ISSN 1364-0380 (on line) 1465-3060 (printed) **1032**

Geometry & Topology
Volume 8 (2004) 1032–1042
Published: 7 August 2004

Weighted L^2–cohomology of Coxeter groups based on barycentric subdivisons

Boris L Okun

Department of Mathematical Sciences
University of Wisconsin–Milwaukee
Milwaukee, WI 53201, USA

Email: okun@uwm.edu

Abstract

Associated to any finite flag complex L there is a right-angled Coxeter group W_L and a contractible cubical complex Σ_L (the Davis complex) on which W_L acts properly and cocompactly, and such that the link of each vertex is L. It follows that if L is a generalized homology sphere, then Σ_L is a contractible homology manifold. We prove a generalized version of the Singer Conjecture (on the vanishing of the reduced weighted $L^2_{\mathbf{q}}$–cohomology above the middle dimension) for the right-angled Coxeter groups based on barycentric subdivisions in even dimensions. We also prove this conjecture for the groups based on the barycentric subdivision of the boundary complex of a simplex.

AMS Classification numbers Primary: 58G12

Secondary: 20F55, 57S30, 20F32, 20J05

Keywords: Coxeter group, aspherical manifold, barycentric subdivision, weighted L^2–cohomology, Tomei manifold, Singer conjecture

Proposed: Benson Farb Received: 15 March 2004
Seconded: Walter Neumann, Steven Ferry Revised: 3 August 2004

1 Introduction

A construction of Davis ([1], [2], [4]), associates to any finite flag complex L, a "right-angled" Coxeter group W_L and a contractible cubical cell complex Σ_L on which W_L acts properly and cocompactly. W_L has the following presentation: the generators are the vertices of L, each generator has order 2, and two generators commute if the span an edge in L. The most important feature of this construction is that the link of each vertex of Σ_L is isomorphic to L. A simplicial complex L is a *generalized homology m–sphere* (for short, a GHS^m) if it is a homology m–manifold having the same homology as a standard sphere $\mathbb{S}^m$ (the homology is with real coefficients.) It follows, that if L is a GHS^{n-1}, then Σ_L is a homology n–manifold.

If L is a simplicial complex, bL will denote the barycentric subdivision of L. bL is a flag simplicial complex. Let $\partial\Delta^n$ denote the boundary complex of the standard n–dimensional simplex.

We study a certain weighted L^2–cohomology theory $L^2_{\mathbf{q}}\mathcal{H}^*$, described in [7], [5]. Suppose, for each vertex of $v \in L$ we are given a positive real number q_v, and let $\mathbf{q}$ denote the vector with components q_v. Given a minimal word $w = v_1 \dots v_n \in W_L$, let $\mathbf{q}^w = q_{v_1} \dots q_{v_n}$. For each W_L–orbit of cubes pick a representative σ_0 and let $w(\sigma) = w$ if $\sigma = w\sigma_0$. (The ambiguity in the choices will not matter in our discussion.) Let $L^2_{\mathbf{q}}C^i(\Sigma_L) = \{\Sigma c_\sigma \sigma \mid \Sigma c_\sigma^2 \mathbf{q}^{w(\sigma)} < \infty\}$ be the Hilbert space of infinite i–cochains, which are square-summable with respect to the weight $\mathbf{q}^w$. The usual coboundary operator d is then a bounded operator, and we define the reduced weighted $L^2_{\mathbf{q}}$–cohomology to be $L^2_{\mathbf{q}}\mathcal{H}^i(\Sigma_L) = \mathrm{Ker}(d^i)/\overline{\mathrm{Im}(d^{i-1})}$. Similarly, one can define the reduced weighted $L^2_{\mathbf{q}}$–homology, except, instead of the usual boundary operator one uses the adjoint of d. It follows from the Hodge decomposition that the resulting homology and cohomology spaces are naturally isomorphic. These spaces are Hilbert modules over the Hecke–von Neumann algebra $\mathcal{N}_{\mathbf{q}}$ (an appropriately completed Hecke algebra of W_L.) This allows us to introduce the weighted $L^2_{\mathbf{q}}$ Betti numbers — the dimension of $L^2_{\mathbf{q}}\mathcal{H}^i$ over $\mathcal{N}_{\mathbf{q}}$. If $\mathbf{q} = \mathbf{1} = (1, \dots, 1)$, we obtain the usual reduced L^2–cohomology, and we omit the index $\mathbf{q}$. We write $\mathbf{q} \le \mathbf{1}$, if each component of $\mathbf{q}$ is ≤ 1.

