, S).

The proof of Theorem 7.5 The proof is a modification of the proof in [15, Theorem 13.8 in Section 13] and follows a series of lemmas.

Lemma 7.9. (i) *The map $\pi : V \to W$ induces an isometric embedding $\pi^* : L^2_{\mathbf{p}}(W) \to L^2_{\mathbf{q}}(V)$.*

(ii) *For each $s \in S$, $\pi^*(a_s) = a_{T_s}$. Moreover, for each spherical subset $J \subset S$, $\pi^*(a_J) = a_{\pi^{-1}(J)}$.*

(iii) *The map $\pi^* : L^2_{\mathbf{p}}(W) \to L^2_{\mathbf{q}}(V)$ induces a monomorphism of von Neumann algebras $\pi^* : \mathcal{N}_{\mathbf{p}}(W) \to \mathcal{N}_{\mathbf{q}}(V)$. (In particular, π^* commutes with the $*$ anti-involutions on $\mathcal{N}_{\mathbf{p}}(W)$ and $\mathcal{N}_{\mathbf{q}}(V)$.)*

Proof. To prove (i), notice that as w varies over W, the vectors $\pi^*(e_w)$ are orthogonal to each other, and equation (7.1) implies that $\|\pi^*(e_w)\|_{\mathbf{q}} = \|e_w\|_{\mathbf{p}}$.

Statement (ii) follows immediately from the definitions.

The idempotents a_s and a_r, with $r, s \in S$, commute if and only if r and s commute. So, if a_s commutes with a_r, then a_{T_s} commutes with a_{V_r}. Since the a_s generate the Hecke algebra, statement (iii) follows from (i) and (ii). □

Similarly to the equation (7.1), the measures $\mu_{\mathbf{q}}$ and $\mu_{\mathbf{p}}$ on the cells of $\mathcal{B}(V, M)$ and $\mathcal{U}(W, M)$ are related by

$$\sum_{c' \in \pi^{-1}(c)} \mu_{\mathbf{q}}(c') = \mu_{\mathbf{p}}(c). \tag{7.2}$$

By Lemma 7.9, the map $\pi : \mathcal{B}(V, M) \to \mathcal{U}(W, M)$ induces a cochain map $\pi^* : L^2_{\mathbf{p}}C^*(\mathcal{U}(W, M)) \to L^2_{\mathbf{q}}C^*(\mathcal{B}(V, M))$. We also have a "transfer map" $t : L^2_{\mathbf{q}}C^*(\mathcal{B}(V, M)) \to L^2_{\mathbf{p}}C^*(\mathcal{U}(W, M))$ defined by

$$t(f)(c) := \sum_{c' \in \pi^{-1}(c)} f(c') \frac{\mu_{\mathbf{q}}(c')}{\mu_{\mathbf{p}}(c)}.$$

Lemma 7.10. (i) *$t \circ \pi^* = id : L^2_{\mathbf{p}}C^i(\mathcal{U}(W, M)) \to L^2_{\mathbf{p}}C^i(\mathcal{U}(W, M))$.*

(ii) *The maps π^* and t are adjoint to each other.*

(iii) *These maps take harmonic cocycles to harmonic cocycles.*

Proof. Statement (i) is obvious.

(ii) For $f \in L^2_{\mathbf{p}}C^i(\mathcal{U}(W,M))$ and $f' \in L^2_{\mathbf{q}}C^i(\mathcal{B}(V,M))$, we have

$$\begin{aligned}\langle \pi^*(f), f' \rangle_{\mathbf{q}} &= \sum_{c' \in \mathcal{B}^{(i)}} [\pi^*(f)(c')][f'(c')]\mu_{\mathbf{q}}(c') \\ &= \sum_{c' \in \mathcal{B}^{(i)}} f(\pi(c'))f'(c')\mu_{\mathbf{q}}(c') \\ &= \sum_{c \in \mathcal{U}^{(i)}} f(c) \sum_{c' \in \pi^{-1}(c)} f'(c')\mu_{\mathbf{q}}(c') \\ &= \sum_{c \in \mathcal{U}^{(i)}} [f(c)][t(f')(c)]\mu_{\mathbf{p}}(c) \\ &= \langle f, t(f') \rangle_{\mathbf{p}},\end{aligned}$$

where $\mathcal{B}^{(i)}$ and $\mathcal{U}^{(i)}$ denote the set of i-cells in $\mathcal{B}(V,M)$ and $\mathcal{U}(W,M)$, respectively.

