

Indivisibilities, Lotteries, and Monetary Exchange¹

Aleksander Berentsen

Department of Economics, University of Bern, Bern, Switzerland
aleksander.berentsen@vwi.unibe.ch

Miguel Molico

Department of Economics, University of Western Ontario, London, Canada
mmolico@julian.uwo.ca

and

Randall Wright

Department of Economics, University of Pennsylvania, Philadelphia, Pennsylvania
rwright@econ.sas.upenn.edu

Received December 29, 1998; final version received May 13, 2000;
published online January 25, 2001

We introduce lotteries (randomized trading) into search-theoretic models of money. In a model with indivisible goods and fiat money, we show goods trade with probability 1 and money trades with probability τ , where $\tau < 1$ iff buyers have sufficient bargaining power. With divisible goods, a nonrandom quantity q trades with probability 1 and, again, money trades with probability τ where $\tau < 1$ iff buyers have sufficient bargaining power. Moreover, q never exceeds the efficient quantity (not true without lotteries). We consider several extensions designed to get commodities as well as money to trade with probability less than 1, and to illuminate the efficiency role of lotteries. Journal of Economic Literature Classification Numbers: E40, D83. © 2002 Elsevier Science (USA)

Key Words: indivisibilities; lotteries; search; money.

¹ We have benefitted from the comments of Gabriel Camera, Dean Corbae, John Moore, Martin Schindler, and Neil Wallace, as well as participants in presentations at the Cornell–Penn State Macro Workshop, the Penn State Conference on Exchange in Search Equilibrium, the 1999 Society for Economic Dynamics, European Economic Association and European Econometric Society meetings, the Universities of Pennsylvania, Vienna and Western Ontario, and the London School of Economics. A referee and an associate editor also made several important suggestions. The National Science Foundation and Schweizerische Nationalfonds provided financial support. Last, for their hospitable and stimulating environment, we thank Café Lutecia in Philadelphia, where we did much of the work on this project.

1. INTRODUCTION

In this paper we introduce *lotteries*—that is, randomized trading—into search-theoretic models of monetary exchange. There are several reasons for doing so. First, consider the most basic version of the model, with indivisible goods and money and a storage technology that allows agents to inventory at most one object at a time (Kiyotaki and Wright [9, 10]). Although this model is simplistic, it does have virtues. In particular, since every trade is a one-for-one swap, one can relatively easily study certain aspects of the exchange process and illustrate certain interesting features of money without having to determine exchange rates or the distribution of inventories. To the extent that this model is useful, one would like to understand its properties. It is well known from the study of various economic environments with indivisibilities or other nonconvexities that agents can often do better using randomized rather than deterministic trading mechanisms, and so it is interesting to ask if there is a role for lotteries in this model, too.²

In this simple model with indivisible goods and money, bargaining over lotteries means bargaining over the joint probability distribution of (q, m) , where $q \in \{0, 1\}$ is the amount of the good and $m \in \{0, 1\}$ the amount of money to be exchanged. We show that monetary equilibria exist iff buyers (agents with money) have bargaining power θ above some threshold. If θ is above the threshold but not too large, then when a buyer meets a seller with a good he desires he gives the seller money with probability 1 and receives the good with probability 1. If θ is larger, however, the good still changes hands with probability 1, but now the buyer gives up the money with probability $\tau < 1$. Hence, there is a role for lotteries. Moreover, lotteries allow us to discuss a notion of prices, even with indivisible goods and money, since τ is the average amount of currency that trades for a good.³

² Previous analyses of nonconvexities and lotteries include Prescott and Townsend [13, 14], Rogerson [15], Diamond [6], Shell and Wright [18], and Chatterjee and Corbae [4].

³ We emphasize here that lotteries are different from mixed strategies. In particular, with indivisible goods and money and no lotteries, suppose that, as is typically the case, there exists both an equilibrium where money is accepted with probability 1 and an equilibrium where it is accepted with probability 0. Then there is also a mixed strategy equilibrium where money is accepted with probability in $(0, 1)$. In a mixed strategy equilibrium sellers are indifferent between getting and not getting the money; in the lottery model, by contrast, sellers strictly prefer getting the money. Once we allow lotteries, mixed strategy equilibria of the above variety no longer exist. Thus, another reason to introduce lotteries into the simplest (indivisible goods) model is that this serves to eliminate the somewhat unnatural mixed equilibria.

Now consider the model with divisible consumption goods, where even if we continue to assume that money is indivisible and agents have a unit storage capacity, prices can be determined by letting agents bargain over the quantity of goods buyers get for a unit of currency (Shi [19]; Trejos and Wright [21]). Agents again bargain over the joint probability distribution of (q, m) , but now $q \in [0, \infty)$. In this model, there is a unique monetary equilibrium for all parameters, and when a buyer meets a seller with a good he desires, he gives him money with probability τ where again τ is strictly less than 1 iff θ is above some threshold, and gets q units of the good with probability 1 where q is deterministic and independent of whether the money changes hands. Hence, there is again a role for lotteries, even when goods are perfectly divisible.

Furthermore, we show that q may be less than but can never exceed the efficient quantity q^* (defined below). One reason this is interesting is the following. It is natural to expect q will be less than q^* in a monetary model, as argued in Trejos and Wright [21], for example; but one can only rule out $q > q^*$ for some parameters in the model in that paper. Once lotteries are introduced, we can show $q \leq q^*$ for all parameters. Moreover, this immediately implies that welfare is higher, and strictly higher for some parameters, when lotteries are allowed. This is not necessarily so in the indivisible goods model, where allowing lotteries can actually reduce welfare for some values of the bargaining power parameter; however, if we solve the social planner's problem of maximizing ex ante utility, as opposed to looking for equilibria for arbitrary bargaining parameters, we show that lotteries also increase welfare in the indivisible goods model.

One perhaps surprising feature of both models described above is the asymmetry between goods and money: goods always change hands with probability 1 while money may change hands with probability $\tau < 1$. To investigate what is behind this result, we discuss some alternative models, including ones where agents barter consumption goods directly, where there is commodity rather than fiat money, and where the bargaining powers of agents varies across meetings. In each of these cases goods may trade with probability less than 1, and we discuss the reasons. Also on the subject of commodity money, we find that if a commodity money is sufficiently intrinsically valuable then equilibria are necessarily efficient. Moreover, the introduction of lotteries allows us to derive a new version of Gresham's Law, which says that a sufficiently valuable commodity money will be withdrawn from circulation in a probabilistic sense, but this will not harm the efficiency of exchange.

