ON ATIYAH CONJECTURE FOR RIGHT-ANGLED HECKE–VON NEUMANN ALGEBRAS

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ABSTRACT. A version of the Strong Atiyah Conjecture predicts possible weighted L^2-Betti numbers for Coxeter group actions. We prove it for a dense set of rational weights.

In the setting of weighted L^2-cohomology of right-angled Coxeter groups the appropriate form of the Atiyah Conjecture seems to be the following:

Weighted Atiyah Conjecture. Suppose a right-angled Coxeter system (W, S) acts by reflections on a CW complex Y. Then the weighted $L^2_{\mathbf{q}}$-Betti numbers are piecewise rational functions of $\mathbf{q}$ of the form

$$L^2 b_{\mathbf{q}}(Y) = \sum_{J \in \mathcal{S}} \frac{n_J(\mathbf{q})}{W_J(\mathbf{q})},$$

where $n_J(\mathbf{q})$ are piecewise constant integer functions of the multiparameter $\mathbf{q}$.

The usual argument shows that this conjecture is implied by the following algebraic version:

Weighted Atiyah Conjecture. Let (W, S) be a right-angled Coxeter system and let A be a matrix over $\mathbb{Z}W$. The the dimension of the kernel of the right multiplication by A is a piecewise rational function of $\mathbf{q}$ of the form

$$\dim_{\mathbf{q}} \ker A = \sum_{J \in \mathcal{S}} \frac{n_J(\mathbf{q})}{W_J(\mathbf{q})},$$

where $n_J(\mathbf{q})$ are piecewise constant integer functions of the multiparameter $\mathbf{q}$.

Let $\mathcal{B}(W, \mathbf{q})$ denote the integer span of the set $\{W_J(\mathbf{q})^{-1} | J \in \mathcal{S}\}$. The goal of this note is to prove that for $\dim_{\mathbf{q}} \ker A \in \mathcal{B}(W, \mathbf{q})$ for a dense set of weights.

We will call a subset $\mathcal{T}$ of positive real numbers *good* if for any right-angled Coxeter system (W, S), any multiparameter $\mathbf{q} \in \mathcal{T}^S$, and any $\mathbb{Z}W_L$ matrix A, $\dim_{\mathbf{q}} \ker A \in \mathcal{B}(W, \mathbf{q})$. We will prove the following:

Theorem 1. *There is a good set which is dense in* $\mathbb{R}_+$.

Remark. We know that weighted $L^2_{\mathbf{q}}$-Betti numbers are continuous functions of $\mathbf{q}$ [1]. However it appears that the combination of these two results is not enough to deduce the full Weighted Atiyah Conjecture.

The proof will follow a series of Lemmas. First we make an elementary observation.

Lemma 2. *If* $r \in \mathbb{Q}$ *then for* $k < n$, r^k *is in the integer span of* 1 *and* r^n.

Proof. Write $r = a/b$, $\gcd(a, b) = 1$. Then $\gcd(a^2, b^2) = 1$ and we can solve the equation

$$xa^n + yb^n = a^k b^{n-k}$$

in integers. Dividing both sides by b^n proves the claim. □

Lemma 3. *Let* (W, S) *be a spherical right-angled Coxeter system,* $W = (\mathbb{Z}/2)^{|S|}$. *Let* $\mathbf{q}^{-1} = (q_s^{-1})_{s\in S}$. *Then* $W(\mathbf{q}^{-1})^{-1} \in \mathcal{B}(W, \mathbf{q})$.

Proof. We have

$$W(\mathbf{q}^{-1})^{-1} = \prod_{s\in S} \frac{1}{1 + q_s^{-1}} = \prod_{s\in S} \frac{q_s}{1 + q_s} = \sum_{J\subset S} \frac{(-1)^{|J|}}{W_J(\mathbf{q})}.$$

□

Given a right-angled Coxeter system (W, S) let L denote its nerve. L is a flag simplicial complex with the vertex set S whose edges correspond to the pairs of commuting generators. One can think of W_L as a graph product of the groups $\mathbb{Z}/2$ over (1-skeleton of) L. More generally, suppose $\{G_s\}_{s\in S}$ is a family of groups indexed by S. The *graph product* G of the G_s over (1-skeleton of) L is the quotient of the free product of the G_s, $s \in S$, by the normal subgroup generated by all commutators of the form, $[g_s, g_t]$, where $\{s, t\}$ is an edge of L, $g_s \in G_s$ and $g_t \in G_t$.