(iii) Since $\pi^* : L^2_{\mathbf{p}}C^*(\mathcal{U}(W,M)) \to L^2_{\mathbf{q}}C^*(\mathcal{B}(V,M))$ is induced by the cellular map $\pi : \mathcal{B}(V,M) \to \mathcal{U}(W,M)$, it takes cocycles to cocycles. We must show it also takes cycles to cycles. If $c' \in \mathcal{B}^{(i-1)}$ and $d' \in \mathcal{B}^{(i)}$ and if the incidence number $[c' : d']$ is nonzero, then it is equal to $[\pi(c') : \pi(d')]$. Hence,

$$\partial^{\mathbf{q}}(\pi^*(f))(c') = \sum_{d'} [c' : d'] \frac{\mu_{\mathbf{q}}(d')}{\mu_{\mathbf{q}}(c')} f(\pi(c')) = \sum_{d} [c : d] \frac{\mu_{\mathbf{p}}(d)}{\mu_{\mathbf{p}}(c)} f(c) = \partial^{\mathbf{p}}(f)(c),$$

where $c = \pi(c')$, $d = \pi(d')$, the first and the last equality come from the definition, and the middle equality comes from equation (7.2). So, $\partial^{\mathbf{p}}(f) = 0$ implies that $\partial^{\mathbf{q}}(\pi^*(f)) = 0$. Since t is the adjoint of π^*, it also must take cocycles to cocycles and cycles to cycles. □

Consider the diagram:

$$\begin{array}{ccccc} \bigoplus L^2_{\mathbf{p}}(W) & \xrightarrow{\oplus a_{S(c)}} & \bigoplus A_{S(c)} = L^2_{\mathbf{p}}C^*(\mathcal{U}(W,M)) & \xrightarrow{P} & L^2_{\mathbf{p}}\mathcal{H}^*(\mathcal{U}(W,M)) \\ \downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle \pi^*} \\ \bigoplus L^2_{\mathbf{q}}(V) & \xrightarrow{\oplus a_{\pi^{-1}(S(c))}} & \bigoplus A_{\pi^{-1}(S(c))} = L^2_{\mathbf{q}}C^*(\mathcal{B}(V,M)) & \xrightarrow{P} & L^2_{\mathbf{q}}\mathcal{H}^*(\mathcal{B}(V,M)) \end{array}$$

where P denotes the orthogonal projection onto harmonic cocycles.

Lemma 7.11. *The above diagram commutes.*

Proof. The commutativity of the first square follows from Lemma 7.9.

Let $x \in L^2_{\mathbf{p}}C^*(\mathcal{U}(W,M))$. To prove commutativity of the second square, it is enough to show that $P\pi^*(x)-\pi^*P(x)$ is orthogonal to any harmonic cocycle $h \in L^2_{\mathbf{q}}\mathcal{H}^*(\mathcal{B}(V,M))$. We have: $\langle P\pi^*(x), h\rangle_{\mathbf{q}} = \langle \pi^*(x), P(h)\rangle_{\mathbf{q}} = \langle \pi^*(x), h\rangle_{\mathbf{q}}$. Hence,

$$\langle P\pi^*(x) - \pi^*P(x), h\rangle_{\mathbf{q}} = \langle \pi^*(x - P(x)), h\rangle_{\mathbf{q}} = \langle x - P(x), t(h)\rangle_{\mathbf{p}} = 0,$$

where the second and third equalities follow, respectively, from parts (ii) and (iii) of Lemma 7.10. □