The rest of the paper is organized as follows. In Section 2 we present the basic assumptions underlying all of the models. In Section 3 we analyze the version with indivisible goods. In Section 4 we analyze the version

with divisible goods. Section 5 considers the various extensions. Section 6 concludes.⁴

2. THE GENERAL MODEL

The economy is populated by a $[0, 1]$ continuum of infinitely-lived agents who specialize in consumption and production. Assume consumption goods are non-storable so that they cannot serve as money. Let X_i be the set of goods that agent i consumes. No agent i produces a good in X_i . Moreover, for a pair of agents i and j selected at random, the probability that i produces a good in X_j and j also produces a good in X_i is 0 (there is never a double coincidences of wants), while the probability that i produces a good in X_j but j does not produce a good in X_i is $x \in (0, 1)$. For example, if there are N goods and N types, $N > 2$, and each type i agent consumes only good i and produces only good $i + 1 \pmod{N}$, then $x = 1/N$. Let Q denote the set of feasible quantities that agents can produce. We will consider two cases: the indivisible goods model, where $Q = \{0, 1\}$, and the divisible goods model, where $Q = \mathfrak{R}_+$.

Preferences are described as follows. Every agent i derives utility $u(q)$ from q units of any good in X_i . He also derives disutility $c(q)$ from producing q units. We always assume $u(0) = c(0) = 0$. For the divisible goods model, we assume that both u and c are C^2 , and that $u'(q) > 0$, $c'(q) > 0$, $u''(q) \leq 0$ and $c''(q) \geq 0$, with at least one of the weak inequalities holding strictly, for all $q > 0$. We also assume $u'(0) > c'(0) = 0$, and that there exists a $\bar{q} > 0$ such that $u(\bar{q}) = c(\bar{q})$. For the indivisible goods model, let $u(1) = U$ and $c(1) = C$ and assume $U > C > 0$. The rate of time preference is $r > 0$.

In addition to the consumption goods described above, there is also an object that cannot be produced or consumed called *fiat money*. We assume that money is indivisible and that individuals have a single unit storage capacity, so that if a fraction $M \in (0, 1)$ of the population are each initially endowed with one unit of money then there will always be M agents with and $1 - M$ agents without money. We call agents with money *buyers* and agents without *sellers*. Agents meet randomly according to a Poisson process with arrival rate α . Thus, the probability per unit time that buyer i meets a seller j such that j produces a good in X_i is $\alpha(1 - M)x$, and the probability that seller j meets a buyer i such that j produces a good in X_i is αMx . Without loss of generality, we normalize $\alpha x = 1$.

⁴ In this paper we do not consider models where both money and goods are divisible, or where money is indivisible but agents can hold multiple units in inventory; see Molico [11], Green and Zhou [7], Zhou [23], Camera and Corbae [3], Taber and Wallace [20] or Berentsen [1]. The role for lotteries in those environments is an open question.

We want to consider exchanges that may be random. Define an event to be a pair (q, m) , where $q \in Q$ denotes the quantity of the good and $m \in \{0, 1\}$ the amount of money that is traded. Let $E \equiv Q \times \{0, 1\}$ denote the space of such events and \mathcal{E} denote the Borel σ -algebra. Define a *lottery* to be a probability measure λ on the measurable space (E, \mathcal{E}) . One can always write $\lambda(q, m) = \lambda_m(m) \lambda_{q|m}(q)$, where λ_m is the marginal probability measure of m and $\lambda_{q|m}$ is the conditional probability measure of q given m . Then to reduce notation let $\lambda_m(m=1) = \tau$ and $\lambda_m(m=0) = 1 - \tau$. Thus, $\tau \in [0, 1]$ is the probability that the money changes hands. A lottery can be completely described by the probability τ and the two probability measures $\lambda_{q|0}$ and $\lambda_{q|1}$.⁵

Let V_m denote the value function for an agent with $m \in \{0, 1\}$ units of money. The expected payoffs from a lottery for a buyer and a seller are given by

$$\begin{aligned} \Pi_1 &= \tau \left[V_0 + \int u(q) \lambda_{q|1}(dq) \right] + (1 - \tau) \left[V_1 + \int u(q) \lambda_{q|0}(dq) \right] \\ \Pi_0 &= \tau \left[V_1 - \int c(q) \lambda_{q|1}(dq) \right] + (1 - \tau) \left[V_0 - \int c(q) \lambda_{q|0}(dq) \right]. \end{aligned}$$

We focus on symmetric equilibria, where in any meeting between a buyer and a seller that produces a good the buyer wants, they agree to the same lottery. Then we can write Bellman's equations as follows:

$$\begin{aligned} rV_1 &= (1 - M)(\Pi_1 - V_1) \\ rV_0 &= M(\Pi_0 - V_0). \end{aligned} \tag{1}$$

For example, the first of these equations sets the flow value to being a buyer, rV_1 , equal to the rate at which he meets sellers who produce a good in X_i , which is simply $1 - M$ given the normalization $\alpha x = 1$, times the net gain from playing the lottery, which is $\Pi_1 - V_1$.

We employ the generalized Nash bargaining solution. That is, we determine τ , $\lambda_{q|0}$ and $\lambda_{q|1}$ by solving

$$\max(\Pi_1 - T_1)^\theta (\Pi_0 - T_0)^{1-\theta}, \tag{2}$$

⁵ One may question how agents can commit to the outcome of a lottery. For example, suppose that we agree to randomize so that you give me the good for sure and we flip a coin to see whether I give you the money. If the coin comes up so that I keep the money, will you still give me the good? Of course, in any exchange some notion of commitment is required, but perhaps it is more delicate when objects are not exchanged simultaneously. To the extent that one might worry about this, there are devices to get around the problem. For example, if I am supposed to give you the money with probability n/m , I can put it in one of m boxes and shuffle them, and then we can *simultaneously* swap n of the boxes for the good.