The map π which sends G_s to s extends to a folding map $\pi : G \to W$, which gives G the structure of a right-angled building of type (W, S). When all the vertex groups are right-angled Coxeter groups (V_{T_s}, T_s), the graph product V also has the natural structure of a right-angled Coxeter group (V, T), where $T = \bigcup_{s\in S} T_s$ and the nerve P is the polyhedral join over L of the nerves L_s of the vertex groups.

Given the weights $\mathbf{q}$ on T such that all the growth series $V_{T_s}(\mathbf{q})$ of vertex groups are convergent, put the weights $p_s = V_{T_s}(\mathbf{q}) - 1$ on $s \in S$. Then we have:

Lemma 4 ([2, Lemma 7.9])**.** (i) *The map $\pi : V \to W$ induces an isometric embedding $\pi^* : L^2_{\mathbf{p}}(W) \to L^2_{\mathbf{q}}(V)$.*

(ii) *For each $s \in S$, $\pi^*(a_s) = a_{T_s}$. Moreover, for each spherical subset $J \subset S$, $\pi^*(a_J) = a_{\pi^{-1}(J)}$.*

(iii) *The map $\pi^* : L^2_{\mathbf{p}}(W) \to L^2_{\mathbf{q}}(V)$ induces a monomorphism of von Neumann algebras $\pi^* : \mathcal{N}_{\mathbf{p}}(W) \to \mathcal{N}_{\mathbf{q}}(V)$. (In particular, π^* commutes with the $*$ anti-involutions on $\mathcal{N}_{\mathbf{p}}(W)$ and $\mathcal{N}_{\mathbf{q}}(V)$.)*

The following statements must be well-known.

Lemma 5. *Suppose $i : \mathcal{N} \to \mathcal{M}$ is an isometric monomorphism of von Neumann algebras. Then:*

(i) *i preserves the trace: $\operatorname{tr}_{\mathcal{N}}(a) = \operatorname{tr}_{\mathcal{M}}(i(a))$ for any $a \in \mathcal{N}$.*

(ii) *i preserves dimension of the kernels of (right) multiplication by $a \in \mathcal{N}$:*

$$\dim_{\mathcal{N}} \operatorname{Ker} a = \dim_{\mathcal{M}} \operatorname{Ker} i(a).$$

Proof. (i) is immediate: $\operatorname{tr}_{\mathcal{N}}(a) = \langle a, 1\rangle = \langle i(a), 1\rangle = \operatorname{tr}_{\mathcal{M}}(i(a))$.

Next we prove (ii). Let $p \in \mathcal{N}$ denote the projection onto the closure of the image of the multiplication by a. Among all the projections in $\mathcal{N}$, p is characterized uniquely by two properties: $ap = a$ and $p \in \overline{\operatorname{Im} a}$. Since i is a continuous homomorphism, both properties are preserved by i, and therefore $i(p) \in \mathcal{M}$ is the projection onto the closure of the image of the multiplication by $i(a)$. Now the claim follows, since i preserves the trace. □

Lemma 6. *Suppose $\mathcal{T}$ is good and $t \in \mathcal{T}$. Then:*

(i) *$\mathcal{T} \cup \{t^{-1}\}$ is good.*

(ii) *If $t \in \mathbb{Q}$ and $n \in BN$, then $\mathcal{T} \cup \{(1+t)^n - 1\}$ is good.*

Proof. Let (W, S) be a right-angled system, and let L be its nerve, and let A be a $\mathbb{Z}W$ matrix. Let $\mathcal{T}_+$ denote one of the sets in (i) and (ii). Given a multiparameter $\mathbf{p} \in \mathcal{T}_+^S$, let U denote the set of vertices with the known good weights, $U = \{s \in S \mid p_s \in \mathcal{T}\}$.

To prove (i), we consider the corresponding partial j-homomorphism (see [4] for details). Recall that $j_U : \mathcal{N}_{\mathbf{p}} \to \mathcal{N}_{j_U(\mathbf{p})}$ is an isometric homomorphism, defined by

$$j_U(s) = \begin{cases} s & \text{if } s \in U, \\ -s & \text{if } s \notin U. \end{cases}$$

$$j_U(\mathbf{p})_s = \begin{cases} p_s & \text{if } s \in U, \\ p_s^{-1} & \text{if } s \notin U. \end{cases}$$

Thus j_U preserves the trace, and we have

$$\dim_{\mathbf{p}} \operatorname{Ker} A = \dim_{j_U(\mathbf{p})} \operatorname{Ker} j_U(A).$$

