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# Fiat money in the Kiyotaki–Wright model

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**Summary.** We study versions of the Kiyotaki–Wright (1989) model with fiat money and show that: (1) The use of a low storage cost fiat money may be necessary for specialization and trade, (2) there can be valued fiat money steady states which are indeterminate, (3) there are no nontrivial steady-states in which all trades consist of fiat money for goods, (4) fiat money may be valued even if it is not the least costly-to-store object, and lastly, (5) two fiat monies with different storage costs may both be valued.

It is usual to define a monetary economy as one in which the pattern of exchange displays the familiar monetary characteristic that one object or a claim to that object appears in a relatively large number of trades, if not in all trades. That being so, one of the tasks of monetary theory ought to be the construction of models with an endogenous pattern of exchange and the identification of features of the models that imply that the pattern is monetary. So far as we know, Kiyotaki–Wright (1989) is the only model with an imaginable physical environment that has an endogenous transaction pattern. In this paper, we use a version of the model to study the transactions role of a fiat object, an object which is neither consumed nor used in production.

The model is one in which people meet other people pairwise and at random and start with one unit of some object – an object which is indivisible, durable, and costly-to-store. In their original paper, Kiyotaki–Wright describe steady-state patterns of exchange among objects in versions with 3 consumption goods and a fiat object, with the fiat object being a least costly-to-store object. They consider only pure strategies and find parameters for which there are no pure-strategy steady-states. In a companion paper, Aiyagari–Wallace (1991), we allow mixed strategies in an $N$ good and fiat object version and establish some general existence

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results. One result is that if storage costs are sufficiently low relative to the utility of consuming and if there is an object which is least costly-to-store for everyone, then there is a positive consumption steady-state in which that object is always accepted in trade.

Strategies aside, the version of the model we use here differs from the original Kiyotaki–Wright model in a way that has consequences for comparisons between equilibria in which the fiat object is and is not accepted in trade. In the Kiyotaki–Wright version, subject to being able to store at most one unit of some object, an agent can always choose to store his produced good from one period to the next. A consequence is that in an active trade equilibrium in which the fiat object is not used, any initial stock of it is discarded and replaced by produced goods. That makes their model resemble one in which any object, including a fiat object, is a potential input into production. We avoid that by making consumption an input into production. Our assumption is consistent with stationarity and permits us to drop the storage capacity assumption. In our version, if the fiat object is not accepted, then it may be discarded but there is no accompanying increase in the amount of other goods. In this respect, our version is comparable to other models with a fiat object that is not a potential input into production and which have equilibria in which that object is accepted in trade and equilibria in which it is not – for example, standard versions of overlapping generations models. As we show, our version is consistent with a welfare enhancing role for a fiat object when $N = 2$ and with its use being the only alternative to autarky. Neither is possible in the original version.

The body of the paper proceeds as follows. The model is set out in Sect. 1. The definition of equilibrium is given in Sect. 2. In contrast to both the original Kiyotaki–Wright paper and our companion paper, the focus here is not exclusively on steady-states. Our results appear in Sects. 3–6. In Sect. 3, we present a fairly complete analysis of a version with 2 goods and a single fiat object. In addition to the results noted above, we show that an equilibrium in which that object is used exists only if it is a least costly-to-store object. We also demonstrate that there can be indeterminate steady-states – a continuum of equilibrium paths converging to the same steady-state – for a robust class of examples. This feature is well known to be present in other models of fiat money. In Sect. 4, for general $N$, we consider whether there exist steady-states in which trade is purely monetary (satisfies the dictum that goods do not trade for goods). We show that there is no positive consumption steady state in which this happens. Sections 5 and 6 are devoted to showing that there are equilibria in which a not least costly-to-store fiat object is used. In Sect. 5, we present two examples with 3 goods and a single fiat object which is not a least costly-to-store object and show that they have steady-states in which the fiat object circulates. In Sect. 6, we consider a version with 2 goods and 2 fiat objects. The fiat objects, while being least costly-to-store relative to the goods, have different storage costs. We show that both may circulate.

## 1 The model with one fiat object (money)

Time is discrete. There are $N + 1$ indivisible and storable objects – $N$ (consumption) goods and a fiat object, which, from now on, we call fiat money. There is a $[0, 1/N]$

continuum of each of $N$ types of infinitely lived agents. (We often index agent types by Greek letters – $\alpha, \beta, \ldots$, taking integer values $1, 2, \ldots, N$, and objects by Latin letters – $i, j, \ldots$, taking integer values $0, 1, 2, \ldots, N$ with object 0 being fiat money.)

An agent of type $\alpha$ maximizes the expected value of discounted utility, with discount factor $\rho \in (0, 1)$ not dependent on $\alpha$. Date $t$ utility of type $\alpha$ is an increasing time independent function of $\alpha$'s date $t$ consumption of good $i = \alpha$ and a decreasing function of the amounts of objects $\alpha$ stores from $t$ to $t+1$. It turns out that we need to evaluate date $t$ utility at only the following $N+3$ points. We let $u_\alpha > 0$ be the date $t$ utility for type $\alpha$ of consuming one unit of good $i = \alpha$ at $t$ and not storing anything from $t$ to $t+1$; we let $-c_{\alpha i} \leqq 0$ be the date $t$ utility for type $\alpha$ of not consuming at $t$ and of storing one unit of object $i$ from $t$ to $t+1$; and we let 0 be the date $t$ utility of neither consuming nor storing.

For each unit of good $i = \alpha$ consumed by a type $\alpha$ agent at date $t$, the agent produces one unit of good $j = \alpha + 1$ (modulo $N$) which appears at date $t+1$. That is, there is a linear technology of the form: one unit of consumption of good $i$ at $t$ gives rise to one unit of good $i+1$ at $t+1$.

At each date, each agent is paired at random with one other agent. Moreover, it is assumed that paired agents do not know each other's trading histories. This is plausible because with a continuum of agents the probability is zero that these agents have met before or have met others who directly or indirectly have met the current trading partner. Paired agents are assumed to know each other's type and current inventory. Finally, the initial condition is that each agent begins with exactly one unit of some object and that a fraction $m$ of agents starts with a unit of fiat money (instead of with a good).

## 2 Definition of equilibrium

We use a notation that presumes that each person enters each period with one unit of some object. Thus, the notation presumes that agents never dispose of any object nor give one away to another agent for nothing. It also presumes that agents consume whenever they get possession of their consumption good. And, it presumes that trading strategies are nondiscriminatory (willingness to trade does not depend on the type of agent one meets) and symmetric (all agents of the same type in the same trading situation use the same strategy). As we will see, the equilibria we describe below remain equilibria even if agents are permitted to freely dispose and not to consume and even if discriminatory and nonsymmetric strategies are permitted.

We let $s^j_{\alpha i}(t) \in I \equiv [0, 1]$ be the strategy of agent $\alpha$ holding object $i$ who meets with an opportunity to trade $i$ for $j$ at date $t$. We interpret it as the probability that $\alpha$ is willing to trade. We let $s(t) \in I^{(N+1)(N+1)N}$ denote the vector of such strategies.