Proof of Theorem 7.5. Let $e_c \in \bigoplus L^2_{\mathbf{q}}(V)$ denote the unit vector $e_1 \in L^2_{\mathbf{q}}(V)$ in the summand corresponding to a cell $c \in M^{(i)}$, and similarly for $L^2_{\mathbf{p}}(W)$. Note that $\pi^*(e_c) = e_c$. Let θ denote the compositions of the maps in the top and the bottom rows of the above diagrams, i.e., θ is the orthogonal projection of the free Hecke–von Neumann module onto harmonic cocycles. Using Lemmas 7.11 and 7.10, we get

$$\begin{aligned}
b^i_{\mathbf{q}}(\mathcal{B}(V,M)) &:= \dim_{\mathcal{N}_{\mathbf{q}}(V)} L^2_{\mathbf{q}}H^i(\mathcal{B}(V,M)) \\
&= \sum \langle \theta(e_c), e_c\rangle_{\mathbf{q}} = \sum \langle \theta\pi^*(e_c), \pi^*(e_c)\rangle_{\mathbf{q}} \\
&= \sum \langle \pi^*\theta(e_c), \pi^*(e_c)\rangle_{\mathbf{q}} = \sum \langle \theta(e_c), t\pi^*(e_c)\rangle_{\mathbf{p}} \\
&= \sum \langle \theta(e_c), (e_c)\rangle_{\mathbf{p}} = \dim_{\mathcal{N}_p(W)} L^2_{\mathbf{p}}H^i(\mathcal{U}(W,M)) \\
&:= b^i_{\mathbf{p}}(\mathcal{U}(W,M)).
\end{aligned}$$

□

8 Octahedralization

Suppose L is a simplicial complex. Its *octahedralization*, OL, is defined by

$$OL := \ast_L S^0. \tag{8.1}$$

Next we work out an example which motivated most of the the calculations in this paper. For each $s \in S$, V_s is the infinite dihedral group with generating set, $T_s := \{s^+, s^-\}$. Suppose (W_L, S) is the RACS associated to the graph Γ and (V,T) is the graph product of the infinite dihedral groups (so that (V,T) is also a RACS. By (5.8), $L(V,T) = OL$. So, in this special case we shall write W_{OL} for V and OS for S and call the RACS, (W_{OL}, OS), the *octahedralization* of (W,S).

Theorem 4.5 gives the following calculation of the cohomology of W_{OL} with group ring coefficients.

Theorem 8.1.

$$\operatorname{Gr} H^n(W_{OL}; \mathbf{Z}W_{OL}) = \bigoplus_{J \in \mathcal{S}(W,S)} H^{n-|J|}(K_J, \partial K_J) \otimes \mathbf{Z}(W_{OL}/W_{OJ}).$$

Proof. Since $\mathbf{D}_\infty$ acts properly and cocompactly on $\mathbf{R}^1$, it is a 1-dimensional virtual Poincaré duality group. It follows that the cohomology of $W_{OJ} = (\mathbf{D}_\infty)^{|J|}$ with group ring coefficients is given by

$$H^j(W_{OJ}; \mathbf{Z}W_{OJ}) = \begin{cases} \mathbf{Z}, & \text{if } j = |J|, \\ 0, & \text{otherwise.} \end{cases}$$

Substituting this into the formula in Theorem 4.5 gives the result. □

It is proved in [16] that the cubical complex $\widetilde{T}_L$ can be identified with $\mathcal{U}(W_{OL}, \mathcal{K})$, and that W_{OL} and A_L are commensurable. Hence, Theorem 8.1 gives a calculation of $H^*(A_L; \mathbf{Z}A_L)$. (In fact this was the method used by Jensen and Meier in their proof of Theorem 3.3.)

Remark. Since $\mathbf{Z} \subset \mathbf{D}_\infty$, there is an obvious inclusion of graph products, $A_L \subset W_{OL}$. However, whenever L is not a simplex, the image of A_L is of infinite index in W_{OL}. In [16] it is proved that A_L and W_{OL} are both isomorphic to subgroups of index $2^{|S|}$ in a larger RACG.