The following conjecture, attributed to Singer, goes back to 1970's.

The Singer Conjecture If M^n is a closed aspherical manifold, then

$$L^2\mathcal{H}^i(\widetilde{M^n}) = 0 \text{ for all } i \ne n/2.$$

As explained in [5, Section 14], the appropriate generalization of the Singer Conjecture to the weighted case is the following conjecture:

The Generalized Singer Conjecture Suppose L is a flag GHS^{n-1}. Then $L^2_{\mathbf{q}}\mathcal{H}^i(\Sigma_L) = 0$ for $i > n/2$ and $\mathbf{q} \le \mathbf{1}$.

(Poincaré duality shows that for $\mathbf{q} = \mathbf{1}$ this conjecture implies the Singer Conjecture for Σ.)

This conjecture holds true for $n \le 4$ by [5]. One of the main results of this paper is a proof of this conjecture for barycentric subdivisions in even dimensions. The proof uses a reduction to a very special case $L = b\partial\Delta^{2k-1}$.

We prove this case as Theorem 5.2. It turns out (Theorem 5.3), that this result implies the vanishing of the $L^2_{\mathbf{q}}$–cohomology in a certain range for arbitrary right-angled Coxeter groups based on barycentric subdivisions. (For $\mathbf{q} = \mathbf{1}$, this implication is proved in [6].) In particular, it follows that the Generalized Singer Conjecture is true for all barycentric subdivisions in even dimensions (Theorem 5.4), and for $b\partial\Delta^n$ in all dimensions (Theorem 5.6).

This paper relies very heavily on [5]. In the inductive proofs we mostly omit the first steps, they are easy exercises using [5].

The author wishes to thank Mike Davis, Jan Dymara and Tadeusz Januszkiewicz for several illuminating discussions.

2 Vanishing conjectures

We will follow the notation from [5]. Given a flag complex L and a full subcomplex A, set:

$$\begin{aligned}
\mathfrak{h}_i^{\mathbf{q}}(L) &= L^2\mathcal{H}_i(\Sigma_L)\\
\mathfrak{h}_i^{\mathbf{q}}(A) &= L^2\mathcal{H}_i(W_L\Sigma_A)\\
\mathfrak{h}_i^{\mathbf{q}}(L,A) &= L^2\mathcal{H}_i(\Sigma_L, W_L\Sigma_A)\\
\mathrm{b}^i_{\mathbf{q}}(L) &= \dim_{\mathcal{N}_{\mathbf{q}}}(\mathfrak{h}_i^{\mathbf{q}}(L))\\
\mathrm{b}^i_{\mathbf{q}}(L,A) &= \dim_{\mathcal{N}_{\mathbf{q}}}(\mathfrak{h}_i^{\mathbf{q}}(L,A))
\end{aligned}$$

The dimension of Σ_L is one greater than the dimension of L. Hence, $\mathrm{b}^i_{\mathbf{q}}(L) = 0$ for $i > \dim L + 1$.

We will use the following three properties of $L^2_{\mathbf{q}}$–homology.

Proposition 2.1 (See [5, Section 15])

The Mayer–Vietoris sequence *If $L = L_1 \cup L_2$ and $A = L_1 \cap L_2$, where L_1 and L_2 (and therefore, A) are full subcomplexes of L, then*

$$\to \mathfrak{h}_i^{\mathbf{q}}(A) \to \mathfrak{h}_i^{\mathbf{q}}(L_1) \oplus \mathfrak{h}_i^{\mathbf{q}}(L_2) \to \mathfrak{h}_i^{\mathbf{q}}(L) \to$$

is weakly exact.