where T_1 and T_0 are the threat points of the buyer and the seller, respectively, and $\theta \in [0, 1]$ is the bargaining power of the buyer. It is well known that this is equivalent to an explicit strategic bargaining model of the sort developed by Rubinstein [16] when the time between offers and counter offers vanishes, where θ and T_j depend on details of the strategic environment. In what follows we allow θ to take on any value in $[0, 1]$, and consider two cases for the threat points that have been used in the literature: $T_j = V_j$, which follows from the strategic model if one assumes individuals continue to meet other potential trading partners between bargaining rounds; and $T_j = 0$, which follows from the strategic model if one assumes they cannot meet other trading partners between rounds.⁶ We also impose the incentive compatibility conditions

$$\Pi_1 \geq V_1 \quad \text{and} \quad \Pi_0 \geq V_0. \quad (3)$$

A steady state equilibrium for this economy is a list $(V_1, V_0, \tau, \lambda_{q|0}, \lambda_{q|1})$ such that: the value functions satisfy the Bellman equations in (1) taking the lottery as given; and the lottery solves the maximization problem in (2) subject to the constraints in (3) taking the value functions as given. If $\lambda_{q|0}(0) = \lambda_{q|1}(0) = 1$ or $\tau = 0$ the equilibrium is called nonmonetary, and otherwise it is called monetary. It is clear that a nonmonetary equilibrium always exists. From now on we focus on monetary equilibria. In any monetary equilibrium, the second constraint in (3) can be rearranged to yield

$$V_1 - V_0 \geq \int c(q) \lambda_{q|1}(dq) + \frac{1-\tau}{\tau} \int c(q) \lambda_{q|0}(dq) > 0. \quad (4)$$

Hence, $V_1 > V_0$. In the next two sections we analyze in turn the two models, with $Q = \{0, 1\}$ and $Q = \mathfrak{R}_+$.

3. THE INDIVISIBLE GOODS MODEL

When $q \in \{0, 1\}$, a lottery is completely described by τ , plus two numbers, $\lambda_1 \equiv \lambda_{q|1}(q=1)$ and $\lambda_0 \equiv \lambda_{q|0}(q=1)$, which give the probabilities that the good changes hands conditional on money changing hands and conditional on money not changing hands, respectively (of course, λ_0 is irrelevant if $\tau = 1$ and λ_1 is irrelevant if $\tau = 0$). Given any lottery one can solve

⁶ See Binmore, Rubinstein and Wolinsky [2] or Osborne and Rubinstein [12]; see Coles and Wright [5] for an exposition in the context of monetary search theory.

(1) for the value functions, substitute into (3), and verify that the first constraint holds for all parameters, while the second holds iff

$$rC \leq \tau(1 - M)(U - C). \quad (5)$$

Notice that λ_1 and λ_0 do not appear in this expression. So that monetary equilibria have a chance to exist we assume

$$C < \left(\frac{1 - M}{r + 1 - M} \right) U, \quad (6)$$

since otherwise (5) could not be satisfied for any $\tau \leq 1$ (we ignore the non-generic case where the condition holds with equality).

We begin by briefly reviewing the standard model, where lotteries are ruled out. Let Ω denote the probability that money is accepted by a seller. Then

$$\begin{aligned} rV_1 &= (1 - M)\Omega(U + V_0 - V_1) \\ rV_0 &= M\Omega(V_1 - V_0 - C). \end{aligned}$$

In this model, which is essentially the one in Kiyotaki and Wright [10], there is nothing to bargain over, and an equilibrium is simply a list (V_1, V_0, Ω) such that either: $V_1 - V_0 - C \geq 0$ and $\Omega = 1$; $V_1 - V_0 - C \leq 0$ and $\Omega = 0$; or $0 < \Omega < 1$ and $V_1 - V_0 - C = 0$. Given (6), it is easy to see that there exists an equilibrium with $\Omega = 0$, an equilibrium with $\Omega = 1$, and an equilibrium with $\Omega = rC/(1 - M)(U - C) \in (0, 1)$.

We claim that the equilibrium with $\Omega \in (0, 1)$ is an artifact of ruling out lotteries. To see this, notice that in such an equilibrium the seller is indifferent between trading and not trading, $V_1 - V_0 - C = 0$, while the buyer strictly prefers to trade, $U + V_0 - V_1 > 0$. This means that trading with probability less than 1 is inconsistent with efficient bargaining. To see this, think about the strategic game of alternating offers that underlies the Nash solution, and suppose that buyer i makes seller j the following offer: i will give j the money with probability 1 and j will give i the good with probability λ_1 . Clearly, there are values $\lambda_1 < 1$ such that both i and j strictly prefer to trade. Consequently, there are no equilibria with $\Omega \in (0, 1)$ once we allow lotteries and bargaining, and we can set $\Omega = 1$.⁷

⁷ This is essentially the same argument that rules out mixed strategy monetary equilibria in the divisible goods model, except that there the buyer offers to take a slightly smaller quantity while here he offers to take the indivisible quantity with a slightly lower probability. Note that we will actually show below that in any monetary equilibrium the good changes hands with probability $\lambda_1 = 1$ in this model; setting $\lambda_1 < 1$ was only used to show that $\Omega < 1$ is *not* an equilibrium.

Given lotteries are allowed, the following proposition characterizes the set of equilibria when the threat points are equal to the continuation values.

PROPOSITION 1. *Assume $T_j = V_j$. Then there are critical values $\underline{\theta}_1$ and $\bar{\theta}_1$ constructed in the proof, with $0 < \underline{\theta}_1 < \bar{\theta}_1 < 1$, such that the following is true: if $\theta < \underline{\theta}_1$ there is no monetary equilibrium; if $\theta \in (\underline{\theta}_1, \bar{\theta}_1]$ there exists a unique monetary equilibrium and it entails $\tau = 1$ and $\lambda_1 = 1$; and if $\theta > \bar{\theta}_1$ there exists a unique monetary equilibrium and it entails $\lambda_1 = \lambda_0 = 1$ and $\tau = \tau_1 \in (0, 1)$, where*

$$\tau_1 = \frac{r[\theta C + (1 - \theta) U]}{(\theta - M)(U - C)}.$$

Proof. In this model (2) reduces to choosing $(\tau, \lambda_0, \lambda_1) \in [0, 1] \times [0, 1] \times [0, 1]$ to solve

$$\max(\Pi_1 - V_1)^\theta (\Pi_0 - V_0)^{1-\theta}$$

where $\Pi_1 = \tau(\lambda_1 U + V_0) + (1 - \tau)(\lambda_0 U + V_1)$ and $\Pi_0 = \tau(-\lambda_1 C + V_1) + (1 - \tau)(-\lambda_0 C + V_0)$, taking V_1 and V_0 as given. Necessary and sufficient conditions for a solution are

$$\begin{aligned} \theta[V_0 - V_1 + (\lambda_1 - \lambda_0) U](\Pi_0 - V_0) \\ + (1 - \theta)[V_1 - V_0 - (\lambda_1 - \lambda_0) C](\Pi_1 - V_1) - \eta_\tau \leq 0, \quad &= \text{if } \tau > 0 \end{aligned} \tag{7}$$

$$\theta\tau U(\Pi_0 - V_0) - (1 - \theta)\tau C(\Pi_1 - V_1) - \eta_1 \leq 0, \quad = \text{if } \lambda_1 > 0$$

$$\theta(1 - \tau)U(\Pi_0 - V_0) - (1 - \theta)(1 - \tau)C(\Pi_1 - V_1) - \eta_0 \leq 0, \quad = \text{if } \lambda_0 > 0,$$

where the η_j 's are nonnegative multipliers for the constraints that the choice variables cannot exceed 1.