Weighted L^2-cohomology of $\boldsymbol{W_{OL}}$. We have $OS = \{s^+, s^-\}_{s \in S}$. Let $\mathbf{q} = (q_{s^\pm})_{s \in S}$. The growth series of the infinite dihedral group is easy to compute. (For example, see [11, Ex. 17.1.2].) We have

$$V_s(\mathbf{q}) = \frac{(1+q_{s^+})(1+q_{s^-})}{1 - q_{s^+}q_{s^-}} \quad \text{and} \quad \frac{1}{V_s(\mathbf{q}^{-1})} = \frac{q_{s^+}q_{s^-} - 1}{(1+q_{s^-})(1+q_{s^+})},$$

and

$$p_s = V_s(\mathbf{q}) - 1 = \frac{q_{s^+} + q_{s^-} + 2q_{s^+}q_{s^-}}{1 - q_{s^+}q_{s^-}}.$$

Write $\mathbf{q} < \mathbf{1}$ (resp. $\mathbf{q} > \mathbf{1}$) to mean that each $q_{s^\alpha} < 1$ (resp. > 1), for $\alpha \in \{+, -\}$. The following is a corollary of the results in §7.

Theorem 8.2. *Suppose* (W_{OL}, OS) *is the octahedralization of* (W, S).

(i)
$$W_{OL}(\mathbf{q}) = W(\mathbf{p}).$$

(ii) *If* $\mathbf{q} < \mathbf{1}$*, then*
$$L^2_{\mathbf{q}} b^n(W_{OL}) = L^2_{\mathbf{p}} b^n(W_L).$$

If, in addition, $\mathbf{p}^{-1} \in \overline{\mathcal{R}(W)}$ *(i.e.,* $\mathbf{q}$ *is sufficiently close to* $\mathbf{1}$*), then*

$$L^2_{\mathbf{q}} b^j(W_{OL}) = \sum_{J \in \mathcal{S}(W,S)} b^k(K, K^{S-J}) \dim_{\mathcal{N}_{\mathbf{p}}} D^J,$$

where a formula for $\dim_{\mathcal{N}_{\mathbf{p}}} D^J$ *is defined in Theorem 6.1.*

(iii) *If* $\mathbf{q} = \mathbf{1}/3$*, then*
$$L^2_{1/3} b^n(W_{OL}) = L^2 b^n(W_L),$$

(iv) *If* $\mathbf{q} > \mathbf{1}$*, then*

$$L^2_{\mathbf{q}} b^n(W_{OL}) = \sum_{J \in \mathcal{S}(W,S)} b^{n-|J|}(K_J, \partial K_J) \prod_{s \in J} \frac{q_{s^+}q_{s^-} - 1}{(1 + q_{s^-})(1 + q_{s^+})}.$$

(v) *If* $\mathbf{q} = \mathbf{1}$*, then*

$$L^2 b^n(W_{OL}) = b^n(K, \partial K) = \overline{b}^{n-1}(L),$$

where $\overline{b}^*(\ )$ *refers to the reduced Betti-number.*

Proof. (i),(ii) and (iii) are immediate from the formula for p_s, Lemma 7.1, and Theorems 7.7 and 6.3.

The region of convergence of the dihedral group $\mathcal{R}(\mathbf{D}_\infty)$ is given by $q_{s^+}q_{s^-} < 1$. For a spherical J, V_J is the J-fold product of $\mathbf{D}_\infty$. Thus, if $\mathbf{q} > \mathbf{1}$, then $L^2_{\mathbf{q}} H^*(V_J)$ is concentrated in degree $|J|$ and

$$L^2_{\mathbf{q}} b^{|J|}(V_J) = \frac{1}{V_J(\mathbf{q}^{-1})} = \prod_{s \in J} \frac{q_{s^+}q_{s^-} - 1}{(1 + q_{s^-})(1 + q_{s^+})}.$$

and we apply Theorem 7.2 to obtain (iv). Finally, if $\mathbf{q} = \mathbf{1}$, then all the terms with nonempty J in (iv) (or (ii)) vanish and we obtain (v). (Since $L^2 b^n(W_{OL}) = L^2 b^n(A_L)$, it also follows from Theorem 3.7.) □