We let $p_{\alpha i}(t)$ be the proportion who are type $\alpha$ agents and who hold object $i$ at the start of date $t$, before trade. By agents meeting pairwise at random, we mean that for any agent, the probability at the start of date $t$ of meeting a type $\alpha$ agent holding object $i$ is $p_{\alpha i}(t)$. We let $p(t) \in I^{N(N+1)}$ denote the vector of $p_{\alpha i}(t)$'s.

The components of $p(t)$ evolve as follows:

$$(2.1\text{a}) \quad a_{\alpha i}(t) = p_{\alpha i}(t) - p_{\alpha i}(t) \sum_{\beta} \sum_{j \neq i} p_{\beta j}(t) s^j_{\alpha i}(t) s^i_{\beta j}(t) + \sum_{\beta} \sum_{j \neq i} p_{\alpha j}(t) p_{\beta i}(t) s^i_{\alpha j}(t) s^j_{\beta i}(t)$$

(2.1b) $$p_{\alpha i}(t+1)=(1-\delta_{\alpha i})a_{\alpha i}(t)+\delta_{i,\alpha+1}a_{\alpha\alpha}(t)$$

where $\delta_{ij}=1$ if $i=j$ and 0 otherwise. Note that $\sum_{\alpha} p_{\alpha 0}(1)=m$ since a fraction $m$ of agents start out holding money. In (2.1a), $a_{\alpha i}(t)$ denotes the fraction of agents who are type $\alpha$ and end up with object $i$ immediately after trading at $t$. As we have written this expression, the first term is the proportion who are type $\alpha$ and held $i$ at $t$ before trade. The second term is the proportion who are type $\alpha$ and held object $i$ at $t$ before trade and traded for a different object. The third term is the proportion who are type $\alpha$ and who did not hold object $i$ at $t$ before trade and who traded for it. In (2.1b), if $i$ is neither $\alpha$ nor $\alpha+1$ ($\alpha$'s produced good), then $p_{\alpha i}(t+1)=a_{\alpha i}(t)$. If $i=\alpha$, then $p_{\alpha i}(t+1)=0$, because the good is consumed. Finally, if $i=\alpha+1$, then $p_{\alpha i}(t+1)=a_{\alpha i}(t)+a_{\alpha\alpha}(t)$, where $a_{\alpha\alpha}(t)$ equals production of good $\alpha+1$.

We now define the individual decision problems in terms of optimal values (expected discounted utilities). Let $w_{\alpha i}(t)\in R$ be the optimal value for type $\alpha$ who begins period $t$ with object $i$; let $r_{\alpha i}^{\beta j}(t)\in R$ be the optimal value of type $\alpha$ with object $i$ who meets a type $\beta$ with object $j$ at $t$; and let $v_{\alpha i}(t)\in R$ be the optimal value for type $\alpha$ ending with object $i$ after trade at $t$, but before consuming or storing. The values satisfy the following version of Bellman's equation

(2.2a) $$\begin{aligned} r_{\alpha i}^{\beta j}(t)&=\max_{x\in I}[xs_{\beta j}^{i}(t)v_{\alpha j}(t)+(1-xs_{\beta j}^{i}(t))v_{\alpha i}(t)]\\ &=v_{\alpha i}(t)+s_{\beta j}^{i}(t)\max_{x\in I}[x(v_{\alpha j}(t)-v_{\alpha i}(t))]\end{aligned}$$

(2.2b) $$w_{\alpha i}(t)=\sum_{\beta}\sum_{j}p_{\beta j}(t)r_{\alpha i}^{\beta j}(t)$$

(2.2c) $$v_{\alpha i}(t)=\begin{cases}\rho w_{\alpha i}(t+1)-c_{\alpha i} & \text{if} \quad i\neq\alpha \quad \text{or} \quad 0\\ \max[0,\rho w_{\alpha i}(t+1)-c_{\alpha i}] & \text{if} \quad i=0\\ u_{\alpha}+\rho w_{\alpha,i+1}(t+1) & \text{if} \quad i=\alpha\end{cases}$$

The notation for expected discounted utilities and proportions are related to activities within a period as follows:

| $t$ | $A$ | $B$ | $C$ | $t+1$ |
|---|---|---|---|---|
| $p_{\alpha i}(t)$ | $r_{\alpha i}^{\beta j}(t)$ | $a_{\alpha k}(t)$ | | $p_{\alpha\ell}(t+1)$ |
| $w_{\alpha i}(t)$ | | $v_{\alpha k}(t)$ | | $w_{\alpha\ell}(t+1)$ |
| an agent starts with one unit of some object | meets another agent to trade | ends up with some object after trading | stores or consumes and produces | starts with one unit of some object |

At date $t$, the start of period $t$, agent $\alpha$ begins with one unit of object $i$. At time $A$, he meets agent $\beta$ with object $j$ and they may trade. At time $B$, agent $\alpha$ ends up with an object after trading ($k$ is either $i$ or $j$). At time $C$, if the object is not $\alpha$'s consumption good, then he stores it and begins the next period with it; if the object is $\alpha$'s consumption good, then it is consumed and he begins the next period with his produced good ($\ell$ is either $k$ or $\alpha+1$ modulo $N$).

We then have the following definitions.

**Definition.** An *equilibrium* in which no one disposes of any object consists of a sequence $\{p(t+1), s(t), v(t)\}$ for $t = 1, 2, \ldots$, that for given $p(1)$ satisfies (2.1) and (2.2) and

(2.3a) $x = s_{\alpha i}^{k}(t)$ attains $r_{\alpha i}^{\beta k}(t)$ in (2.2a) for all $\beta$ (the strategies are optimal)

(2.3b) $u_\alpha + \rho w_{\alpha,\alpha+1}(t+1) \geqq \rho w_{\alpha\alpha}(t+1) - c_{\alpha\alpha}$ (consuming is optimal)

(2.3c) $v_{\alpha i} \geqq 0$ (not freely disposing is optimal).

**Definition.** A *steady-state* is a constant $(p, s, v)$ such that $\{p(t+1), s(t), v(t)\} = (p, s, v)$ for all $t \geqq 1$ is an equilibrium for the initial condition $p(1) = p$ and $p$ satisfies $\sum_\alpha p_{\alpha 0} = m$.

Two remarks need to be made about these definitions. First, if no one else gives objects away, an agent's expected discounted utility from disposing of any object is zero. Therefore, condition (2.3c) implies that there is no gain from disposing of any object. Second, in the case of an equilibrium in which fiat money is not accepted in trade, $p_{\alpha 0}(t)$ must be interpreted as the proportion of agents who are type $\alpha$ and who hold at most one unit of money, because in such an equilibrium agents who hold money want to dispose of it if it is costly-to-store. The above definitions, including (2.1), are valid in such situations provided $p_{\alpha 0}(t)$ is given that interpretation.

In what follows, we use the term monetary equilibrium to refer to an equilibrium in which at every date each holder of fiat money has a positive probability of trading it for a good and use nonmonetary equilibrium for one in which that probability is zero. Also, unless the context indicates otherwise, we use the term money to refer to fiat money.