The Künneth Formula *The Betti numbers of the join of two complexes are given by:*

$$\mathrm{b}_{\mathbf{q}}^k(L_1 * L_2) = \sum_{i+j=k} \mathrm{b}_{\mathbf{q}}^i(L_1)\, \mathrm{b}_{\mathbf{q}}^j(L_2).$$

Poincaré Duality *If L is a flag GHS^{n-1}, then $\mathrm{b}_{\mathbf{q}}^i(L) = \mathrm{b}_{\mathbf{q}^{-1}}^{n-i}(L)$.*

If σ is a simplex in L, let L_σ denote the link of σ in L. To simplify notation we will write bL_v instead of $(bL)_v$ to denote the link of the vertex v in bL. Let $\mathcal{C}$ be a class of GHS's closed under the operation of taking link of vertices, i.e. if $S \in \mathcal{C}$ and v is a vertex of S then $S_v \in \mathcal{C}$. Following Section 15 of [5] we consider several variations of the Generalized Singer Conjecture for the class $\mathcal{C}$.

$\mathbf{I}(n)$ *If $S \in \mathcal{C}$ and $\dim S = n-1$, then $\mathrm{b}_{\mathbf{q}}^i(S) = 0$ for $i > n/2$ and $\mathbf{q} \le \mathbf{1}$.*

$\mathbf{III}'(2k+1)$ *Let $S \in \mathcal{C}$ and $\dim S = 2k$. Let v be a vertex of S. Then the map $i_*\colon\ \mathfrak{h}_k^{\mathbf{q}}(S_v) \to \mathfrak{h}_k^{\mathbf{q}}(S)$, induced by the inclusion, is the zero homomorphism for $\mathbf{q} \ge \mathbf{1}$.*

$\mathbf{V}(n)$ *Let $S \in \mathcal{C}$ and $\dim S = n-1$. Let A be a full subcomplex of S.*

- *If $n = 2k$ is even, then $\mathrm{b}_{\mathbf{q}}^i(S, A) = 0$ for all $i > k$ and $\mathbf{q} \le \mathbf{1}$.*
- *If $n = 2k+1$ is odd, then $\mathrm{b}_{\mathbf{q}}^i(A) = 0$ for all $i > k$ and $\mathbf{q} \le \mathbf{1}$.*

The argument in Section 16 of [5] goes through without changes if we consider only GHS's from a class $\mathcal{C}$ to give the following:

Theorem 2.2 (Compare [5, Section 16]) *If we only consider* GHS*'s from a class $\mathcal{C}$, then the following implications hold.*

(1) $\mathbf{I}(2k+1) \implies \mathbf{III}'(2k+1)$.
(2) $\mathbf{V}(n) \implies \mathbf{I}(n)$.
(3) $\mathbf{V}(2k-1) \implies \mathbf{V}(2k)$.
(4) $[\mathbf{V}(2k)$ *and* $\mathbf{III}'(2k+1)] \implies \mathbf{V}(2k+1)$.

Let $\mathcal{JD}$ denote the class of finite joins of the barycentric subdivisions of the boundary complexes of standard simplices:

$$\mathcal{JD} = \{b\partial\Delta^{n_1} * \cdots * b\partial\Delta^{n_i}\}.$$

Lemma 2.3 *The class $\mathcal{JD}$ is closed under the operation of taking link of vertices.*

Proof Let $S = b\partial\Delta^{n_1} * \cdots * b\partial\Delta^{n_j}$ and $v \in S$. We can assume that $v \in b\partial\Delta^{n_1}$. Then $S_v = b\partial\Delta^{n_1}_v * b\partial\Delta^{n_2} * \cdots * b\partial\Delta^{n_j} = b\partial\Delta^{\dim(v)} * b\partial\Delta^{n_1-\dim(v)-1} * b\partial\Delta^{n_2} * \cdots * b\partial\Delta^{n_j}$, and therefore $S \in \mathcal{JD}$. □

Next, consider the following statement:

III$''(2k+1)$ *Let v be a vertex of $b\partial\Delta^{2k+1}$. Then the map $i_*\colon\ \mathfrak{h}^{\mathbf{q}}_k(b\partial\Delta^{2k+1}_v) \to \mathfrak{h}^{\mathbf{q}}_k(b\partial\Delta^{2k+1})$, induced by the inclusion, is the zero homomorphism for $\mathbf{q} \geq \mathbf{1}$.*

Lemma 2.4 **III**$''(2k+1) \implies$ **III**$'(2k+1)$ *for the class $\mathcal{JD}$.*

Proof By induction, we can assume that the lemma holds for all odd numbers $< 2k+1$. Then it follows from the Theorem 2.2 that $\mathbf{V}(m)$ and therefore $\mathbf{I}(m)$ hold for all $m < 2k+1$.