We note that $j_U(A)$ is still a $\mathbb{Z}W_L$ matrix. Since $j_U(\mathbf{p}) \in \mathcal{T}^S$ it suffices to show that $\mathcal{B}(W, j_U(\mathbf{p})) \subset \mathcal{B}(W, \mathbf{p})$, i.e., that for any spherical subset J of S, $W_J(j_U(\mathbf{p}))^{-1} \in \mathcal{B}(W, \mathbf{p})$. Indeed, we have

$$W_J(j_U(\mathbf{p})) = W_{J\cap U}(\mathbf{p}) W_{J-U}(\mathbf{p}^{-1}).$$

Thus, it follows from Lemma 3 that $W_J(j_U(\mathbf{p}))^{-1} \in \mathcal{B}(W_J, \mathbf{p})$ and we proved (i).

To prove (ii), we consider the graph product V of V_s over L, where the groups V_s are given by:

$$V_s = \begin{cases} \mathbb{Z}/2, & \text{if } s \in U, \\ (\mathbb{Z}/2)^n, & \text{if } s \notin U. \end{cases}$$

This is a right-angled Coxeter group, whose nerve P is the polyhedral join over L of points t_s (over vertices $s \in U$) and $n-1$-simplices $\langle t_{s1} \dots t_{sn} \rangle$ (over vertices $s \notin U$). Let T denote the set of vertices of P and let $\mathbf{q}$ be the multiparameter for T given by $q_{t_s} = p_s$ for $s \in U$, and $q_{t_{si}} = t$ for $s \notin U$, $i = 1 \dots n$. Thus, $\mathbf{q} \in \mathcal{T}^T$.

Since the growth series of each $(\mathbb{Z}/2)^n$ is $(1+t)^n$, $\pi(\mathbf{q}) = \mathbf{p}$. Thus we can apply Lemmas 4 and 5 to conclude that

$$\dim_{\mathbf{p}} \operatorname{Ker} A = \dim_{\mathbf{q}} \operatorname{Ker} \pi^*(A) \in \mathcal{B}(V, \mathbf{q}).$$

To finish the proof we show that $\mathcal{B}(V, \mathbf{q}) \subset \mathcal{B}(W, \mathbf{p})$. It's enough to show that for any spherical subset J of T, $V_J(\mathbf{q})^{-1} \in \mathcal{B}(W, \mathbf{p})$.

Let $I = \pi(J)$. We have the following splittings:

$$W_I(\mathbf{p}) = W_{I\cap U}(\mathbf{p}) W_{I-U}(\mathbf{p}) = W_{I\cap U}(\mathbf{p})(1+t)^{n|I-U|} = W_{I\cap U}(\mathbf{p})(1+t)^{|T_{I-U}|},$$

$$V_J(\mathbf{q}) = V_{J\cap T_U}(\mathbf{q}) V_{J-T_U}(\mathbf{q}) = W_{I\cap U}(\mathbf{p})(1+t)^{|J-T_U|}.$$

We claim that $V_J(\mathbf{q})^{-1}$ is in the integer span of $W_{I\cap U}(\mathbf{p})^{-1}$ and $W_I(\mathbf{p})^{-1}$. This is equivalent to $(1+t)^{-|J-T_U|}$ being in the span of 1 and $(1+t)^{-|T_{I-U}|}$, which follows from lemma 2, since $J - T_U \subset T_{I-U}$. □

Now we are ready to prove the main result.

Proof of Theorem 1. We start by noting that the set $\{1\}$ is good since for $q = 1$ the claim becomes $2^{\dim L+1} \dim_W \operatorname{Ker} A \in \mathbb{Z}$. In other words, it becomes the standard Strong Atiyah Conjecture for right-angled Coxeter groups, which was proved in [3].

Let $\mathcal{T}$ be a maximal good set which contains 1. Then it is closed under operations (i) (inversion) and (ii) of Lemma 6. If $q \in \mathcal{T}$, then by repeatedly applying operation (ii) of Lemma 6 to q we see that

$((1+q)^n - 1) \in \mathcal{T}$ for any $n \in \mathbb{N}$. In particular, $\mathcal{T}$ contains arbitrarily large numbers: $(2^n - 1) \in \mathcal{T}$. Since it is closed under inversion, $\mathcal{T}$ also contains arbitrarily small positive numbers. It follows that $\mathcal{T}$ is dense in $\mathbb{R}_+$: given $x > 0$, we obtain x as a limit point of $\mathcal{T}$ by writing $x = \lim_{q\to 0}((1+q)^n - 1)$ where $n = [\log(x+1)/q]$ and $q \in \mathcal{T}$. □

References

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