## 3 Two goods

Here we explore the role of money in the simplest nontrivial case, when there are only two goods.

### *3.1 A necessary condition for a monetary equilibrium*

With two goods, an agent of type $i$ who has his produced good at $t$ cannot improve his opportunities for trade in the future by acquiring money. If he acquires money, he, at best, acquires good $i$ at $t+1$ with probability $p_{ji}(t+1)$ – that is, by meeting the other type holding his produced good. But if type $j$ always trades for his consumption good, then the type $i$ agent will acquire his consumption good with this same probability if he holds on to his produced good. Therefore, it seems that a necessary condition for type $i$ agent to acquire money at $t$ is that money be no more costly-to-store than his produced good. This is the idea behind the following proposition.

**Proposition 3.1.** *If $N = 2$ and $c_{i0} > c_{ij}$ for some $i \neq j$, then there does not exist an equilibrium in which agents always trade for and consume their consumption good and type $i$ agents at some time trade good $j$ for money.*

*Proof.* Suppose to the contrary. Then, for all $t$ and $j \neq i, 0$,

$$(3.1) \quad v_{ij}(t) \geqq -c_{ij} + p_{ji}(t+1)\rho[u_i + \rho w_{ij}(t+2)] + [1 - p_{ji}(t+1)]\rho v_{ij}(t+1).$$

Here, the second term on the right results from a double coincidence, while the third term results from not trading being an option. Also, for all $t$,

$$(3.2) \quad v_{i0}(t) \leqq -c_{i0} + p_{ji}(t+1)\rho[u_i + \rho w_{ij}(t+2)] + [1 - p_{ji}(t+1)]\rho v_{i0}(t+1).$$

This comes from noting that a type $i$ agent with money either gets to consume or keeps the money. The latter is the only alternative because any person with good $j$ is also a type $i$ person who trades only if $v_{i0}(t+1) \geqq v_{ij}(t+1)$. Finally, the consumption alternative occurs with probability no greater than $p_{ji}(t+1)$.

Now, letting $x_t = v_{ij}(t) - v_{i0}(t)$ and $z_{t+1} = [1 - p_{ji}(t+1)]\rho$, (3.1) and (3.2) imply $x_t \geqq \Delta + z_{t+1}x_{t+1}$, where $\Delta = c_{i0} - c_{ij} > 0$. Therefore, $x_{t+1} \leqq (x_t - \Delta)/z_{t+1}$. But this implies that if $x_t \leqq 0$ for some $t$, then $x_{t+k}$ approaches $-\infty$. Since $v(t)$ is bounded, this is a contradiction. □

In the remainder of Sect. 3, we assume that this necessary condition holds.

### *3.2 Pure strategy monetary equilibrium*

We now characterize equilibrium in which agents always accept money and always trade for and consume their consumption good and never dispose of an object or give an object away. Our procedure is to deduce the $p(t)$ and $v(t)$ sequences implied by these strategies from (2.1) and (2.2), respectively, and then to check conditions (2.3).[1] From (2.1a), these strategies imply

$$a_{i0}(t) = p_{i0}(t)m + p_{i0}(t)p_{ij}(t) + p_{ij}(t)p_{j0}(t) = p_{i0}(t)m + p_{ij}(t)m = m/2$$

where $j \neq i, 0$. Therefore, for any initial condition, the implied $p$ sequence, in matrix form (with rows pertaining to types and columns to objects ordered by 0, 1, 2), is given by

$$(3.3) \quad P(t) = 0.5\begin{bmatrix} m & 0 & 1-m \\ m & 1-m & 0 \end{bmatrix}, \quad \text{for all} \quad t > 1.$$

It follows that the implied $v(t)$ sequence is constant and satisfies, for $j \neq i, 0$,

$$(3.4) \quad v_{ij} = -c_{ij} + p_{ji}\rho[u_i + \rho w_{ij}] + p_{j0}\rho v_{i0} + \rho v_{ij}/2$$

and

$$(3.5) \quad v_{i0} = -c_{i0} + p_{ji}\rho[u_i + \rho w_{ij}] + p_{j0}\rho v_{i0} + \rho v_{i0}/2.$$

Subtracting (3.4) from (3.5), we have

$$(3.6) \quad v_{i0} - v_{ij} = (c_{ij} - c_{i0})/(1 - \rho/2).$$

[1] This is the general procedure we adopt for constructing all of our examples. It is easy to show that there is a unique sequence $\{v_t\}_{t=1}^{\infty}$ implied by given sequences $\{p_t\}_{t=1}^{\infty}$ and $\{s_t\}_{t=1}^{\infty}$. In addition, if $(p_t, s_t) \to (p, s)$ then $v_t \to v$ where $v$ is the vector of expected discounted utilities implied by $(p, s)$.

Therefore, the necessary condition on the $c$'s implies $v_{i0}(t) \geqq v_{ij}(t)$ for all $t$ and $j \neq i$. This verifies condition (2.3a) for $k=0$.

Next, we display the condition assuring that (2.3c), the free disposal condition, holds. We solve for $v_{ij}$ in (3.4) after substituting for $v_{i0}$ from (3.6) and $\rho w_{ij} = v_{ij} + c_{ij}$. The result for $j \neq i, 0$ can be written

$$v_{ij}(1-\rho)(2-\rho) = p_{ji}\rho(2-\rho)(u_i + c_{ij}) - c_{ij}(2-\rho) + 2p_{j0}\rho(c_{ij} - c_{i0})$$

or, using (3.3)

$$v_{ij}(1-\rho) = [(1-m)/2]\rho(u_i + c_{ij}) - c_{ij} + m\rho(c_{ij} - c_{i0})/(2-\rho). \tag{3.7}$$

Therefore, we have

**Proposition 3.2.** *If the* RHS *of (3.7) is positive and if $c_{i0} \leqq c_{ij}$ for $j \neq i$, then there exists a pure strategy monetary equilibrium.*

Finally, we note that if $v_{ij}$ in (3.7) is positive for some $m$, then it is decreasing in $m$. Suppose not. Then, because $v_{ij}$ is linear in $m$, it attains a maximum at $m=1$. But the RHS of (3.7) at $m=1$ is not positive, a contradiction.

The fact that (3.7) is decreasing in $m$ (if it is positive for any $m$) implies that, among economies identical except for $m$, everyone is worse off the higher is $m$. This is not surprising since more $m$ means fewer people holding goods and fewer opportunities for consuming. That feature of (3.7) also implies that Kiyotaki–Wright would never find a welfare enhancing role for fiat money with $N=2$ because they would compare (3.7) for positive $m$ to (3.7) for $m=0$. We find such a role below, when we compare different equilibria for an economy with a given and positive $m$.