We are looking for monetary equilibria, which means that $\tau > 0$, and the first condition in (7) holds with equality. First consider the case $\tau < 1$, which implies $\eta_\tau = 0$. If $\lambda_1 \in [0, 1)$ then $\eta_1 = 0$ and $\theta\tau U(\Pi_0 - V_0) \leq (1 - \theta)\tau C(\Pi_1 - V_1)$, and combining this with the first condition in (7) yields $U \leq C$, which is a contradiction. A similar contradiction results if $\lambda_0 \in [0, 1)$. Hence, $\tau < 1$ implies $\lambda_1 = \lambda_0 = 1$. Given this, we can solve (1) for the V_j 's, substitute them into first condition in (7) at equality, and solve for $\tau = \tau_1$, where τ_1 is given above. Notice that $\tau_1 \in (0, 1)$ iff $\theta > \bar{\theta}_1$, where

$$\bar{\theta}_1 = \frac{(r + M)U - MC}{(1 + r)(U - C)}.$$

One can easily check that the incentive condition (5) is satisfied at $\tau = \tau_1$. We conclude that there exists an equilibrium with $\lambda_1 = \lambda_0 = 1$ and $\tau = \tau_1 \in (0, 1)$ iff $\theta > \bar{\theta}_1$.

Now consider the case where $\tau = 1$. This means $\lambda_1 > 0$ in a monetary equilibrium, and λ_0 is irrelevant so we simply set $\lambda_0 = \lambda_1$ (nothing actually depends on this but it facilitates the argument). Inserting the V_j 's into the second equation in (7) at equality and rearranging, we arrive at:

$$\begin{aligned} & \lambda_1 \{ \theta U [(1 - M) U - (r + 1 - M) C] - (1 - \theta) C [(r + M) U - MC] \} \\ & = (1 + r) \eta_1. \end{aligned} \tag{8}$$

Suppose $\lambda_1 < 1$; then $\eta_1 = 0$, and (8) can be satisfied only for the nongeneric parameter value $\theta = \underline{\theta}_1$ where

$$\underline{\theta}_1 = \frac{C(1 + r)}{(1 - M) U + MC} \bar{\theta}_1.$$

Hence, except for the nongeneric case $\theta = \underline{\theta}_1$, the only solution to (8) with $\lambda_1 < 1$ is $\lambda_1 = 0$. Therefore, in any monetary equilibrium we have $\lambda_1 = 1$. But this means that (8) holds iff the left hand side is non-negative, which is true iff $\theta \geq \underline{\theta}_1$. So monetary equilibria are only possible if $\theta \geq \underline{\theta}_1$ and $\lambda_1 = 1$. Given this, one can check that $\tau = 1$ satisfies the first condition in (7) iff $\theta \leq \bar{\theta}_1$. One can also check that (5) is satisfied at $\tau = 1$. We conclude that there exists an equilibrium with $\lambda_1 = 1$ and $\tau = 1$ iff $\underline{\theta}_1 \leq \theta \leq \bar{\theta}_1$.

Summarizing, an equilibrium with $\tau \in (0, 1)$ exists iff $\theta > \bar{\theta}_1$ and an equilibrium with $\tau = 1$ exists iff $\underline{\theta}_1 \leq \theta \leq \bar{\theta}_1$, and in either case we have $\lambda_0 = \lambda_1 = 1$. Finally, one can verify that $0 \leq \underline{\theta}_1 < \bar{\theta}_1 < 1$ using (6). This completes the proof. ■

For completeness we also describe the model with $T_j = 0$. However, since the argument and results are qualitatively the same as in Proposition 1, we omit the proof.

PROPOSITION 2. *Assume $T_j = 0$. Then there are critical values $\underline{\theta}_0$ and $\bar{\theta}_0$, with $0 < \underline{\theta}_0 < \bar{\theta}_0 < 1$, such that the following is true: if $\theta < \underline{\theta}_0$ there exists no monetary equilibrium; if $\theta \in (\underline{\theta}_0, \bar{\theta}_0]$ there exists a unique monetary equilibrium and it entails $\tau = 1$ and $\lambda_1 = 1$; and if $\theta > \bar{\theta}_0$ there exists a unique monetary equilibrium and it entails $\lambda_1 = \lambda_0 = 1$ and $\tau = \tau_0 \in (0, 1)$, where*

$$\tau_0 = \frac{r[\theta(r + M) C + (1 - \theta)(r + 1 - M) U]}{[r(\theta - M) + M(1 - M)(2\theta - 1)](U - C)}.$$

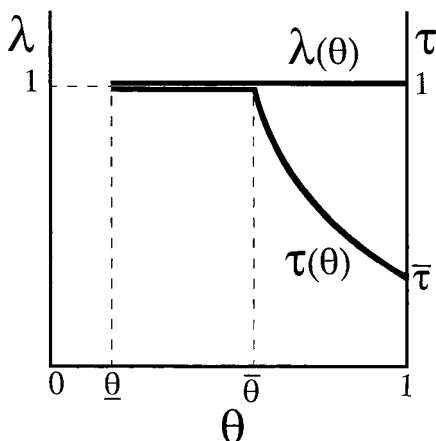


FIG. 1. Monetary equilibrium, indivisible goods model.

Several comments are in order concerning these results. First, since $\bar{\theta} < 1$, we have $\tau \in (0, 1)$ in a region of parameter space with positive measure. Hence, the implicit restriction made in the previous literature, that lotteries are not allowed, is indeed restrictive. Second, there is an asymmetry in the model: money may change hands randomly, but goods either change hands with probability 1 or not at all. This is depicted in Fig. 1, which plots τ and λ as functions of θ . As is clear, for $\theta > \bar{\theta}$ goods trade with probability 1 and money trades randomly, for intermediate $\theta \in [\underline{\theta}, \bar{\theta}]$ both objects trade with probability 1, and for $\theta < \underline{\theta}$ monetary equilibria do not exist. We discuss this asymmetry further below.