Remark 8.3. If $\mathbf{q}$ is such that $q_{s^+} = q_{s^-}$ $(= q_s)$, then in Corollary 8.2(i) the formula becomes

$$L^2_{\mathbf{q}} b^n(W_{OL}) = \sum_{J \in \mathcal{S}} b^{n-|J|}(K_J, \partial K_J) \prod_{s \in J} \frac{q_s - 1}{1 + q_s},$$

and in (ii) the formula for p_s simplifies to

$$p_s = \frac{2q_s}{1 - q_s}.$$

Remark 8.4. Our original motivation for computing the weighted L^2-cohomology of the octahedralization of W_L was to compute the ordinary L^2-cohomology of the Bestvina–Brady group BB_L (which we did by the spectral sequence method in Theorem 4.4). Although we were never able to make complete sense of the calculation, it was supposed to go something like this. Suppose that the multiparameter $\mathbf{q}$ is a positive constant q. Then it should be possible to define the weighted L^2-cohomology of A_L and BB_L. Moreover, the L^2_q-Betti numbers of A_L should equal those of W_{OL} and L^2_q-Betti numbers of BB_L should behave as if the Davis complex for W_{OL} were to split as a product of the complex Z_L with the real line (with $\mathbf{D}_\infty$-action) and as if the Künneth formula were true, i.e.,

$$L^2_q b^n(W_{OL}) = L^2_q b^n(BB_L) \cdot L^2_q b^0(\mathbf{D}_\infty) = L^2_q b^n(BB_L) \left[\frac{1-q}{q+1}\right], \quad \text{for } q < 1$$

$$L^2_q b^{n+1}(W_{OL}) = L^2_q b^n(BB_L) \cdot L^2_q b^1(\mathbf{D}_\infty) = L^2_q b^n(BB_L) \left[\frac{q-1}{q+1}\right], \quad \text{for } q > 1.$$

These can be rewritten as

$$L^2_q b^n(BB_L) = L^2_q b^n(W_{OL}) \left[\frac{q+1}{1-q}\right], \quad \text{for } q < 1, \text{ and} \tag{8.2}$$

$$= L^2_q b^{n+1}(W_{OL}) \left[\frac{q+1}{q-1}\right], \quad \text{for } q > 1. \tag{8.3}$$

Next we want to find the ordinary L^2-Betti numbers of BB_L by taking the limit of either of these formulas as $q \to 1$. Since

$$W_I(p) = (1+p)^{|I|} = \left(\frac{2q}{1-q} + 1\right)^{|I|} = \left(\frac{1+q}{1-q}\right)^{|I|},$$

the formula in Theorem 6.1 becomes

$$\dim_{\mathcal{N}_{\mathbf{p}}} D^J = (-1)^{|J|} \sum_{I \in \mathcal{S}(W,S)_{\geq J}} \left(\frac{q-1}{1+q} \right)^{|I|}. \tag{8.4}$$

After multiplying through by $(1-q)/(1+q)$, the only terms on the right hand side of (8.4) which will be nonzero when $q = 1$ are those with $|I| = 1$ and hence, $|J| = 0$ or 1. Since $\partial K = K^S$ is acyclic, the term for $J = \emptyset$ in Theorem 8.2, namely, $b^n(K, K^S)$, vanishes. So, using Theorem 8.2 (i), formula (8.2) gives, for $q < 1$,

$$L^2 b^n(BB_L) = \lim_{q \to 1} L^2_q b^n(BB_L) = \sum_{s \in S} b^n(K, K^{S-s}) = \sum_{s \in S} b^n(K_s, \partial K_s)$$

where the last equation follows from the fact that ∂K is acyclic and the excision, $H^*(\partial K, K^{S-s}) \cong H^*(K_s, \partial K_s)$. Similarly, for $q > 1$, Theorem 8.2 (ii) can be written as

$$L^2_q b^{n+1}(W_{OL}) = \sum_{J \in \mathcal{S}(W,S)} b^{n-|J|+1}(K_J, \partial K_J) \left(\frac{q-1}{1+q} \right)^{|J|}$$

and (8.3) gives, for $q > 1$,

$$L^2 b^n(BB_L) = \lim_{q \to 1} L^2_q b^n(BB_L) = \sum_{s \in S} b^n(K_s, \partial K_s),$$

since when $q = 1$ only the terms with $|J| = 1$ are nonzero. So, in Theorem 8.2 both the formulas (i) and (ii) give the same answer as Theorem 4.4.