### 3.3 *A continuum of (mixed strategy) monetary equilibria*

We now demonstrate the existence of a continuum of (mixed strategy) monetary equilibria in a substantial (and open) region of the parameter space. Specifically, we show the existence of mixed strategy monetary steady states which are locally indeterminate – that is, for initial $p$'s in some neighborhood of the steady state $p$'s there exist a continuum of equilibrium paths converging to the steady state.[2] The equilibria are mixed in the sense that everyone trades for and consumes their own consumption good, but plays mixed strategies at every date when offered money – that is, people with a good accept money only with some probability.[3] Of course, in such equilibria, $v_{ij}(t) = v_{i0}(t)$ for all $t$ and $i \neq j$. The intuition behind the existence of mixed strategy equilibria is as follows. Suppose money is strictly less costly-to-store. Then, if money is accepted for sure, we saw above that $v_{i0}(t) > v_{ij}(t)$. But if the other person only accepts money with some probability, that would tend

[2] Kehoe, Kiyotaki and Wright (1990) have demonstrated the existence of indeterminate steady states in a version with 3 goods and no fiat money.

[3] Note that pure strategy steady states cannot be indeterminate. For initial $p$'s in a small enough neighborhood of the steady states $p$'s, the steady state (pure) strategies will remain individually optimal along any path of $p$'s converging to the steady state. Equations (2.1) then imply that if such a path of $p$'s exists it is unique. This is why the study of a continuum of equilibria naturally leads to looking at mixed strategy equilibria.

to drive $v_{i0}(t)$ down and, perhaps, into equality with $v_{ij}(t)$. The demonstration of the existence of such equilibria proceeds as follows. We first obtain some necessary conditions for the existence of such equilibria. We then use these to develop candidates for the equilibria. Finally, we show that these are, in fact, equilibria.

The intuition behind the existence of a continuum of mixed strategy equilibria is as follows. Both $v_{ij}(t)$ and $v_{i0}(t)$ for $i \neq j$ depend on strategies and proportions at $(t+1)$. Therefore, the condition for having a mixed strategy at $t$, namely $v_{ij}(t) = v_{i0}(t)$, imposes a restriction on $s(t+1)$ and $p(t+1)$ but not directly on $s(t)$. Hence, it may be possible to alter $s(t)$ slightly (which also alters $p(t+1)$ via Eqs. (2.1)) and maintain the equality of $v_{ij}(t)$ and $v_{i0}(t)$ by altering $s(t+1)$ slightly. This, in turn, will alter $p(t+2)$ and will require altering $s(t+2)$ in order to maintain mixed strategies at $(t+1)$. In what follows we demonstrate that such a construction is possible.

If $v(t+1)$ is consistent with mixed strategies, then for $j \neq i, 0$

$$v_{ij}(t) = -c_{ij} + p_{ji}(t+1)\rho[u_i + \rho w_{ij}(t+2)] + [1 - p_{ji}(t+1)]\rho v_{ij}(t+1) \tag{3.8}$$

and

$$v_{i0}(t) = -c_{i0} + s_{ji}^0(t+1)p_{ji}(t+1)\rho[u_i + \rho w_{ij}(t+2)] + [1 - s_{ji}^0(t+1)p_{ji}(t+1)]\rho v_{ij}(t+1). \tag{3.9}$$

Substituting $\rho w_{ij}(t+2) = v_{ij}(t+1) + c_{ij}$ and equating the RHS's of (3.8) and (3.9), we get

$$[1 - s_{ji}^0(t+1)]p_{ji}(t+1) = (c_{ij} - c_{i0})/\rho(u_i + c_{ij}) \equiv A_i, \quad t \geqq 1. \tag{3.10}$$

If agents trade for and consume their consumption goods, from (2.1), we have for $j \neq i, 0$

$$p_{ij}(t+1) = p_{ij}(t) - p_{ij}(t)s_{ij}^0(t)p_{j0}(t) + p_{i0}(t)p_{ji}(t)s_{ji}^0(t). \tag{3.11}$$

Now, imposing (3.10) on the RHS for $t > 1$,

$$p_{ij}(t+1) = p_{ij}(t) - p_{j0}(t)[p_{ij}(t) - A_j] + p_{i0}(t)[p_{ji}(t) - A_i], \quad t > 1. \tag{3.12}$$

Then, using $p_{i0}(t) = 0.5 - p_{ij}(t)$ for $j \neq i, 0$, we have

$$p_{ij}(t+1) = p_{ij}(t)(0.5 + A_i) + p_{ji}(t)(0.5 - A_j) + 0.5(A_j - A_i), \quad t > 1. \tag{3.13}$$

Finally, using $p_{ji}(t) = 1 - m - p_{ij}(t)$, we get

$$p_{ij}(t+1) = p_{ij}(t)(A_i + A_j) + B_i, \quad t > 1 \tag{3.14}$$

where $B_i = 0.5[1 - (A_i + A_j)] - m(0.5 - A_j)$. Therefore, if $A_i + A_j < 1$, then $p_{ij}(t)$ from (3.14) converges monotonically to $B_i/[1 - (A_i + A_j)]$. It is easy to see that if the $A$'s are each less than 0.5 and $m < [1 - (A_i + A_j)]$, then $B_i/[1 - (A_i + A_j)] > A_j$. This insures consistency in the limit with a mixed strategy (see (3.10)).

From (3.8), the limiting or steady-state $v_{ij}(t)$, call it $v_{ij}$, is

$$v_{ij}(1 - \rho) = \{B_j/[1 - (A_i + A_j)]\}\rho(u_i + c_{ij}) - c_{ij}. \tag{3.15}$$

If fiat money has a sufficiently low storage cost, then the conditions just given imply that $v_{ij}$ is positive. These results allow us to prove the following.

**Proposition 3.3.** *If $c_{i0} < c_{ij}$ for $j \neq i$, $A_i < 0.5$, $m < [1-(A_i + A_j)]$, and the right side of (3.15) is positive, then there is a neighborhood of the steady-state implied by (3.14) such that for $p(1)$ in the neighborhood there exists a continuum of mixed strategy monetary equilibria all of which converge monotonically to the steady-state implied by (3.14) and (3.10).*

*Proof.* Proposition 3.3 holds for any $p(1)$ for which there exists a mixed $s(1)$ that when inserted on the RHS of (3.11) implies a $p(2)$ in a small neighborhood of the steady-state implied by (3.14). In particular, since the steady-state satisfies (3.11) and since the RHS of (3.11) is continuous in $p(1)$ and $s(1)$, it follows that if $p(1)$ is in a small neighborhood of the steady-state, then any $s(1)$ in a small neighborhood of the steady-state strategies implies that $p(2)$ from (3.11) is in a small neighborhood of the steady-state. Under the assumptions, any such $p(2)$, by way of (3.14), implies a monotone sequence for $p(t)$ that converges to the steady-state. Therefore, by way of (3.10), the entire sequence is consistent with mixed strategies. Moreover, since all the other equilibrium conditions (see (2.3b) and (2.3c)) are defined by inequalities, it follows that if these hold as strict inequalities at the steady-state (which is implied by the assumptions), then by continuity they also hold along any such path. (See note 1.) □

### *3.4 Welfare in monetary and nonmonetary equilibria*

We begin by describing the conditions for a nonmonetary equilibrium in which agents trade for and consume their consumption goods. In such an equilibrium, $p(t) = p(1)$ for all $t$. It follows that the $v(t)$ sequence is constant and that for $j \neq i, 0$, $v_{ij}$ is the solution to

$$v_{ij} = -c_{ij} + p_{ji}\rho(u_i + v_{ij} + c_{ij}) + (1 - p_{ji})\rho v_{ij}$$

or

$$v_{ij}(1-\rho) = p_{ji}\rho(u_i + c_{ij}) - c_{ij}. \tag{3.16}$$

Nonnegativity of the RHS of (3.16) for $j \neq i, 0$ is necessary and sufficient for the existence of such a nonmonetary equilibrium.

