The complex exponential function

1 Comment

You will not need this material in Math 231, but you will need it in later course in mathematics, physics and electrical engineering.

2 Why a complex exponential function should exist

Recall that by definition,

$$\exp(r) = \lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n$$

where r is any real number. It will be shown in Math 232 that

$$\exp(r) = \lim_{n\to\infty}\left(1+r+\frac{r^2}{2!}+\frac{r^3}{3!}+\cdots+\frac{r^n}{n!}\right).$$

(We could do this now with the binomial theorem, but this topic is not a part of this course.)

In either case it is the case that we could consider r to be a complex number and have an infinite sequence of complex numbers. In the case where $r = it$ where t is a real number and $i^2 = -1$ we would get

$$\begin{aligned}\exp(it) &= \lim_{n\to\infty}\left(1-\frac{t^2}{2!}+\frac{t^4}{4!}+\cdots+(-1)^n\frac{t^{2n}}{(2n)!}\right)\\ &\quad + i\lim_{n\to\infty}\left(t-\frac{t^3}{3!}+\frac{t^5}{5!}+\cdots+(-1)^n\frac{t^{2n-1}}{(2n-1)!}\right)\\ &= f(t)+ig(t).\end{aligned}$$

The existence of the two limits, which I have called $f(t)$ and $g(t)$ is easily established via the theorem on limits of monotone functions, and we can show that

$$\begin{aligned}f(0) &= 1\\ g(0) &= 0\\ f'(t) &= -g(t)\\ g'(t) &= f(t)\end{aligned}$$

all of which allows us to conclude that $f(t) = \cos(t)$ and $g(t) = \sin(t)$, as pretty remarkable result:

Theorem 1 *For each real number t,*

$$\exp(it) := \lim_{n\to\infty}\left(1+\frac{it}{n}\right)^n = \cos(t) + i\sin(t).$$

For example,

$$\exp(2\pi i) = 1.$$

It follows from the addition formulae for sine and cosine that for any two real numbers t and s that

$$\begin{aligned}\exp(it+is) &= \cos(s+t)+i\sin(s+t)\\ &= (\cos(t)\cos(s)-\sin(t)\sin(s))+i\,(\sin(t)\cos(s)+\sin(s)\cos(t))\\ &= (\cos(t)+i\sin(t))\times(\cos(s)+i\sin(s))\\ &= \exp(it)\exp(is)\end{aligned}$$

so the exponential function property is still valid. In fact, for any complex number $a+bi$, we may define

$$\exp(a+bi)=\exp(a)\exp(bi)$$

and we get an exponential function defined on all complex numbers. By this we mean that $\exp(x)\exp(y)=\exp(x+y)$ for x and y complex numbers, not just real numbers.

3 Applications to trigonometric identities

We have for any real numbers A and B:

$$\begin{aligned}& (\cos(A)\cos(B)-\sin(A)\sin(B))+i\,(\sin(A)\cos(B)+\sin(B)\cos(A))\\ =\;& (\cos(A)+i\sin(A))\,(\cos(B)+i\sin(B))\\ =\;& \exp(iA)\exp(iB)\\ =\;& \exp((A+B)i)\\ =\;& (\cos(A)\cos(B)-\sin(A)\sin(B))+i\,(\sin(A)\cos(B)+\sin(B)\cos(A))\\ =\;& \cos(A+B)+i\sin(A+B)\end{aligned}$$

so the identity $\exp(iA)\exp(iB)=\exp(i(A+B)$ encapsulates both the sine and cosine addition formulae. In fact, it follows by induction that

$$(\cos(A)+i\sin(A))^N=\cos(NA)+i\sin(NA)$$

for any real number A and any integer (even negative integers!) N. For example, if we want to find the triple angle formulae for sine and cosine:

$$\begin{aligned}\cos(3A)+i\sin(3A) &= (\cos(A)+i\sin(A))^3\\ &= \cos^3(A)+3i\sin(A)\cos^2(A)-3\sin^2(A)\cos(A)-i\sin^3(A)\end{aligned}$$

so

$$\begin{aligned}\cos(3A) &= \cos^3(A)-3\sin^2(A)\cos(A)\\ \sin(3A) &= 3\sin(A)\cos^2(A)-\sin^3(A)\end{aligned}$$

Observe that we also have

$$\begin{aligned}\cos(A) &= \frac{\exp(iA)+\exp(-iA)}{2}\\ \sin(A) &= \frac{\exp(iA)-\exp(-iA)}{2i}\end{aligned}$$

We can then see that

$$\begin{aligned}
\cos(A)\cos(B) &= \frac{1}{4}(\exp(iA)+\exp(-iA))(\exp(iB)+\exp(-iB)) \\
&= \frac{1}{4}\left(\exp(i(A+B))+\exp(-i(A+B))+\exp(i(B-A))+\exp(-i(A-B))\right) \\
&= \frac{1}{2}(\cos(A+B)+\cos(A-B)),
\end{aligned}$$

$$\begin{aligned}
\sin(A)\sin(B) &= -\frac{1}{4}(\exp(iA)-\exp(-iA))(\exp(iB)-\exp(-iB)) \\
&= -\frac{1}{4}(\exp(i(A+B))+\exp(-i(A+B))-[\exp(i(B-A))+\exp(-i(A-B))]) \\
&= \frac{1}{2}(\cos(A-B)-\cos(A+B)),
\end{aligned}$$