Note that τ measures the price level—it is the average number of units of money that it takes to buy a good. One can show τ is decreasing in θ , increasing in r , increasing in C , and decreasing in U , for both the model with $T_j = V_j$ and the model with $T_j = 0$. The effects of changes in M depend on which version of the model we use, however: one can show $\partial \tau_1 / \partial M > 0$, but, perhaps surprisingly, $\partial \tau_0 / \partial M > 0$ iff r and M are not too small. Also, as $r \rightarrow 0$ we have $\tau \rightarrow 0$ for all $\theta > \bar{\theta}$ (the τ curve in Fig. 1 becomes vertical at $\theta = \bar{\theta}$); thus, if θ is big and agents are very patient buyers get the good virtually for free, which sellers are willing to go along with since on the small but positive chance they get the money it will convey exactly the same benefit to them.⁸

⁸ The thresholds also depend on r ; in particular, as $r \rightarrow 0$, we have $\bar{\theta}_1 \rightarrow M$ and $\underline{\theta}_1 \rightarrow CM / [(1 - M)U + MC]$ in the model with $T_j = V_j$, and $\bar{\theta}_0 \rightarrow 1/2$ and $\underline{\theta}_0 \rightarrow C / (U + C)$ in the model with $T_j = 0$. Other differences between the models with different threat points include the following. When $T_j = 0$, $\bar{\theta}_0 > 1/2$ for all $r > 0$, and so lotteries are not used when buyers and sellers have equal bargaining power; but when $T_j = V_j$, it is possible to have $\tau < 1$ when $\theta = 1/2$. Also, as long as τ_1 and τ_0 are in $(0, 1)$, we have $\tau_0 < \tau_1$ iff $M > 1/2$.

Finally, we mention welfare. For low θ there is no monetary equilibrium, and the only possible equilibrium is autarchy, where $V_1 = V_0 = 0$. If lotteries are ruled out then there is an equilibrium where money is accepted and $V_1 > V_0 > 0$ for all parameters satisfying (6). Hence, allowing lotteries can actually reduce welfare. This should not be too surprising, however, as it simply says that agents may be better off if they can *commit* to $\lambda = 1$ rather than bargaining in each bilateral meeting. In any case, we will see below that there is a welfare-improving role for lotteries in a slight variant of this model, where we consider commodity money, and where rather than looking for equilibria for an arbitrary value of the bargaining weight θ we look for incentive-feasible allocations that a social planner might choose. Also, we will soon see that lotteries can only enhance welfare in the divisible goods version of the model presented in the next section.

4. THE DIVISIBLE GOODS MODEL

When $q \in \mathfrak{R}_+$, a lottery is generally described by τ and two conditional probability distributions, $\lambda_{q|0}$ and $\lambda_{q|1}$. However, we claim the amount of goods that changes hand is degenerate and independent of whether money changes hands.

PROPOSITION 3. *There is a value for q , say $q = q'$ (that depends on parameter values), such that $\lambda_{q|0}(q') = \lambda_{q|1}(q') = 1$.*

Proof. The Nash bargaining problem is to choose $\tau \in [0, 1]$ and probability measures $\lambda_{q|0}$ and $\lambda_{q|1}$ to solve

$$\begin{aligned} & \max \left\{ \tau \left[\int u(q) \lambda_{q|1}(dq) + V_0 \right] + (1 - \tau) \left[\int u(q) \lambda_{q|0}(dq) + V_1 \right] - T_1 \right\}^\theta \\ & \times \left\{ \tau \left[- \int c(q) \lambda_{q|1}(dq) + V_1 \right] \right. \\ & \left. + (1 - \tau) \left[- \int c(q) \lambda_{q|0}(dq) + V_0 \right] - T_0 \right\}^{1-\theta} \end{aligned}$$

subject to the incentive constraints in (3), taking V_0 and V_1 as given. Suppose that the solution implies $\lambda_{q|0}$ and $\lambda_{q|1}$ are nondegenerate, and let $q_0 = \int q \lambda_{q|0}(dq)$ and $q_1 = \int q \lambda_{q|1}(dq)$. Since $u(q)$ is concave and $c(q)$ convex, and at least one is strictly so, by Jensen's inequality the incentive constraints are still satisfied and the Nash product is higher when $\lambda_{q|0}(q_0) = \lambda_{q|1}(q_1) = 1$, which is a contradiction. Hence, $\lambda_{q|0}$ and $\lambda_{q|1}$ are degenerate at q_0 and q_1 , respectively. Now suppose $q_0 \neq q_1$, and let

$Eq = \tau q_1 + (1 - \tau) q_0$. Again by Jensen's inequality the incentive constraints are still satisfied and the Nash product is higher at Eq , another contradiction. So q must be degenerate at some value q' . ■

The above result makes the analysis much simpler because we can now restrict attention to lotteries that are completely characterized by two numbers, τ and q (in what follows we use q to denote the quantity rather than the random variable). Given any such lottery, one can, as in the previous section, solve (1) for the value functions, substitute in (3), and verify that the first constraint is never binding and the second is satisfied iff

$$rc(q) - \tau(1 - M)[u(q) - c(q)] \leq 0. \quad (9)$$

In particular, if $\tau = 1$, then (9) holds iff

$$\varphi(q) \equiv rc(q) - (1 - M)[u(q) - c(q)] \leq 0. \quad (10)$$

It is easy to see that $\varphi(0) = 0$, $\varphi'(0) < 0$, $\varphi''(q) \geq 0$ for all q , and $\varphi(q) > 0$ for large q ; hence, if $\tau = 1$ the constraints are satisfied iff q is below some critical value \hat{q} . Also, let q^* be the *efficient* quantity, defined by $u'(q^*) = c'(q^*)$. It is easy to verify that q^* is the quantity that maximizes welfare, $W = MV_1 + (1 - M)V_0$. If $q = q^*$ then (9) holds iff

$$\tau \geq \hat{\tau} \equiv \frac{rc(q^*)}{(1 - M)[u(q^*) - c(q^*)]}, \quad (11)$$

which can hold iff r is not too big.

The following proposition characterizes the set of equilibria for the model with threat points equal to continuation values.