Let $S = b\partial\Delta^{n_1} * \cdots * b\partial\Delta^{n_j}$ with $n_1 + \cdots + n_j = 2k+1$ and $v \in S$. We assume that $v \in b\partial\Delta^{n_1}$. Then $S_v = b\partial\Delta^{n_1}_v * b\partial\Delta^{n_2} * \cdots * b\partial\Delta^{n_j}$ and, by the Künneth formula, the map in question decomposes as the direct sum of maps of the form

$$(\mathfrak{h}^{\mathbf{q}}_{k_1}(b\partial\Delta^{n_1}_v) \to \mathfrak{h}^{\mathbf{q}}_{k_1}(b\partial\Delta^{n_1})) \otimes \bigotimes_{i=2}^{j} (\mathfrak{h}^{\mathbf{q}}_{k_i}(b\partial\Delta^{n_i}) \to \mathfrak{h}^{\mathbf{q}}_{k_i}(b\partial\Delta^{n_i}))$$

where $k_1 + \cdots + k_j = k$. Since $n_1 + \cdots + n_j = 2k+1$ it follows that $k_i < n_i/2$ for some index i. If $n_i < 2k+1$, then the range of the corresponding map in the above tensor product is 0 by $\mathbf{I}(n_i)$ and Poincaré duality, and therefore the tensor product map is 0. If $n_i = 2k+1$ then, in fact, $i = 1$ (the join is a trivial join) and the result follows from **III**$''(2k+1)$. □

Thus, it follows from Theorem 2.2, Lemmas 2.3 and 2.4, and induction on dimension, that in order to prove the Generalized Singer Conjecture for the class $\mathcal{JD}$ all we need is to prove **III**$''(2k+1)$.

3 Removal of an odd-dimensional vertex

Let L be a simplicial complex and bL be its barycentric subdivision. The vertices of bL are naturally graded by "dimension": each vertex v of bL is the barycenter of a unique cell (which we still denote v) of the complex L, and we call the dimension of this cell the *dimension* of the vertex v. Let E_L denote the subcomplex of bL spanned by the even dimensional vertices. Let $\mathcal{A}_L$ denote the set of full subcomplexes A of L containing E_L, which have the following property: if A contains a vertex of odd dimension $2j+1$, then A contains all vertices of bL of dimensions $\leq 2j$. In other words, any such A can be obtained inductively from bL by repeated removal of an odd-dimensional vertex of the highest dimension.

If $L = \partial\Delta^n$ we will use the notation $E_n = E_L$ and $\mathcal{A}_n = \mathcal{A}_L$.

Lemma 3.1 *Assume* $\mathbf{III}''(2m+1)$ *holds for* $2m+1 < n$*. Then for any* $(n-1)$*–dimensional simplicial complex* L *and any complex* $A \in \mathcal{A}_L$ *we have:*

$$\mathrm{b}^i_{\mathbf{q}}(A) = \mathrm{b}^i_{\mathbf{q}}(bL) = 0 \text{ for } i > (n+1)/2 \text{ and } \mathbf{q} \leq \mathbf{1}.$$

Proof By induction, we can assume that the lemma holds for all $m < n$. First, we claim that removal of odd-dimensional vertices does not change the homology above $(n+1)/2$. Let $A \in \mathcal{A}_L$ and let $B = A - v$ where v is a vertex of the highest odd dimension of A. We let $\dim(v) = 2d-1$, $1 \leq d \leq k$. We want to prove that $\mathrm{b}^i_{\mathbf{q}}(A) = \mathrm{b}^i_{\mathbf{q}}(B)$ for $i > (n+1)/2$. Consider the Mayer–Vietoris sequence of the union $A = B \cup_{A_v} CA_v$:

$$\to \mathfrak{h}^{\mathbf{q}}_i(A_v) \to \mathfrak{h}^{\mathbf{q}}_i(B) \oplus \mathfrak{h}^{\mathbf{q}}_i(CA_v) \to \mathfrak{h}^{\mathbf{q}}_i(A) \to \mathfrak{h}^{\mathbf{q}}_{i-1}(A_v)$$

Suppose $i > (n+1)/2$. Since $A_v = B \cap bL_v = B \cap (b\partial\Delta^{2d-1} * b(L_v))$, and since $B \in \mathcal{A}_L$, it follows, by construction, that A_v splits as the join:

$$A_v = b\partial\Delta^{2d-1} * A_1,$$

with $A_1 \in \mathcal{A}_{(L_v)}$. By inductive assumption the lemma holds for L_v, i.e. $\mathrm{b}^i_{\mathbf{q}}(A_1) = \mathrm{b}^i_{\mathbf{q}}(b(L_v)) = 0$ for $i > (n+1)/2 - d$.