While this condition and money being least costly-to-store are sufficient for the existence of a pure strategy monetary equilibrium (see Proposition 3.2), they are not necessary. There exist parameters for which there are pure strategy monetary equilibria, but which violate nonnegativity of the RHS of (3.16). For such parameters, the only nonmonetary equilibrium is a no-trade equilibrium in which people dispose of objects that are not their consumption good. Thus, there are parameters for which monetary trade, the use of a low storage cost fiat object, is necessary for specialization in production and trade. Note that the no-trade equilibrium need not be interpreted as a zero consumption equilibrium, because the model would not be changed if it were assumed that every person is endowed each period with some nondurable amount of consumption which does not serve as an input and the consumption of which, by itself, implies zero utility.

As regards welfare, note that a type $i$ person with good $j$ is better off the higher is $p_{ji}$, or the greater is the proportion of initial money holders who are type $i$. Not surprisingly, unambiguous welfare comparisons among kinds of equilibria emerge only if these and other asymmetries are assumed away.

If we assume that $p(1)$ is given by the RHS of (3.3) and that $A_1 = A_2$ in (3.10), then there exist parameters for which there are pure strategy monetary, mixed strategy monetary, and nonmonetary equilibria all of which are constant equilibria. It is straightforward to compare welfare across such constant equilibria. In making the comparisons, we distinguish among people by their type and by the good they hold initially.

**Proposition 3.4.** *If $p(1)$ is given by the RHS of (3.3) and $A_1 = A_2$ in (3.10), then, if they exist, the constant pure strategy monetary equilibrium is Pareto superior to the constant mixed strategy monetary equilibrium, which, in turn, is Pareto superior to the constant nonmonetary equilibrium.*

*Proof.* Under the symmetry hypotheses of the proposition, both (3.15) and (3.16) reduce to

$$v_{ij}(1-\rho) = [(1-m)/2]\rho(u_i + c_{ij}) - c_{ij}. \tag{3.17}$$

That is, a goods holder is equally well-off in a nonmonetary constant equilibrium and in a mixed strategy constant equilibrium; in the latter he consumes less frequently than in the former, but has lower storage costs. The proposition follows from comparing this and (3.7) and noting what happens to people who start holding money. Under the hypotheses assuring existence of all three equilibria, the RHS of (3.7) exceeds the RHS of (3.17). That is, people who start with goods are best off in a pure strategy monetary equilibrium and are equally well off in mixed monetary and nonmonetary equilibria. People who start with money are best off in the pure strategy monetary equilibrium (they are better off than they would be starting with their produced good), next best off in a mixed strategy monetary equilibrium (as well off as they would be starting with their produced good), and worst off in a nonmonetary equilibrium.[4] □

## 4 The nonexistence of steady-states with purely monetary trade

By purely monetary trade we mean that goods never trade for goods. Here we show that if storage costs of different objects are ordered the same way by everyone, then there is no steady-state in which trade is purely monetary and in which expected discounted utilities are strictly positive, implying positive consumption for everyone. That is, we prove

**Proposition 4.1.** *If $c_{\alpha j} \geqq c_{\alpha k}$ implies $c_{\beta j} \geqq c_{\beta k}$, then there is no steady-state satisfying* (i) *$p_{\alpha j} > 0$ implies $v_{\alpha j} > 0$ and* (ii) *$p_{\alpha i} s^j_{\alpha i} s^i_{\beta j} p_{\beta j} = 0$ for $i$ and $j \neq 0$.*

*Proof.* We proceed by contradiction. The first step is to show that (i) and (ii) imply

[4] If $p(1) \neq p$ (in 3.3) and $A_1 \neq A_2$, then there may exist nonconstant equilibrium paths some of which converge to a pure strategy monetary steady state and some of which converge to a mixed strategy monetary steady state. We suspect that the conclusion of Proposition 3.4 will continue to hold for initial conditions and parameter values in a neighborhood of those assumed in the proposition.

that $p$ satisfies $P_{i0} = m/N$ and $p_{i,i+1} = (1-m)/N$ (there is symmetry and no one holds anything but money or their produced good). We then show that (ii) contradicts individual optimization.

Suppose $p_{ij} > 0$ for $j \neq 0$ and $i+1$. Then, by (i), $v_{ij} > 0$. This implies that people leave this state and, since we have a steady-state, that people enter it. However, by (ii), it must be that people enter it and leave it by trading for money. Therefore, it must be that $v_{ij} = v_{i0}$ and

$$v_{ij} = -c_{ij} + \rho[qv_{i0} + (1-q)v_{ij}]$$

for some $q \in [0,1]$. It follows that $v_{ij}(1-\rho) = -c_{ij}$, which contradicts $v_{ij} > 0$. Essentially, the value of having good $j$ equals the value of storing it forever because the only alternative is trading it for money which yields exactly the same value.

We are now ready to derive the symmetry. First note that $s^0_{i,i+1} = 1$ for all $i \neq 0$. This follows because the only possible trade is a trade for money and hence, $v_{i,i+1} = -c_{i,i+1} + \rho[qv_{i0} + (1-q)v_{i,i+1}]$ for some $q \in [0,1]$. This implies $v_{i,i+1} < v_{i0}$. Now suppose $p_{i0} \neq m/N$ for some $i$. Then there must be a $k$ such that

$$p_{k+1,0} < p_{k0} = \max_i p_{i0}. \tag{4.1}$$

The flow into the state $(k,0)$ is given by $p_{k,k+1}p_{k+1,0}$ while the flow out of the state $(k,0)$ is given by $p_{k0}p_{k-1,k}$. In view of (4.1) and the fact that $p_{i0} + p_{i,i+1} = 1/N$ we have that $p_{k,k+1} \leqq p_{k-1,k}$. Therefore,

$$p_{k,k+1}p_{k+1,0} < p_{k0}p_{k-1,k}. \tag{4.2}$$

That is, the flow into $(k,0)$ is strictly less than the flow out of $(k,0)$ which is inconsistent with a steady state. This contradiction establishes that $p_{i0} = m/N$ for all $i$ and hence that $p_{i,i+1} = (1-m)/N$ for all $i$.

Having established the above properties of $p$, we are now ready to derive a contradiction concerning strategies. First, note that the properties of $p$ and the assumed strategies imply that

$$v_{ij} \geqq -c_{ij} + \rho(m/N)v_{i0} + \rho(1-m/N)v_{ij} \tag{4.3}$$

and with equality if $j = i+1$. Therefore,

$$[1-\rho(1-m/N)]v_{ij} \geqq -c_{ij} + \rho(m/N)v_{i0} \tag{4.4}$$

and with equality if $j = i+1$.

