Calculus Survival Facts: Algebra

Eric Key

Abstract

This is the bare minimum of algebra you need to be successful in the first semester of calculus.

1 Distance and Midpoint

1. The absolute value of x, denoted by $|x|$ is the distance from x to 0.
2. $|x - y|$ is the distance from x to y.
3. The distance from (a, b) to (c, d) is $\sqrt{|a - c|^2 + |b - d|^2}$.
4. The circle with radius R and center (a, b) has an equation of the form $(x-a)^2+(y-b)^2 = R^2$. Conversely, each such equation describes a circle of radius R and center (a, b).
5. If $0 \leq t \leq 1$ and (a, b) and (c, d) are given, then the point

$$((1 - t)a + tc, (1 - t)b + td)$$

lies t of the way from (a, b) to (c, d). If $t = 1/2$ this is the midpoint of the line segment joining (a, b) and (c, d).

2 Linear Equations

1. A line is determined by 2 distinct points.
2. Every line has an equation of the form $Ax + By + C = 0$. This is the general equation of a line. Conversely, if $A^2 + B^2 \neq 0$ then $Ax + By + C = 0$ is the equation of a line. This equation is called the **general equation of a line**.
3. The distance from the point (a, b) to the line with equation $Ax + By + C = 0$ is given by

$$\frac{|Aa + Bb + C|}{\sqrt{A^2 + B^2}}.$$

4. Every line parallel to the line $Ax+By+C = 0$ has an equation of the form $Ax+By+D = 0$.
5. Every line perpendicular to the line $Ax + By + C = 0$ has an equation of the form $Bx - Ay + D = 0$.
6. If a line passes through the points (a, b) and (c, d) and $a \neq c$ then it has an equation of the form

$$y = b\frac{x - c}{a - c} + d\frac{x - a}{c - a}.$$

This is the **interpolation equation** of the line.

7. If a line passes through the points (a, b) and (c, d) and $a \neq c$ then the quantity

$$\frac{d - b}{c - a}$$

is called the **slope** of the line. Every line has at most one slope. A line is said to be **vertical** if its slope cannot by defined, and **horizontal** if its slope is 0.

8. If two lines are given and neither is vertical, then they are perpendicular if and only if the product of their slopes is -1.
9. If the slope of a line is m and it passes through (a, b) then $y - b = m(x - a)$ is an equation of this line, called the **point-slope** form of the equation. Conversely, $y - b = m(x - a)$ is an equation of a line with slope m and passing through (a, b).

3 Quadratic Equations

1. If $a \geq 0$ then $\sqrt{a} \geq 0$ and $(\sqrt{a})^2 = a$. If $a < 0$ then $\sqrt{a}$ has no meaning.

2. $\sqrt{x^2} = |x|$ for each real number x.

3. The two solutions of $x^2 + 1 = 0$ are denote i and $-i$.

4. $U^2 - V^2 = (U + V)(U - V)$.

5. $(U + V)^2 = U^2 + 2UV + V^2$.

6. Completing the square. Suppose that $A \neq 0$. Then

$$\begin{aligned} Ax^2 + Bx + C &= \frac{(Ax)^2 + BAx}{A} + C \\ &= \frac{(Ax)^2 + B(Ax) + (B/2)^2 - (B/2)^2}{A} + C \\ &= \frac{(Ax + (B/2))^2}{A} + C - \frac{B^2}{4A} \\ &= \frac{(Ax + (B/2))^2}{A} + \frac{4AC - B^2}{4A} \end{aligned}$$

   In the special case where $A = 1$:

$$x^2 + Bx + C = \left(x + \frac{B}{2}\right)^2 + C - \frac{B^2}{4}$$

7. Quadratic formula. Suppose that $A \neq 0$. If

$$Ax^2 + Bx + C = 0$$

   and $B^2 - 4AC \geq 0$ then either

$$x = \frac{-B + \sqrt{B^2 - 4AC}}{2A}$$

   or

$$x = \frac{-B - \sqrt{B^2 - 4AC}}{2A}.$$

   If $B^2 - 4AC \leq 0$ then either

$$x = \frac{-B + \sqrt{|B^2 - 4AC|}i}{2A}$$

   or

$$x = \frac{-B - \sqrt{|B^2 - 4AC|}i}{2A}.$$

8. Given a line in the plane and a point not on this line, a parabola is the set of all points in the plane that are equidistant from this point and this line. This line is called the directrix of the parabola and the this point is called the focus. If the directrix is horizontal then the the parabola has an equation of the form $y - b = A(x - b)^2$ where (a, b) is the midpoint of the perpendicular line segment from the focus to directrix. (a, b) is called the vertex of the parabola.

9. If $a < c < e$ and (a, b), (c, d) and (e, f) are given points in the plane that do not lie on the same straight line, then

$$y = b\frac{(x - c)(x - e)}{(a - c)(a - e)} + d\frac{(x - a)(x - e)}{(c - a)(c - e)} + f\frac{(x - a)(x - c)}{(e - a)(e - c)}$$

   is an equation for the unique parabola passing through these three points.