9 Duality groups

An m-dimensional simplicial complex L is *Cohen–Macaulay* if for each $J \in \mathcal{S}(L)$, $\overline{H}^*(\mathrm{Lk}(J))$ is concentrated in degree $m - |J|$ and is torsion-free. For $J = \emptyset$, this means that $\overline{H}^*(L)$ is concentrated in degree m. It also implies that any maximal simplex J has dimension m, since, when J is maximal, $\mathrm{Lk}(J) = \emptyset$ and our convention is that $\overline{H}^*(\emptyset)$ is concentrated in degree -1 (where it is $= \mathbf{Z}$).

A group G of type FP is an n-dimensional *duality group* if $H^*(G; \mathbf{Z}G)$ is concentrated in degree n and is torsion-free.

An immediate consequence of Corollary 8.1 is the following.

Proposition 9.1 (Brady–Meier [5] and Jensen–Meier [26]). *Suppose* (W, S) *is a RACS with nerve* L. *Then the octahedralization,* W_{OL}, *is a virtual duality group if and only if* L *is Cohen–Macaulay (and consequently, the same is true for the associated RAAG,* A_L*).*

Brady and Meier asked if these conditions are equivalent for a general Artin group (Question 2 of [5]) and attributed the question to the first author of this paper. The equivalence follows immediately from Theorem 4.1 whenever the $K(\pi, 1)$ Conjecture holds for A. We state this as the following.

Proposition 9.2. *Suppose* X *is the Salvetti complex associated to a Coxeter system with nerve* L. *Then* $H^*(X; \mathbf{Z}A)$ *is concentrated in degree* n *and is torsion-free if and only if* L *is an* $(n-1)$*-dimensional Cohen–Macaulay complex.*

(The "if" direction was also proved by Brown–Meier in [7] by using a different spectral sequence.)

Similarly, by Theorem 4.3, for Bestvina–Brady groups we have the following.

Proposition 9.3. *Suppose* L *is an acyclic flag complex. Then* BB_L *is a duality group if and only if* L *is Cohen–Macaulay. (For example,* L *could be a acyclic, compact manifold with boundary.)*

As explained in [18, Sec. 6], for graph products of finite groups, Theorem 3.5 leads to a slightly different condition. An m-dimensional simplicial complex L has *punctured homology concentrated in dimension* m (abbreviated PH^m) if for each closed simplex σ of L, $\overline{H}_*(L - \sigma)$ is torsion free and concentrated in degree m. The PH^m condition implies that L is Cohen–Macaulay but is not equivalent to it (cf. [18, Cor. 6.9]).

Proposition 9.4 ([18, Theorem 6.2] and also cf. [24]). *Let* G *be the graph product of a collection of nontrivial finite groups. Then* G *is a* n*-dimensional duality group if and only if* L *is* PH^{n-1}.

References

[1] P. Abramenko and K. Brown, *Buildings: Theory and Applications*, GTM **248**, Springer-Verlag, New York, 2008.

[2] A. Bahri, M. Bendersky, F.R. Cohen and S. Gitler, *Decompositions of the polyhedral product functor with applications to moment-angle complexes and related spaces*, Proc. Natl. Acad. Sci. USA **106** (2009), 12241–12244.

[3] M. Bestvina, *Non-positively curved aspects of Artin groups of finite type*, Geometry & Topology **3** (1999), 269–302.

[4] M. Bestvina and N. Brady, *Morse theory and finiteness properties of groups*, Invent. Math. **129** (1997), 123–139.

[5] N. Brady and J. Meier, *Connectivity at infinity for right angled Artin groups*, Transactions of the AMS **353** (2001) 117-132.