Now, suppose that all goods are not equally costly-to-store. Then there must be a good $k$ such that it is least costly-to-store and is strictly less costly-to-store than good $k-1$. Consider a type $k-1$ with $k$ who meets a type $k-2$ with $k-1$. The former want to trade because he would get his consumption good. By (4.4), the latter wants to trade since good $k$ is strictly less costly-to-store than good $k-1$. Essentially, for type $k-1$, the trading opportunities are exactly the same whether he holds good $k-1$ or good $k$. In either case he can obtain money with the same probability by trading either with $(k-1,0)$ or with $(k,0)$. Since good $k$ is less costly-to-store than good $k-1$ he would prefer to trade for good $k$. This contradicts (ii).

So, suppose that all goods are equally costly-to-store. Then a type $N$ person with good 1 can attain his consumption good by trading for good 2 if he meets type 1 with his produced good, then trading that for good 3 if he meets type 2 with his produced good, and so on, until he ends up with good $N-1$. At all points along this trading route type $N$ faces the same trading opportunities (he can trade for money with probability $m/N$) and the same storage costs. However, at the end of the trading route when he holds good $N-1$, he can not only trade for money with $(N-1,0)$ but he can also directly obtain his consumption good from $(N-1, N)$, due to the double coincidence of wants. Therefore, such a trading strategy must dominate the alternative of not trading for any nonconsumption goods. This can be seen rigorously as follows by working backwards from $v_{N,N-1}$. Letting $c_N$ stand for the common storage cost of goods for type $N$, we have

$$v_{N,N-1} \geqq -c_N + \rho(m/N)v_{N0} + \rho[(1-m)/N]v_{NN} + \rho(1-1/N)v_{N,N-1}. \tag{4.5}$$

Since $v_{NN} > v_{N,N-1}$, this implies

$$[1-\rho(1-m/N)]v_{N,N-1} > -c_N + \rho(m/N)v_{N0} \tag{4.6}$$

which implies from (4.4) that $v_{N,N-1} > v_{N1}$. For the induction step, suppose that $v_{N,N-k+1} > v_{N1}$ and note that

$$v_{N,N-k} \geqq -c_N + \rho(m/N)v_{N0} + \rho[(1-m)/N]v_{N,N-k+1} + \rho(1-1/N)v_{N,N-k}, \tag{4.7}$$

or

$$v_{N,N-k} > -c_N + \rho(m/N)v_{N0} + \rho[(1-m)/N]v_{N1} + \rho(1-1/N)v_{N,N-k}, \tag{4.8}$$

or

$$[1-\rho(1-1/N)]v_{N,N-k} > -c_N + \rho(m/N)v_{N0} + \rho[(1-m)/N]v_{N1}. \tag{4.9}$$

Since $v_{N1}$ satisfies (4.4) with equality, it follows from (4.9) that $v_{N,N-k} > v_{N1}$. Since $v_{N,N-1} > v_{N1}$ it follows that $v_{Nk} > v_{N1}$ for all $k > 1$ and, in particular, that $v_{N2} > v_{N1}$. That is, when $(N,1)$ meets $(1,2)$ the former should trade. This contradicts (ii). Thus, the strategy of trading only for money and one's consumption good is not a maximizing strategy. □

A question related to the above proposition is whether there exist steady states in which fiat money is the only medium of exchange in the sense that all trades are either goods for money or double coincidence trades. We do not know if such steady states can or cannot exist. Kiyotaki and Wright (1991) have a continuous-time, continuum-of-goods, matching model with this kind of transaction pattern. Such a pattern can emerge from a version of the model studied here by dropping specialization in production; in particular, by assuming that: (i) type $i$ person produces good $k \neq i$ with probability $1/(N-1)$, and (ii) all goods are equally costly-to-store. Such a version can have a steady state in which individuals always accept money and do not accept any good other than their consumption good. In such a steady state, the matrix of the $p_{ij}$'s, with rows pertaining to types and columns to objects starting with object 0 (fiat money), is

$$\begin{bmatrix} q & 0 & p & p & \cdots & p \\ q & p & 0 & p & \cdots & p \\ q & p & p & 0 & \cdots & p \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ q & p & p & p & \cdots & 0 \end{bmatrix}$$

where $q \equiv m/N = p_{i0}$, and $p \equiv (1-m)/N(N-1) = p_{ij}, j \neq i, 0$. Letting $c_i$ stand for the common storage cost of goods for type $i$ persons, the conditions that must be satisfied are $v_{i0} \geqq v_{ij} = v_i \geqq 0$ for all $i$, and $j \neq i$, 0, where $v_{i0}$ and $v_{ij}$ $(j \neq i, 0)$ are the solutions to the following set of linear equations.

$$v_{i0} = -c_{i0} + \rho\{(N-1)p(u_i + v_{i,i+1}) + [1-(N-1)p]v_{i0}\} \tag{4.10}$$

$$v_{ij} = -c_i + \rho\{p(u_i + v_{i,i+1}) + qv_{i0} + (1-p-q)v_{ij}\}, \quad j \neq i, 0. \tag{4.11}$$

These equations do have a solution satisfying the desired conditions if storage costs are not too high .

Since the probability of obtaining one's consumption good holding money is $N-1$ times the same probability holding any other good (in fact, the ratio of trades involving money to double coincidence trades is $N-1$), these conditions are consistent with the storage costs of money being higher than that for goods.[5] In the next section, we show for the original Kiyotaki–Wright (1989) model that money can circulate even if it is not a least-costly-to-store object.

## 5 Monetary steady-states with a not least costly-to-store money

Here we show that there exist steady-states in which fiat money is accepted in trade even if it is not a least costly-to-store object. We saw in Sect. 3 that this cannot happen if $N = 2$. Here we show by example that it can happen if $N > 2$. We present two examples, each suggested by the trading pattern in one of the nonmonetary steady-states displayed by Kiyotaki–Wright. For each example, we first describe schematically the suggestive Kiyotaki–Wright nonmonetary trading pattern and the conjectured monetary trading pattern. Then we give a numerical example that verifies the conjecture. We end with a discussion of the existence of monetary equilibria from initial positions in the neighborhood of these monetary steady-states.

[5] Kiyotaki and Wright (1991) make the same point using transaction costs. They show that the above kind of transaction pattern is consistent with a higher transaction cost for trades involving money than for other trades. The model without specialization in production and equal storage costs for all goods described in the text has other steady states as well in which goods also serve as media of exchange. However, these other steady states are not robust to the introduction of a small transaction cost in each trade whereas the steady state described in the text is. In fact, if transaction costs are sufficiently high, then the steady state described in the text will continue to exist even if all goods are not equally costly-to-store.

| Nonmonetary | Monetary |
| --- | --- |
| $\begin{bmatrix} - & 0 & p_{12} \rightarrow & p_{13} \\ - & 0 & 0 & p_{23} \\ - & p_{31} \rightarrow & p_{32} & 0 \end{bmatrix}$ | $\begin{bmatrix} p_{10} & 0 & p_{12} \rightarrow & p_{13} \\ p_{20} \leftarrow & 0 & 0 & p_{23} \\ p_{30} \leftarrow & p_{31} \rightarrow & p_{32} & 0 \end{bmatrix}$ |

**Fig. 1.** Steady-state trading patterns for Example 1

## 5.1 *Example 1*

This example is suggested by Theorem 2 (and Fig. 6) in Kiyotaki–Wright. The storage costs are ordered as follows: $c_{\alpha 1} < c_{\alpha 0} < c_{\alpha 3} < c_{\alpha 2}$. Figure 1 shows the steady-state nonmonetary trading patterns and our conjectured steady-state monetary trading patterns.