Since $\mathbf{III}''(2d-1)$ holds by hypothesis, by Lemma 2.4 and Theorem 2.2, $\mathbf{I}(2d-1)$ holds for the class $\mathcal{JD}$, and, thus, $\mathrm{b}^i_{\mathbf{q}}(b\partial\Delta^{2d-1}) = 0$ for $i \geq d$.

Then, by the Künneth formula, $\mathrm{b}^{i-1}_{\mathbf{q}}(A_v) = 0$ for $i-1 \geq (n+1)/2$, i.e. for $i > (n+1)/2$. By [5, Proposition 15.2(d)], $\mathrm{b}^i_{\mathbf{q}}(CA_v) = \frac{1}{q_v+1}\mathrm{b}^i_{\mathbf{q}}(A_v)$. Therefore in the above sequence the terms corresponding to A_v and CA_v are 0, and the claim follows. Then it follows by induction, that $\mathrm{b}^i_{\mathbf{q}}(A) = \mathrm{b}^i_{\mathbf{q}}(bL)$ for all $A \in \mathcal{A}_L$ and $i > (n+1)/2$.

To prove the vanishing we note that, in particular, $\mathrm{b}^i_{\mathbf{q}}(E_L) = \mathrm{b}^i_{\mathbf{q}}(bL)$ for $i > (n+1)/2$. Since E_L is spanned by the even-dimensional vertices of bL and since a simplex in bL has vertices of pairwise different dimensions, we have $\dim(E_L) = [(n+1)/2] - 1$. Therefore, $\mathrm{b}^i_{\mathbf{q}}(E_L) = 0$ for $i > (n+1)/2$ and we have proved the lemma. □

In the special case $L = \Delta^{2k+1}$ this lemma admits the following strengthening:

Lemma 3.2 *Let $n = 2k+1$. Assume $\mathbf{III}''(2m+1)$ holds for $2m+1 < n$. Then for any complex $A \in \mathcal{A}_n$, $A \subset b\partial\Delta^n$, we have:*

$$\mathrm{b}^i_{\mathbf{q}}(A) = \mathrm{b}^i_{\mathbf{q}}(b\partial\Delta^n) \text{ for } i > k \text{ and } \mathbf{q} \leq \mathbf{1}.$$

Proof We proceed as in the previous proof. As before, we have $B = A - v$, $\dim(v) = 2d-1$ and $A_v = b\partial\Delta^{2d-1} * A_1$, where now $A_1 \subset b\partial\Delta^{2k+1-2d}$. Therefore, the inductive assumption and the hypothesis on $\mathbf{III}''(2d-1)$ imply that $b_i(A_1) = 0$ for $i > k+d$. The lemma follows as before. □

As explained in [6], when $\mathbf{q} = \mathbf{1}$, the removal of the odd-dimensional vertex does not change homology in *all* dimensions. We record this result below.

Lemma 3.3 *Assume $\mathbf{III}''(2m+1)$ holds for $2m+1 < n$ and $\mathbf{q} = \mathbf{1}$. Then for any $(n-1)$–dimensional simplicial complex L and for any complex $A \in \mathcal{A}_L$, obtained by the repeated removal of highest odd-dimensional vertices, we have:*

$$\mathrm{b}^*(A) = \mathrm{b}^*(bL).$$

Proof Again we repeat the proof of Lemma 3.1. As before, we have the splitting $A_v = b\partial\Delta^{2d-1} * A_1$. The point now is that for $\mathbf{q} = \mathbf{1}$, $\mathbf{I}(2d-1)$ and Poincaré duality imply $\mathrm{b}^*(b\partial\Delta^{2d-1}) = 0$ and therefore $\mathrm{b}^*(A_v) = 0$ by the Künneth formula. □

