Technical details on ‘A Tale of Two Cities’

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Varying-coefficient SIRD model

In my post “A Tale of Two Cities” I used a varying-coefficient SIRD model for curve fitting. This is a variant of the standard SIRD model. Specifically, in a population of size N, at a given time t there are $S(t)$ susceptible individuals (not infected yet), $I(t)$ infectious individuals (individuals who are infected and actively infecting others), $R(t)$ removed individuals (who were infected but are no longer infecting others, because they have either recovered or isolated themselves), and $D(t)$ dead individuals. So $S(t) + I(t) + R(t) + D(t) = N$ at all times. The observable quantities here are $S(t)$ and $D(t)$, which we can get from confirmed positive cases ($(N - S(t))$) and deaths. The other two, $I(t)$ and $R(t)$, are not directly observable. Although some cities and countries report the number of recovered individuals, this is not the same as $R(t)$, because $R(t)$ includes, in addition to the recovered, those who are still sick but in isolation.

The varying-coefficient SIRD model is given by the system of differential equations

$$S'(t) = -\beta(t)I(t)\frac{S(t)}{N}, \tag{1}$$

$$I'(t) = \beta(t)I(t)\frac{S(t)}{N} - \gamma I(t) - \mu I(t), \tag{2}$$

$$R'(t) = \gamma I(t), \tag{3}$$

$$D'(t) = \mu I(t). \tag{4}$$

Here we use a $\beta(t)$ that is a function of time rather than a constant. The reason is that this parameter β, the number of infecting contacts per individual, is the one that quarantine-like measures are intended to modify, so it is expected to vary with time. The other two parameters, the removal rate γ and the death rate μ, can be assumed constant, in principle.

The function $\beta(t)$ itself can be modeled, for example, as a spline function. I used cubic splines in my analysis. This way the model calibration problem becomes a common multivariate estimation problem, and the parameters can be estimated by least squares from the data.

Since $\beta(t)$ is a function of time, the basic reproduction number $R_0(t)$ will also be a function of time,

$$R_0(t) = \frac{\beta(t)}{\gamma + \mu}.$$

When does the curve turn around?

From the second equation above we see that the inflection point $I'(t) = 0$ occurs when

$$R_0(t)\frac{S(t)}{N} - 1 = 0,$$

so $I(t)$ decreases when

$$R_0(t)\frac{S(t)}{N} < 1.$$

This occurs **either** if $R_0(t) < 1$, regardless of the proportion of susceptibles $S(t)/N$, **or** if the proportion of susceptibles $S(t)/N$ satisfies

$$\frac{S(t)}{N} < \frac{1}{R_0(t)}.$$

Since $S(t)$ decreases in time, epidemics **always** come to end, sooner of later: some because $R_0(t) < 1$ and some because the herd-immunity threshold $1/R_0(t)$ is attained.

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