Here components not set at zero are to be understood to be positive. The trading strategies are shown by arrows, except that we omit arrows corresponding to people wanting to trade for their consumption good. Thus, consider type 1 with his produced good in the nonmonetary steady state. When presented with an opportunity to trade for his consumption good (an opportunity that in fact arises when he meets type 3 with his produced good), he wants to trade. He also wants to trade when presented with an opportunity to trade for good 3 (indicated by the arrow). Type 1 with good 3 only trades for his consumption good. Note that good 3, the second least costly good, is playing the role of a commodity money in this steady-state in the sense that everyone accepts it when it is offered. Since good 3 is not a least costly object, this is this model's analog of a low return object playing a monetary role.

Our conjecture is that if money is the second least costly-to-store object, then there is a monetary steady-state with the trading strategies denoted by the arrows. These strategies are the nonmonetary ones with willingness to take money added.

We verify the conjecture by finding the $p$ and $v$ implied by this conjecture for an economy with the following parameters: $\rho = 0.9$, $u_i = 30$, $(c_{\alpha 0}, c_{\alpha 1}, c_{\alpha 2}, c_{\alpha 3}) = (1, 0, 3, 2)$, and $m = 0.058$. The implied $p$ and $v$, with rows pertaining to types and columns to objects are as follows:[6]

$$\begin{bmatrix} 0.023 & 0 & 0.135 & 0.175 \\ 0.020 & 0 & 0 & 0.313 \\ 0.015 & 0.227 & 0.092 & 0 \end{bmatrix} \quad \begin{bmatrix} 46.4 & 75.7 & 42.7 & 44.3 \\ 48.9 & 44.2 & 77.8 & 45.8 \\ 64.6 & 51.6 & 52.9 & 81.6 \end{bmatrix}.$$

[6] The calculation of the elements of the $p$-matrix is done by specifying one or more of the $p_{ij}$'s and using the assumed strategies to calculate the rest. The proportion of money holders $m$ is then whatever is implied by the calculated $p$-matrix. This turns out to be much simpler than starting with an assumed $m$ and then calculating the implied $p_{ij}$'s. The same method was used for the next example also.

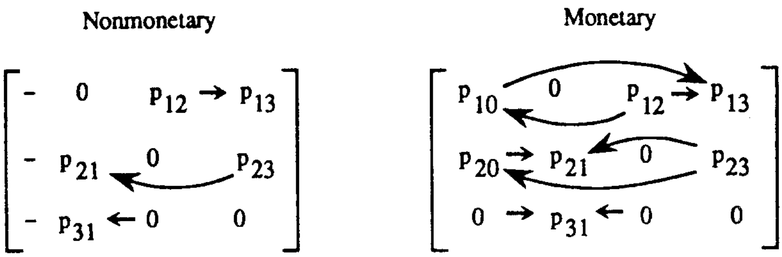


**Fig. 2.** Steady-state trading patterns for Example 2

It is straightforward to check that the $v$ matrix, the second matrix, is consistent with the assumed strategies. First, all the entries are positive. Second, in each row the entry corresponding to the consumption good is the highest, as required in order that trading for one's consumption good be maximizing. Third, the first entry in each row, that corresponding to fiat money, is the second highest entry. This verifies that everyone trading for money is maximizing. The rest of the conjectured trading strategy in Fig. 1 is also seen to be maximizing.

### 5.2 *Example 2*

This example is suggested by what Kiyotaki–Wright call a "speculative" equilibrium in their model 1. The storage costs are ordered as follows: $c_{\alpha 1} < c_{\alpha 0} < c_{\alpha 2} < c_{\alpha 3}$. As above, Fig. 2 shows the trading pattern in their nonmonetary steady-state and our conjectured steady-state monetary trading pattern.

This nonmonetary steady-state has everyone accepting the least costly-to-store good, good 1. It is "speculative" in that type 1 trades his produced good for the more costly-to-store good 3 (because this puts him in a position to trade with type 3). Our conjectured monetary trading pattern preserves the role of good 1 (everyone still accepts it), but it imposes a limited role for fiat money: type 1 take it on the way to getting good 3 (because it is less costly-to-store than good 2), while type 2 always attempt to trade to less costly to store objects.

Existence of such a steady-state is far from obvious. Since neither type 2 nor 3 accept money for good 1, when type 1 acquire money, they give up the possibility of trading for their consumption good at the next date and commit themselves to trying to trade for good 3. After some experimentation, we did find parameters for which such a steady-state exists. They are: $\rho = 0.99$, $u_i = 39$, $(c_{\alpha 0}, c_{\alpha 1}, c_{\alpha 2}, c_{\alpha 3}) = (0.5, 0, 11, 12)$, and $m = 0.0258$. The implied $p$ and $v$ are as follows:

$$\begin{bmatrix} 0.016 & 0 & 0.224 & 0.094 \\ 0.009 & 0.194 & 0 & 0.130 \\ 0 & 0.333 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 32.0 & 75.9 & 25.9 & 38.4 \\ 665.2 & 667.3 & 697.5 & 646.5 \\ 833.4 & 863.0 & 819.2 & 902.0 \end{bmatrix}.$$

The reader can verify that the ordering within rows of the $v$ matrix is consistent with the assumed strategies being maximizing.

For pure strategy steady-states, which, like those found above, are characterized by strict inequalities among the $v_{ij}$ for each $i$, it is easy to check whether there exist convergent equilibrium paths starting from a neighborhood of the steady-state. Such inequalities and continuity imply that the strategies that are maximizing at the steady-state are also maximizing for any $p(t)$ sequence in the neighborhood of the steady-state. (See note 1.) Thus, if Eqs. (2.1) are locally stable in the neighborhood of such a steady-state when the steady-state strategies are imposed, then there exists an equilibrium path converging to the steady-state starting from any $p(1)$ in the neighborhood of the steady-state. Of course, local stability of Eqs. (2.1) evaluated in this way does not imply that *any* equilibrium from such an initial position converges to the given steady-state.

We checked the local stability of Eqs. (2.1) evaluated at the above monetary steady-state strategies in the standard way. We linearized those equations around the steady-state $p$ and computed their eigen values. They are all less than one in absolute value.

The above examples show that fiat money need not be a least costly-to-store object in order to be valued in this model. Nevertheless, in these examples the relatively low storage cost of fiat money seems to play a role. In the first example, only type 3 trade a less costly good, good 1, for fiat money. In Example 2, no one trades a less costly good for fiat money. It is an open question for the general version of this model how relatively bad fiat money can be in terms of storage costs and still have value.