4 Intersection form

Lemma 4.1 *Let L be a GHS^{2k} and let v be a vertex of L. Then the image of the restriction map on L^2–cohomology $i^*\colon\ L^2\mathcal{H}^k(\Sigma_L) \to L^2\mathcal{H}^k(\Sigma_{L_v})$ is an isotropic subspace of the intersection form of Σ_{L_v}.*

Proof Note that the cup product of two L^2–cocycles is an L^1–cocycle. The intersection form is the result of evaluation of the cup product of two middle-dimensional cocycles on the fundamental class, which is L^∞. Since Σ_{L_v} bounds a half-space in Σ_L, $i_*([\Sigma_{L_v}]) = 0$ in L^∞–homology of Σ_L. Thus, if $a, b \in L^2\mathcal{H}^n(\Sigma_L)$, then $\langle i^*(a) \cup i^*(b), [\Sigma_{L_v}]\rangle = \langle a \cup b, i_*([\Sigma_{L_v}])\rangle = 0$. □

Lemma 4.2 *Let G be a group and let A be a bounded G–invariant (with respect to the diagonal action) non-degenerate bilinear form on a Hilbert submodule $M \subset \ell^2(G)$. Then A has no nontrivial G–invariant isotropic subspaces.*

Proof Let us consider the case $M = \ell^2(G)$ first. G–invariance and continuity of the form A implies that A is completely determined by it values $a_g = (g\ A\ 1)$, $g \in G$. It is convenient to think of the form as given by $(x\ A\ y) = \langle x, Ay\rangle$, where $\langle\ ,\ \rangle$ is the inner product and $A = \Sigma_{g\in G} a_g g$ is a bounded G–equivariant operator on $\ell^2(G)$. Non-degeneracy of A means that $Ax = 0$ only if $x = 0$. A is the limit of the group ring elements, and Ax is the limit of the corresponding linear combinations of G–translates of x, i.e. $Ax = \lim \Sigma_{g\in G_n} a_g(gx)$, where G_n is some exhaustion of G by finite sets. It follows that if x belongs to G–invariant isotropic subspace, then Ax belongs to the closure of this subspace. Thus, we have $\langle Ax, Ax\rangle = (Ax\ A\ x) = 0$ by isotropy and continuity, therefore $x = 0$.

The case of general submodule $M \subset \ell^2(G)$ reduces to the above, since the bilinear form A can be extended to $\ell^2(G)$, for example, by taking the orthogonal sum $A \oplus \langle\ ,\ \rangle$ of A on M and the inner product on the orthogonal complement of M. □

5 Vanishing theorems

Our main technical results are the following two theorems.

Theorem 5.1 $\mathbf{III}''(2k+1)$ *is true for all $k > 0$ and $\mathbf{q} = \mathbf{1}$.*

Proof The proof is by induction on k. Suppose the theorem is true for all $m < k$.

Let v be a vertex of $b\partial\Delta^{2k+1}$. We need to show that the restriction map $i^*\colon\ \mathfrak{h}^k(b\partial\Delta^{2k+1}) \to \mathfrak{h}^k(b\partial\Delta_v^{2k+1})$ is the 0–map.

First let us suppose that v is a vertex of dimension 0, i.e. a vertex of Δ^{2k+1}.

Consider the action of the symmetric group $\mathbf{S}_{2k+1}$ on Δ^{2k+1} which fixes the vertex v and permutes other vertices. This action gives a simplicial action of $\mathbf{S}_{2k+1}$ on $b\partial\Delta^{2k+1}$ and therefore, after choosing a base point, lifts to a cubical action of $\mathbf{S}_{2k+1}$ on $\Sigma_{b\partial\Delta^{2k+1}}$ stabilizing $\Sigma_{b\partial\Delta_v^{2k+1}}$. Let G' be the group of cubical automorphisms of $\Sigma_{b\partial\Delta^{2k+1}}$ generated by this action and the standard action of $W_{b\partial\Delta^{2k+1}}$, and let G be the orientation-preserving subgroup of G'. Similarly, let G'_v be the group of cubical automorphisms of $\Sigma_{b\partial\Delta_v^{2k+1}}$ generated by this action and the standard action of $W_{b\partial\Delta_v^{2k+1}}$, and let G_v be the orientation-preserving subgroup of G'_v.