PROPOSITION 4. *Assume $T_j = V_j$. If $\theta = 0$, there does not exist a monetary equilibrium. If $\theta > 0$, then there is a critical value $\tilde{\theta}_1$ constructed in the proof, where $\tilde{\theta}_1 > 0$ for all parameter values and $\tilde{\theta}_1 < 1$ iff $r < (1 - M)[u(q^*) - c(q^*)]/c(q^*)$, such that the following is true: if $\theta < \tilde{\theta}_1$ there exists a unique monetary equilibrium and it entails $\tau = 1$ and $q < q^*$, with $\partial q/\partial \theta > 0$ and $\lim_{\theta \rightarrow \tilde{\theta}_1} q = q^*$; and if $\theta > \tilde{\theta}_1$ there exists a unique monetary equilibrium and it entails $q = q^*$ and $\tau = \tilde{\tau}_1 \in (0, 1)$, where*

$$\tilde{\tau}_1 = \frac{r[\theta c(q^*) + (1 - \theta)u(q^*)]}{(\theta - M)[u(q^*) - c(q^*)]}.$$

Proof. If $\theta = 0$ then the bargaining solution is equivalent to take-it-or-leave-it offers by the seller, which implies $u(q) = \tau(V_1 - V_0)$. Inserting this

into (1), we find $V_1 = 0$, and therefore $V_0 < 0$ by (4). But a seller can always achieve $V_0 = 0$ by not trading. Hence, there cannot exist a monetary equilibrium when $\theta = 0$.

Now assume $\theta > 0$. Then (2) reduces to choosing $(\tau, q) \in [0, 1] \times \mathfrak{R}_+$ to solve

$$\max(\Pi_1 - V_1)^\theta (\Pi_0 - V_0)^{1-\theta},$$

where $\Pi_1 = u(q) + \tau V_0 + (1 - \tau) V_1$ and $\Pi_0 = -c(q) + \tau V_1 + (1 - \tau) V_0$. Necessary and sufficient conditions for a solution are

$$\theta u'(q)(\Pi_0 - V_0) - (1 - \theta) c'(q)(\Pi_1 - V_1) \leq 0, \quad = \text{ if } q > 0 \quad (12)$$

$$\theta(V_0 - V_1)(\Pi_0 - V_0) + (1 - \theta)(V_1 - V_0)(\Pi_1 - V_1) - \eta_\tau \leq 0, \quad = \text{ if } \tau > 0,$$

where η_τ is the nonnegative multiplier on the constraint $\tau \leq 1$. We are looking for monetary equilibria, which implies that both conditions hold with equality.

First consider the case where $\tau < 1$, which implies that $\eta_\tau = 0$. Then combining the two first order conditions yields $u'(q) = c'(q)$, and so $q = q^*$. Solving (1) for the V_j 's and inserting the solutions, as well as $q = q^*$, into the second condition in (12), we can solve for $\tau = \tilde{\tau}_1$ where $\tilde{\tau}_1$ is defined in the statement of the proposition. Notice that $\tilde{\tau}_1 \in (0, 1)$ iff $\theta > \tilde{\theta}_1$ where

$$\tilde{\theta}_1 = \frac{(r + M) u(q^*) - M c(q^*)}{(1 + r)[u(q^*) - c(q^*)]}.$$

One can check that $\tilde{\tau}_1 \geq \hat{\tau}$, where $\hat{\tau}$ is defined in (11), and therefore the incentive condition (9) holds at $\tau = \tilde{\tau}_1$ and $q = q^*$. Hence, we conclude that there exists an equilibrium with $\tau = \tilde{\tau}_1$ and $q = q^*$ iff $\theta > \tilde{\theta}_1$.

Now consider the case where $\tau = 1$, which implies $\eta_\tau \geq 0$. By combining the two conditions in (12), we get $u'(q) \geq c'(q)$, and this implies $q \leq q^*$ in any equilibrium with $\tau = 1$, with strict inequality as long as $\eta_\tau > 0$. Inserting the V_j 's and $\tau = 1$, we can rewrite the first order condition for q as

$$\frac{(1 - \theta) c'(q)}{\theta u'(q)} = \frac{1 - M - (r + 1 - M) c(q)/u(q)}{r + M - M c(q)/u(q)}. \quad (13)$$

The left hand side of (13) is zero at $q = 0$ and it is strictly increasing. As $q \rightarrow 0$, the right hand side approaches $(1 - M)/(r + M) > 0$, because $c(q)/u(q) \rightarrow 0$ by l'Hopital's rule, and it is strictly decreasing and equals 0 when $q = \hat{q}$, where recall that \hat{q} is the solution to (10) at equality. Hence,

there exists a unique solution to (13), call it $\chi = \chi(\theta)$, in $(0, \hat{q})$. Moreover, it is easy to check that $\chi'(\theta) > 0$ and that $\chi(\tilde{\theta}_1) = q^*$. Since we need $\chi(\theta) \leq q^*$ for an equilibrium with $\tau = 1$, an equilibrium of this type cannot exist if $\theta > \tilde{\theta}_1$. If $\theta < \tilde{\theta}_1$ then $\chi(\theta) < q^*$, and we now show that this also implies the first order condition for τ is satisfied at $\tau = 1$. To see this, rearrange the first order condition for τ as

$$\theta \leq \frac{(r + M)u(q) - Mc(q)}{(1 + r)[u(q) - c(q)]}. \quad (14)$$

The right hand side of (14) is decreasing in θ and approaches $(r + M)/(1 + r) > 0$ as $q \rightarrow 0$. Also, (14) is satisfied at equality when $\theta = \tilde{\theta}_1$. Hence, (14) is satisfied iff $\theta \leq \tilde{\theta}_1$. We conclude that $\tau = 1$ and $q = \chi(\theta)$ satisfy the first order conditions iff $\theta \leq \tilde{\theta}_1$. Moreover, since $\chi(\theta) < \hat{q}$, it satisfies the incentive condition (10), and hence satisfies all of the conditions for an equilibrium.

Finally, it is obvious that $\tilde{\theta}_1 > 0$, and that $\tilde{\theta}_1 < 1$ iff $r < (1 - M)[u(q^*) - c(q^*)]/c(q^*)$. This completes the proof. ■

The model with $T_j = 0$ has the same qualitative properties, and so as in the previous section we state the results here without proof.