## 6 Several fiat monies

Here we show that several fiat monies with different intrinsic properties can be traded in this model.[7] In particular, we assume that there are two goods and two monies in fixed supply which have different storage costs and show that under some additional restrictions to be derived there can be equilibria in which both monies are traded. This shows that the result of the last section, where a single fiat money could be traded even if it is not a least costly-to-store object, extends to situations where there is a less costly-to-store fiat object.

With two fiat monies, we must amend our notation. Here we use the subscript $G$ for the good or low storage cost money and the subscript $B$ for the bad or high storage cost money and we let $g(b)$ be the proposition of agents who hold one unit of the good (bad) fiat money and nothing else.

We begin by setting out conjectured steady-state values for both $s$ and $p$. They are shown in Fig. 3. Here the arrows again show individual pure strategies. Thus, both types accept either money for their produced good and accept the good money for the bad money. Note that the latter implies that one money does not trade for the other. Both types are also conjectured to accept their consumption good.

We first show that the steady-state version of (2.1) is satisfied. Given the above strategies, (2.1) for $i,j = 1, 2, j \neq i$, and $X = G, B$, implies

$$p_{iX}(t+1) = p_{iX}(t) - p_{iX}(t)p_{ji}(t) + p_{ij}(t)p_{jX}(t). \tag{6.1}$$

[7] See Matsuyama, Kiyotaki, and Matsui (1991) for a related treatment of multiple fiat currencies.

$$\begin{bmatrix} p_{1G} & p_{1B} & p_{11} & p_{12} \\ p_{2G} & p_{1B} & p_{21} & p_{22} \end{bmatrix} = 0.5 \begin{bmatrix} g \leftarrow b & 0 & 1-g-b \\ g \leftarrow b \leftarrow 1-g-b & 0 & \end{bmatrix}$$

**Fig. 3.** Multiple fiat currencies

Evidently, when the $p$ given in Fig. 3 is inserted on the RHS of (6.1), the result is $p_{iX}(t+1) = p_{iX}(t)$. Therefore, the steady-state version of (2.1) is satisfied.

We next consider the $v$ implied by the above conjecture. First note that

$$v_{iX} = -c_{iX} + p_{ji}\rho(u_i + \rho w_{ij}) + (1-p_{ji})\rho v_{iX}. \tag{6.2}$$

It follows that

$$v_{iG} - v_{iB} = (c_{iB} - c_{iG})/[1-\rho(1-p_{ji})]. \tag{6.3}$$

Next note that

$$v_{ij} = -c_{ij} + p_{ji}\rho(u_i + \rho w_{ij}) + p_{jG}\rho v_{iG} + p_{jB}\rho v_{iB} + (1-p_{ji}-p_{jB}-p_{jG})\rho v_{ij}. \tag{6.4}$$

From (6.2) and (6.4)

$$v_{iB} - v_{ij} = c_{ij} - c_{iB} + (1-p_{ji})\rho v_{iB} - p_{jG}\rho v_{iG} - p_{jB}\rho v_{iB} - (1-p_{ji}-p_{jB}-p_{jG})\rho v_{ij}. \tag{6.5}$$

Adding and subtracting $p_{jG}\rho v_{iB}$ on the RHS and using the conjectured $p$, we get

$$(1-\rho/2)(v_{iB} - v_{ij}) = (c_{ij} - c_{iB}) - (g/2)\rho(v_{iG} - v_{iB}). \tag{6.6}$$

This shows that if the RHS of (6.6) is positive, then it is maximizing to trade the produced good for the bad money. Using (6.3) and the conjectured $p$, the RHS of (6.6) being positive is equivalent to

$$c_{ij} - c_{iB} > g\rho(c_{iB} - c_{iG})/[2-\rho(1+b+g)]. \tag{6.7}$$

Note that $c_{ij} - c_{iB} > 0$ is not sufficient. This is because taking the bad money precludes a subsequent trade for the good money. Finally, note that if $c_{iB} - c_{iG} > 0$ and (6.7) holds, then (6.4) implies

$$v_{ij} \geqq c_{ij} + p_{ji}\rho(u_i + \rho w_{ij}) + (1-p_{ji})\rho v_{ij}. \tag{6.8}$$

Using $\rho w_{ij} = c_{ij} + v_{ij}$ and the conjectured $p$, this becomes

$$(1-\rho)v_{ij} \geqq -c_{ij}[1-\rho(1-b-g)/2] + \rho u_i(1-b-g)/2. \tag{6.9}$$

Assembling these results, we have

**Proposition 6.1.** *If $c_{ij} > c_{iB} > c_{iG}$ for $i \neq j$, (6.7) holds, and the RHS of (6.9) is positive, then there exists a pure strategy monetary steady-state with p and s given by Fig. 3.*

It is straightforward to show that if $p(1)$ is in the neighborhood of the $p$ in Fig. 3 and the hypotheses of Proposition 6.1 hold, then there is an equilibrium with the strategies of Fig. 3 and with monotone convergence of $p(t)$ to the $p$ in Fig. 3. First, substituting $x - p_{iX}(t)$ for $p_{jX}(t)$ in (6.1) (where $x = b(g)$ if $X = B(G)$), we get

$$p_{iX}(t+1) = p_{iX}(t) - p_{iX}(t)(p_{ji}(t) + p_{ij}(t)) + xp_{ij}(t). \tag{6.10}$$

Then, using $p_{ij}(t) + p_{ji}(t) = 1 - b - g$, this becomes

$$p_{iX}(t+1) = p_{iX}(t)(b+g) + xp_{ij}(t). \tag{6.11}$$

Then summing (6.11) for $X = G$ and $X = B$, we get $p_{iG}(t+1) + p_{iB}(t+1) = (b+g)/2$. This implies that $p_{ij}(t+1)$ takes on its steady-state value as given in Fig. 3 at $t = 2$. (Notice the similarity to what we found for 2 goods and a single money.) Then (6.11) implies that for $t > 1$, $p_{iX}(t)$ is given by a convergent first order linear difference equation.

Therefore, if the steady-state strategies are followed and if $p(1)$ is in the neighborhood of the steady-state, then the implied $p(t)$ sequence is in that neighborhood. It follows that $v(t)$ for all $t$ implied by those strategies is in the neighborhood of the steady-state $v$ and, therefore, satisfies the same strict inequalities as does the steady-state $v$. (See note 1.) This implies that those strategies are maximizing at each date and implies that they and the implied $p(t)$ and $v(t)$ sequences are an equilibrium.

## Conclusion

In this paper, we have used a version of Kiyotaki–Wright (1989) to address some standard questions concerning fiat money. Some of the answers we find are reassuring in that they are consistent with what we seem to observe, at least in a loose sense. Thus, as noted in Sect. 3, there are economies in which the use of a low storage-cost fiat object gives rise to an equilibrium with specialization and trade and in which not using it implies autarky. Also, a fiat object can play a role in exchange even if there is another object – a good in Sect. 5, another fiat object in Sect. 6 – which is less costly-to-store. We did, however, find in Sect. 4 that there is no positive consumption steady-state in which goods never trade for goods.

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