We claim that, as a Hilbert G_v–module $L^2\mathcal{H}^k(\Sigma_{b\partial\Delta_v^{2k+1}})$, is a submodule of $\ell^2(G_v)$. Note that $b\partial\Delta_v^{2k+1}$ is naturally isomorphic to $b\partial\Delta^{2k}$.

Using the inductive assumption and Lemma 3.3, we can remove from $b\partial\Delta^{2k}$ all odd-dimensional vertices without changing the L^2-cohomology: $\mathfrak{h}^*(E_{2k}) = \mathfrak{h}^*(b\partial\Delta^{2k})$. Since the action of $\mathbf{S}_{2k+1}$ on $b\partial\Delta_v^{2k+1} = b\partial\Delta^{2k}$ preserves the dimension of the vertices, we have isomorphism $L^2\mathcal{H}^*(G_v\Sigma_{E_{2k}}) = L^2\mathcal{H}^*(\Sigma_{b\partial\Delta^{2k}})$ as Hilbert G_v–modules.

The complex E_{2k} is spanned by the even-dimensional vertices of $b\partial\Delta^{2k}$, which correspond to the proper subsets of vertices of Δ^{2k} of odd cardinality. Thus, the dimension of E_{2k} is $k-1$, and its top-dimensional simplices are chains $v_0 < v_0v_1v_2 < ... < v_0...v_{2k-2}$ of length k of distinct vertices of Δ^{2k}. Therefore $\mathbf{S}_{2k+1}$ acts transitively on $(k-1)$–dimensional simplices of E_{2k} and it follows that G_v acts transitively on k–dimensional cubes of $G_v\Sigma_{E_{2k}}$. Therefore the space of k–cochains is a Hilbert G_v–submodule of $\ell^2(G_v)$, and the claim follows from the Hodge decomposition.

We have, by construction, $G_v = \mathrm{Stab}_G(\Sigma_{b\partial\Delta_v^{2k+1}})$. Then the restriction map $i^*\colon\ L^2\mathcal{H}^k(\Sigma_{b\partial\Delta^{2k+1}}) \to L^2\mathcal{H}^k(\Sigma_{b\partial\Delta_v^{2k+1}})$ is G_v–equivariant and therefore its image is a G_v–invariant subspace of $L^2\mathcal{H}^k(\Sigma_{b\partial\Delta_v^{2k+1}})$. Since G_v acts preserving orientation, the intersection form is G_v–invariant. By Lemma 4.1 the image is isotropic, thus by Lemma 4.2 it is 0. Thus, the map $i^*\colon\ \mathfrak{h}^k(b\partial\Delta^{2k+1}) \to \mathfrak{h}^k(b\partial\Delta_v^{2k+1}) = L^2\mathcal{H}^k(W_{b\partial\Delta^{2k+1}}\Sigma_{b\partial\Delta_v^{2k+1}})$ is the 0–map.

For vertices of the other even dimensions the argument is similar. If $\dim(v) = 2d$, then its link is $b\partial\Delta^{2d} * b\partial\Delta^{2k-2d}$. Again, using Lemma 3.3, we remove, without changing the L^2–cohomology, all odd-dimensional vertices from each factor to obtain $E_{2d} * E_{2k-2d}$. The group $\mathbf{S}_{2d+1} \times \mathbf{S}_{2k-2d+1}$ acts naturally on $b\partial\Delta^{2k+1}$ fixing the vertex v and stabilizing both the link and $E_{2d}*E_{2k-2d}$. This action is again transitive on the top-dimensional simplices of $E_{2d} * E_{2k-2d}$, and the rest of the argument goes through.

Finally, if v is an odd-dimensional vertex, $\dim(v) = 2d+1$, then we have $b\partial\Delta_v^{2k+1} = b\partial\Delta^{2d+1} * b\partial\Delta^{2k-2d-1}$. The hypothesis on $\mathbf{III}''$ and Theorem 2.2 and Lemma 2.4 imply that both $\mathbf{I}(2d+1)$ and $\mathbf{I}(2k-2d-1)$ hold. Therefore, by the Künneth formula $\mathfrak{h}^k(b\partial\Delta_v^{2k+1}) = 0$ in this case. □

Theorem 5.2 *The Generalized Singer Conjecture holds true for $b\partial\Delta^{2k+1}$:*

$$\mathrm{b}^i_{\mathbf{q}}(b\partial\Delta^{2k+1}) = 0 \textit{ for } i > k \textit{ and } \mathbf{q} \leq \mathbf{1}.$$