4 Polynomials

1. If $P(x)$ is a polynomial and $D(x)$ is a polynomial of degree not larger than that of $P(x)$ then there are polynomial $Q(x)$ and $R(x)$ so that
$$P(x) = D(x)Q(x) + R(x).$$
$Q(x)$ is called the quotient of $P(x)$ divided by $D(x)$ and $R(x)$ is called the remainder. If $R(x) = 0$, then we say that $D(x)$ divides $P(x)$ or that $P(x)$ is divisible by $D(x)$.

2. $(x-a)$ divides $P(x)-P(a)$ for any real number a. In particular, $P(x)$ is divisible by $(x-a)$ if and only if $P(a) = 0$.

3. Suppose that
$$P(x) = A_0 + A_1x + \ldots + A_nx^n$$
and A_0, $A_1, \ldots A_n$ are all integers. Then $P(a) = 0$ implies $a = N/D$ where N and D are integers and D divides A_n and N divides A_0.

4. If $P(x)$ is a polynomial with real coefficients and a and b are real numbers, then $P(a+bi) = 0$ implies $P(a-bi) = 0$.

5. Every polynomial of odd degree with real coefficients has at least one real root.

6. Every polynomial with real coeffients factors into a product of quadratics and linear factors, all with real coefficients.

5 Rational Exponents

1. $z^0 = 1$ for every complex number z.

2. If n is a positive integer and z is a complex number then $z^n = z \times z^{n-1}$. This defines positive integer exponents.

3. If n is a negative integer and $z \neq 0$ is a complex number then $z^n = 1/z^{|n|}$.

4. If $n \geq 2$ is an even integer and $a \geq 0$ then $a^{1/n}$ is the unique non-negative solution of $x^n = a$. Thus $a^{1/n}$ is another way of writing $\sqrt[n]{a}$. If $a < 0$ then $a^{1/n}$ has no meaning if n is even. If $n \geq 1$ is an odd integer then $a^{1/n}$ is the unique solution of $x^n = a$.

5. If $n > 0$, $m \geq 0$ and n and m have no common factors then $a^{m/n} = (a^{1/n})^m$.

6. If n and m have no common factors and $m/n < 0$ then
$$a^{m/n} = \frac{1}{a^{|m/n|}}.$$

7. If m/n is not reduced to lowest terms, evaluate $a^{m/n}$ by replacing m/n with an equivalent fraction in lowest terms.

8. With these definitions, if r and s are rational numbers and a^r, b^r, a^s and $(a^r)^s$ are defined, then
$$\begin{aligned} a^r a^s &= a^{r+s} \\ (a^r)^s &= a^{rs} \\ a^r b^r &= (ab)^r \end{aligned}$$

9. If x is not a rational number, a^x cannot be defined without resorting to calculus concepts.

6 Counting, the binomial formula, and geometric sums

1. If n is a non-negative integer, then $n!$ is the number of ways of putting n distinguishable objects in order.

2. $0! = 1$, and if n is a positive integer, then $n! = n \times ((n-1)!)$.

3. If n is a non-negative integer and k is any integer, then

$$\binom{n}{k}$$

is the number of subsets of size k of a set of n elements.

4.

$$\begin{aligned}
\binom{n}{k} &= 0 \text{ if } k < 0; \\
\binom{n}{0} &= 1; \\
\binom{n}{1} &= n; \\
\binom{n}{k} &= \binom{n}{n-k}; \\
\binom{n+1}{k} &= \binom{n}{k-1} + \binom{n}{k}; \\
\binom{n}{k} &= \frac{n!}{k!(n-k)!}
\end{aligned}$$

5. Binomial formula. If n is a non-negative integer then

$$(a+b)^n = \binom{n}{0} a^n b^0 + \binom{n}{1} a^{n-1} b^1 + \binom{n}{2} a^{n-2} b^2 + \cdots + \binom{n}{n} a^0 b^n$$

6. Geometric sum. If $x \neq 0$ and n is a non-negative integer then

$$1 + x + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x}$$

which is equivalent to the factorization:

$$x^{n+1} - 1 = (x-1)(x^n + x^{n-1} + \cdots + x + 1).$$

7. Factoring a difference of powers. If n is a non-negative integer then

$$a^{n+1} - b^{n+1} = (a-b)(a^n b^0 + a^{n-1} b^1 + \cdots + a^1 b^{n-1} + a^0 b^n).$$

This can be derived from the geometric sum formula by dividing the left hand side by b^{n+1}.

8. Three miscellaneous sums:

$$\begin{aligned}
1 + 2 + \ldots + n &= \frac{n(n+1)}{2} \\
1^2 + 2^2 + \ldots + n^2 &= \frac{n(n+1)(2n+1)}{6} \\
1 + 2^3 + \ldots + n^3 &= \left(\frac{n(n+1)}{2}\right)^2
\end{aligned}$$