PROPOSITION 5. *Assume $T_j = 0$. If $\theta = 0$ there does not exist a monetary equilibrium. If $\theta > 0$ then there is a critical value $\tilde{\theta}_0$, where $\tilde{\theta}_0 > 0$ for all parameter values and $\tilde{\theta}_0 < 1$ iff $r < (1 - M)[u(q^*) - c(q^*)]/c(q^*)$, such that the following is true: if $\theta < \tilde{\theta}_0$ there exists a unique monetary equilibrium and it entails $\tau = 1$ and $q < q^*$, with $\partial q / \partial \theta > 0$ and $\lim_{\theta \rightarrow \tilde{\theta}_0} q = q^*$; and if $\theta > \tilde{\theta}_0$ there exists a unique monetary equilibrium and it entails $q = q^*$ and $\tau = \tilde{\tau}_0 \in (0, 1)$, where*

$$\tilde{\tau}_0 = \frac{r[(1 - \theta)(1 - M + r)u(q^*) + \theta(M + r)c(q^*)]}{[r(\theta - M) + M(1 - M)(2\theta - 1)][u(q^*) - c(q^*)]}.$$

The first thing we want to emphasize about the above two propositions is that although there is a monetary equilibrium for all $\theta > 0$ and no monetary equilibrium for $\theta = 0$, there is no real discontinuity because $q \rightarrow 0$ as $\theta \rightarrow 0$ in monetary equilibrium. Next, notice that as in the previous section the two objects are traded asymmetrically: randomization may be used for trading money but not for trading goods. Figure 2 shows τ and q as functions of θ (the figure shows $\tilde{\theta} < 1$, which holds iff r is not too big). Notice that $q \leq q^*$ for all θ with strict inequality iff $\theta < \tilde{\theta}$, where q^* is the efficient quantity: $u'(q^*) = c'(q^*)$. It is argued in Trejos and Wright [21] that it is natural to have q below q^* in a monetary economy, although in the model in that paper, without lotteries, the result does not actually hold

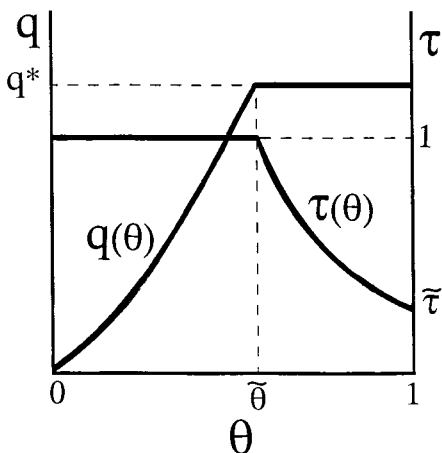


FIG. 2. Monetary equilibrium, divisible goods model.

very generally—it holds when $\theta = 1/2$ in the model where $T_j = 0$, but it may not hold for other values of θ , and it may not hold in the model where $T_j = V_j$ even if $\theta = 1/2$. With lotteries, q can never exceed q^* irrespective of threat points or bargaining power.

Some results for this model are similar to those for the indivisible goods model. For example, the behavior of τ with respect to the parameters is the same. There are also differences. For one thing, in the indivisible goods model we have $\bar{\theta} < 1$, and hence we definitely have $\tau < 1$ for high θ , while in the divisible goods model we can guarantee $\tau < 1$ for high θ iff r is not too big. Also, in the indivisible goods model monetary equilibria do not exist for low θ , while in the divisible goods model a monetary equilibrium exists for all $\theta > 0$. Finally, recall that lotteries could reduce welfare in the indivisible goods model. Lotteries can only improve welfare here: for $\theta \leq \bar{\theta}$, q and therefore welfare are the same with or without lotteries; for $\theta > \bar{\theta}$, $q = q^*$ with lotteries and $q > q^*$ without lotteries, and therefore welfare is strictly higher with lotteries.⁹

5. DISCUSSION

We have seen that although agents may agree to a lottery where money changes hands with probability less than 1, they will never agree to

⁹ Lotteries (sometimes) reduce welfare in the indivisible goods model by causing monetary equilibrium to break down; this never happens in the divisible goods model, and moreover lotteries can enhance welfare by eliminating overproduction—i.e., q is at rather than above the efficient quantity q^* .

a lottery where goods change hands with probability less than 1. What lies behind this asymmetry? It is not due to the assumption that money is indivisible while goods are divisible, because the same asymmetry arises when goods and money are both indivisible. One conjecture is that asymmetry is due to the fiat nature of the money—i.e., to the fact that it has no intrinsic worth and derives value only from its role as a medium of exchange. To investigate this, and some other things, we consider several variations on the basic theme.

One approach is to consider a model with direct barter instead of monetary exchange. For example, suppose some agents consume good 1 and produce good 2, while some consume good 2 and produce good 1. Goods are indivisible, production of good j costs $C_j > 0$, and consumption yields utility U_j . Assume $U_i > C_j$, for all i, j . When two agents of the opposite type meet they bargain over lotteries. Let θ be the bargaining power of type 1, and let τ_j be the probability that type j gives up his production good.¹⁰ It is easy to prove the following (details available upon request): there are critical values $\underline{\theta}$ and $\bar{\theta}$, with $0 < \underline{\theta} < \bar{\theta} < 1$, such that: if $\theta < \underline{\theta}$ then $\tau_1 = 1$ and $\tau_2 \in (0, 1)$; if $\underline{\theta} \leq \theta \leq \bar{\theta}$ then $\tau_1 = \tau_2 = 1$; and if $\theta > \bar{\theta}$ then $\tau_2 = 1$ and $\tau_1 \in (0, 1)$.

Hence, agents get their consumption goods with probability less than 1 iff they have sufficiently low bargaining power, in a model with direct barter, while it was not possible to get goods with probability less than 1 when agents were trading with fiat money. So it seems there is something to the notion that asymmetry is due to the nature of fiat money. To explore things further, consider a model with one indivisible good and money, as in Section 3, but assume now that the money is a *commodity* money, in the sense that it yields a direct utility flow $\gamma > 0$ to an agent holding it.

Let τ and λ be the probabilities that money and goods change hands (we do not need two conditional probabilities λ_0 and λ_1 because one can show, as above, that $\lambda_0 = \lambda_1$). The value functions satisfy

$$\begin{aligned} rV_0 &= M[\tau(V_1 - V_0) - \lambda C] \\ rV_1 &= (1 - M)[\lambda U + \tau(V_0 - V_1)] + \gamma. \end{aligned} \tag{15}$$

The bargaining solution is exactly as above. Then we have the following generalization of Proposition 1 (for brevity we only present results for the case $T_j = V_j$, but the other case is essentially the same).

¹⁰ After trading, one can assume agents return to the market, as in the money model, or that they exit the economy, as in Rubinstein and Wolinsky [17], say. We have explored these and several other sets of assumptions, some that generate models very similar to the one in Rubinstein and Wolinsky; the same qualitative results held for all the barter models we explored. Note that lotteries are *not* useful in the basic model presented by Rubinstein and Wolinsky, but only because that model assumes utility is linear.