Proof We proceed by induction on k. Using the inductive assumption and Lemma 3.2, we can remove all odd-dimensional vertices without changing the weighted $L^2_{\mathbf{q}}$–homology above k. Thus, since the remaining part E_{2k+1} is k–dimensional, the problem reduces to showing that $\mathfrak{h}^{\mathbf{q}}_{k+1}(E_{2k+1}) = 0$ for $\mathbf{q} \leq \mathbf{1}$. Since E_{2k+1} is k–dimensional, the natural map $\mathfrak{h}_{k+1}(E_{2k+1}) \to \mathfrak{h}^{\mathbf{q}}_{k+1}(E_{2k+1})$ is injective and the result follows from the Theorem 5.1. □

Next, we list some consequences. Lemma 3.1 implies:

Theorem 5.3 *Let bL be the barycentric subdivision of an $(n-1)$–dimensional simplicial complex L. Then*

$$\mathrm{b}^i_{\mathbf{q}}(bL) = 0 \textit{ for } i > (n+1)/2 \textit{ and } \mathbf{q} \leq \mathbf{1}.$$

Taking L to be a GHS^{2k-1}, we obtain:

Theorem 5.4 *The Generalized Singer Conjecture holds true for the barycentric subdivision of a GHS^{n-1} for all even n.*

For odd n we obtain a weaker statement:

Theorem 5.5 *Let bL be the barycentric subdivision of a GHS^{2k}. Then*

$$\mathrm{b}^i_{\mathbf{q}}(bL) = 0 \textit{ for } i > k+1 \textit{ and } \mathbf{q} \leq \mathbf{1}.$$

In particular,

$$\mathrm{b}^i(bL) = 0 \textit{ for } i \neq k,\ k+1.$$

Specializing further, and combining with Theorem 5.2, we obtain:

Theorem 5.6 *The Generalized Singer Conjecture holds true for $b\partial\Delta^n$:*

$$\mathrm{b}^i_{\mathbf{q}}(b\partial\Delta^n) = 0 \textit{ for } i > n/2 \textit{ and } \mathbf{q} \leq \mathbf{1},$$

and, therefore, for the class $\mathcal{JD}$.

Finally, let us mention an application of the above result to a more analytic object. Let T_n denote the space of all symmetric tridiagonal $(n+1) \times (n+1)$–matrices with fixed generic eigenvalues, the so-called Tomei manifold. It is proved in [8] that T_n is an n–dimensional closed aspherical manifold.

Theorem 5.7 *The Singer Conjecture holds true for Tomei manifolds T_n.*

Proof The space T_n can be identified with a natural finite index orbifoldal cover of $\Sigma_{b\partial\Delta^n}/W_{b\partial\Delta^n}$ [3]. Thus $\Sigma_{b\partial\Delta^n}$ is the universal cover of T^n, and the claim follows from the previous theorem. □

References

[1] **M W Davis**, *Groups generated by reflections and aspherical manifolds not covered by Euclidean space*, Ann. of Math. 117 (1983) 293–324 MR0690848

[2] **M W Davis**, *Coxeter groups and aspherical manifolds*, from: "Algebraic topology (Aarhus 1982)", Lecture Notes in Math. 1051, Springer, New York (1984) 197–221 MR076580

[3] **M W Davis**, *Some aspherical manifolds*, Duke Math. J. 55 (1987) 105–139 MR0883666

[4] **M W Davis**, *Nonpositive curvature and reflection groups*, from: "Handbook of Geometric Topology", (R Daverman, R Sher, editors), Elsevier, Amsterdam (2002) 373–422 MR1886674

[5] **M W Davis**, **T Januszkiewicz**, **J Dymara**, **B L Okun**, *Weighted L^2–cohomology of Coxeter groups*, `arXiv:math.GT/0402377`

[6] **M W Davis**, **B L Okun**, *ℓ_2–homology of Coxeter groups based on barycentric subdivisions*, Topol. Appl. 140 (2004) 197–202

[7] **J Dymara**, *Thin buildings*, preprint

[8] **C Tomei**, *The topology of isospectral manifolds of tridiagonal matrices*, Duke Math. J. 51 (1984) 981–996 MR0771391