[6] K.S. Brown, *Cohomology of Groups*, Springer-Verlag, Berlin and New York, 1982.

[7] K. Brown and J. Meier, *Improper actions and higher connectivity at infinity*, Comment. Math. Helv. **75** (2000) 171–188.

[8] R. Charney and M.W. Davis, *The $K(\pi,1)$-problem for hyperplane complements associated to infinite reflection groups*, Journal of the AMS **8** (1995), 597–627.

[9] R. Charney and M.W. Davis, *Finite $K(\pi,1)$s for Artin groups*, Prospects in Topology (Frank Quinn ed.), Annals of Math Studies vol. 138, Princeton, 1995, 110–124.

[10] M.W. Davis, *The cohomology of a Coxeter group with group ring coefficients*, Duke Math. J. **91** (1998), 297–314.

[11] M.W. Davis, *The Geometry and Topology of Coxeter Groups*, London Math. Soc. Monograph Series, vol. 32, Princeton Univ. Press, 2008.

[12] M.W. Davis, *Examples of buildings constructed via covering spaces*, Groups Geom. Dyn. **3** (2009), 279–298.

[13] M.W. Davis, J. Dymara, T. Januszkiewicz, J. Meier and B. Okun, *Compactly supported cohomology of buildings*, preprint, arXiv:0806.2412, (2008), to appear in Comment. Math. Helv.

[14] M.W. Davis, J. Dymara, T. Januszkiewicz and B. Okun, *Cohomology of Coxeter groups with group ring coefficients: II*, Algebr. Geom. Topol. **6** (2006), 1289–1318.

[15] M.W. Davis, J. Dymara, T. Januszkiewicz and B. Okun, *Weighted L^2-cohomology of Coxeter groups*, Geometry & Topology **11** (2007), 47–138

[16] M.W. Davis and T. Januszkiewicz, *Right-angled Artin groups are commensurable with right-angled Coxeter groups*, J. of Pure and Appl. Algebra **153** (2000), 229–235.

[17] M.W. Davis and I.J. Leary, *The ℓ^2-cohomology of Artin groups*, J. London Math. Soc. (2) **68** (2003), 493–510.

[18] M.W. Davis and J. Meier, *The topology at infinity of Coxeter groups and buildings*, Comment. Math. Helv. **77** (2002), 746–766.

[19] M.W. Davis and B. Okun, *Vanishing theorems and conjectures for the ℓ^2-homology of right-angled Coxeter groups*, Geometry & Topology **5** (2001), 7–74.

[20] G. Denham and A. Suciu, *Moment-angle complexes, monomial ideals and Massey products*, Pure Appl. Math. Q. **3** (2007), no. 1, part 3, 25–60.

[21] J. Dymara, *Thin buildings*, Geometry & Topology **10** (2006), 667–694.

[22] J. Dymara and T. Januszkiewicz, *Cohomology of buildings and their automorphism groups*, Invent. Math. **150** (2002), 579–627.

[23] A. Haefliger, *Extensions of complexes of groups*, Ann. Inst. Fourier (Grenoble) **42** (1992), 275–311.

[24] J. Harlander and H. Meinert, *Higher generation subgroup sets and the virtual cohomological dimension of graph products of finite groups* (English summary), J. London Math. Soc. (2) **53** (1996), no. 1, 99–117.

[25] T. Januszkiewicz and J. Świątkowski, *Commensurability of graph products*, Algebr. Geom. Topol. **1** (2001), 587–603.

[26] C. Jensen and J. Meier, *The cohomology of right-angled Artin groups with group ring coefficients*, Bulletin of London Math. Soc. **37** (2005), 711–718.

[27] I.J. Leary and M. Saadetoğlu, *The cohomology of Bestvina–Brady groups*, Groups Geom. Dyn. (to appear).

[28] W. Lück, L^2*-invariants and* K*-theory*, Springer-Verlag, Berlin and New York, 2002.

[29] C.C. Squier, *The homological algebra of Artin groups*, Math. Scand. **75** (1994), 5–43.