In particular,

$$\sin\left(\frac{x}{2}\right)\sin(kx) = \frac{1}{2}\left(\cos\left(kx-\frac{x}{2}\right)-\cos\left([k+1]x-\frac{x}{2}\right)\right) \tag{1}$$

Similarly,

$$\begin{aligned}
\sin(A)\cos(B) &= \frac{1}{4i}(\exp(iA)-\exp(-iA))(\exp(iB)+\exp(-iB)) \\
&= \frac{1}{4i}(\exp(i(A+B))-\exp(-i(A+B))+[\exp(i(A-B))-\exp(-i(A-B))]) \\
&= \frac{1}{2}(\sin(A+B)+\sin(A-B)) \\
&= \frac{1}{2}(\sin(A+B)-\sin(B-A))
\end{aligned}$$

$$\sin\left(\frac{x}{2}\right)\cos(kx) = \frac{1}{2}\left(\sin\left((k+1)x-\frac{x}{2}\right)-\sin\left(kx-\frac{x}{2}\right)\right) \tag{2}$$

Another application, a bit fancier, is the following. Suppose that $\cos(A) \neq 1$. Then

$$\begin{aligned}
\sum_{k=0}^{N-1}(\cos(kx)+i\sin(kx)) &= \sum_{k=0}^{N-1}(\cos(x)+i\sin(x))^k \\
&= \frac{1-(\cos(x)+i\sin(x))^N}{1-(\cos(x)+i\sin(x))} \\
&= \frac{1-\cos(Nx)-i\sin(Nx)}{1-\cos(x)-i\sin(x)} \\
&= \frac{1-\cos(Nx)-i\sin(Nx)}{1-\cos(x)-i\sin(x)} \times \frac{1-\cos(x)+i\sin(x)}{1-\cos(x)+i\sin(x)} \\
&= \frac{(1-\exp(iNx))(1-\exp(-ix))}{(1-\cos(x))^2+\sin^2(x)} \\
&= \frac{1-\exp(iNx)-\exp(-ix)+\exp(i(N-1)x)}{2(1-\cos(x))}
\end{aligned}$$

so

$$\begin{aligned}
\sum_{k=0}^{N-1} \cos(kx) &= \frac{1-\cos(Nx)-\cos(x)+\cos((N-1)x)}{2(1-\cos(x))} \\
&= \frac{1}{2} - \frac{\cos(x)-\cos((N-1)x)}{4\sin^2(x/2)} \\
\sum_{k=0}^{N-1} \sin(kx) &= \frac{-\sin(Nx)+\sin(x)+\sin((N-1)x)}{2(1-\cos(x))} \\
&= \frac{-\sin(Nx)+\sin(x)+\sin((N-1)x)}{4\sin^2(x/2)}
\end{aligned}$$

Note that these identities could also be derived from (1) and (2).

4 The relation to the geometric properties of complex numbers

Recall that we may interpret $a + bi$ as a point in the place corresponding to the point (a, b). From the Pythagorean Theorem and the definition of absolute value as the distance from a number to 0 we see that

$$|a + bi|^2 = a^2 + b^2 = (a + bi)(a - bi)$$

so $|a + bi| = \sqrt{a^2 + b^2} = |a - bi|$. Recall that the number $a - bi$ is called the **complex conjugate** of $a - bi$. If $|a + bi| \neq 0$ then

$$a + bi = |a + bi| \left(\frac{a}{|a + bi|} + i\frac{b}{|a + bi|} \right) = |a + bi| \left(\cos(\theta) + i\sin(\theta)\right)$$

where θ is an angle measured (in radians, please) from the ray joining 0 to 1 to the ray joining 0 and $a+bi$. We usually choose $0 \leq \theta < 2\pi$, but we don't have to. Once you choose θ you can replace it by $\theta + 2n\pi$ where n is any integer.

Now that we have the complex exponential function, we see that we can write any non-zero complex number $a + bi$ as $\exp(c + i\theta)$ where θ is as above and $c = \ln(|a + bi|)$. For example,

$$1 + i = \sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = \exp\left(\ln(\sqrt{2}) + i\frac{\pi}{4}\right).$$

5 Derivatives

If c is any complex number it is easy to check that

$$\frac{d}{dx}\exp(cx) = c\exp(cx)$$

by proceeding in two steps. First, write $c = a + bi$ so $\exp(cx) = \exp(ax)\exp(ibx)$. If we can differentiate the second term then we can apply the product rule.

$$\begin{aligned}
\frac{d}{dx}\exp(ibx) &= \frac{d}{dx}(\cos(bx) + i\sin(bx)) \\
&= -b\sin(bx) + ib\cos(bx) \\
&= ib(i\sin(bx) + \cos(bx)) \\
&= ib\exp(bx).
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{d}{dx}\exp(cx) &= \frac{d}{dx}\left(\exp(ax)\exp(ibx)\right) \\
&= a\exp(ax)\exp(ibx) + \exp(ax)(ib\exp(ibx) \\
&= (a+bi)\exp(ax)\exp(ibx) \\
&= c\exp(cx)
\end{aligned}$$

For example, if $f(x) = \exp((2+3i)x)$ then $f'(x) = (2+3i)\exp((2+3i)x)$.