PROPOSITION 6. Let $\bar{\gamma} = (r + M)U - MC$. If $\gamma > \bar{\gamma}$ then for all θ there exists a unique monetary equilibrium and it entails $\lambda = 1$ and $\tau = \tilde{\tau} \in (0, 1)$. If $\gamma \in (0, \bar{\gamma})$ then there are critical values $\underline{\theta}$ and $\bar{\theta}$, with $0 < \underline{\theta} < \bar{\theta} < 1$, such that the following is true: if $\theta < \underline{\theta}$ there exists a unique monetary equilibrium and it entails $\tau = 1$ and $\lambda = \tilde{\lambda} \in (0, 1)$, where

$$\tilde{\lambda} = \frac{\gamma[\theta U + (1 - \theta)C]}{(U - C)[M(1 - \theta)C - \theta(1 - M)U] + rCU};$$

if $\theta \in [\underline{\theta}, \bar{\theta}]$ there exists a unique monetary equilibrium and it entails $\tau = 1$ and $\lambda = 1$; and if $\theta > \bar{\theta}$ there exists a unique monetary equilibrium and it entails $\lambda = 1$ and $\tau = \tilde{\tau} \in (0, 1)$, where

$$\tilde{\tau} = \frac{r[\theta C + (1 - \theta)U]}{\gamma + (\theta - M)(U - C)}.$$

Proof. See the Appendix.

We emphasize first that $\gamma > 0$ implies that a monetary equilibrium exists for all θ , while with fiat money there was no monetary equilibrium for small θ (there is no discontinuity, however, since $\lambda \rightarrow 0$ as $\gamma \rightarrow 0$). However, the key result is that when $\gamma > 0$ we can have $\lambda \in (0, 1)$; i.e., goods can trade with probability less than 1 against commodity money, even though they could not trade with probability less than 1 against fiat money. So it seems that it is not money per se that generates the asymmetry, but fiat money. Before pursuing this issue further, however, we want to highlight some substantive results that emerge from the commodity money model.

First, note that for large γ we must have $\tau \in (0, 1)$, and indeed, as $\gamma \rightarrow \infty$ we have $\tau \rightarrow 0$. This is a version of Gresham's Law: very valuable money will be hoarded, in the sense that the probability it changes hands will be small. However, even though the money is hoarded, the good still changes hands with probability 1. Moreover, one can show that in the divisible goods version of the model with commodity money, with lotteries, for big γ we have $q = q^*$ with probability 1. Hence, a sufficiently valuable money may probabilistically stop circulating, but the outcome is nevertheless efficient. By contrast, without lotteries, a sufficiently valuable money also will be hoarded, but in this case trade will shut down, and welfare will be reduced.¹¹

Returning to the issue of asymmetry, we have seen that goods can trade with probability less than 1 against a money that has some exogenously specified commodity value γ . In fact, we will now show that goods can

¹¹ Velde, Weber and Wright [22] discuss Gresham's Law in search models with heterogeneous currencies, without allowing lotteries.

trade with probability less than 1 against a fiat money if there is some value to the fiat money that arises endogenously due to, in this case, heterogeneity in bargaining. In particular, consider a generalization of the model with indivisible goods and fiat money where we now assume that with probability ω the buyer gets to make a take-it-or-leave-it offer to the seller, while with probability $1 - \omega$ they bargain according to the Nash solution, with bargaining power θ and threat points $T_j = V_j$, as above. The chance of meeting a seller and making a take-it-or-leave-it offer generates an endogenous return to holding fiat money that will play a role analogous to the role of γ in the commodity money model.¹²

Let (τ, λ) be the lottery that results from the Nash solution and $(\hat{\tau}, \hat{\lambda})$ that which results from the take-it-or-leave-it offer (again we do not need conditional probabilities λ_0 and λ_1 , or $\hat{\lambda}_0$ and $\hat{\lambda}_1$, since it turns out that $\lambda_0 = \lambda_1 = \lambda$ and $\hat{\lambda}_0 = \hat{\lambda}_1 = \hat{\lambda}$). The value functions satisfy

$$\begin{aligned} rV_1 &= (1 - M)\{\omega[\hat{\lambda}U + \hat{\tau}(V_0 - V_1)] + (1 - \omega)[\lambda U + \tau(V_0 - V_1)]\} \\ rV_0 &= M(1 - \omega)[\tau(V_1 - V_0) - \lambda C]. \end{aligned}$$

In the equation for V_0 we have used the fact that the seller gets no surplus when the buyer makes a take-it-or-leave-it offer: $\hat{\tau}(V_1 - V_0) - \hat{\lambda}C = 0$. Moreover, this condition also implies that either $\hat{\tau} = 1$ and $\hat{\lambda} = \frac{V_1 - V_0}{C}$, or $\hat{\tau} = \frac{C}{V_1 - V_0}$ and $\hat{\lambda} = 1$. In the Appendix we show that generically we cannot have $\hat{\tau} = 1$ and $\hat{\lambda} = \frac{V_1 - V_0}{C}$; hence we proceed to characterize equilibria with $\hat{\tau} = \frac{C}{V_1 - V_0}$ and $\hat{\lambda} = 1$.

In the interest of manageability, and since we are only interested in showing that it is possible to have $\lambda \in (0, 1)$ we assume in what follows that $\omega \geq \underline{\omega} = (U - C)[(1 - M)U + CM]/C[rU + M(U - C)]$, which serves to guarantee that for all parameter values monetary equilibria exists.

PROPOSITION 7. *Assume $T_j = V_j$, and $\omega \geq \underline{\omega}$. Then for all parameters there exists an equilibrium with $\hat{\tau} < 1$, $\hat{\lambda} = 1$, and τ and λ described as follows. There are critical values $\underline{\theta}$ and $\bar{\theta}$ constructed in the proof, with $0 < \underline{\theta} < \bar{\theta} < 1$, such that: if $\theta < \underline{\theta}$ then $\tau = 1$ and $\lambda < 1$; if $\underline{\theta} < \theta < \bar{\theta}$ then $\tau = 1$ and $\lambda = 1$; and if $\theta > \bar{\theta}$ then $\tau < 1$ and $\lambda = 1$.*

Proof. See the Appendix.

The key part of this result is that we can have $\lambda < 1$ in this model, just like in the commodity money model. One way to understand what is going

¹² This model was inspired by comments of the associate editor, although his suggestion was actually to endow different agents with permanently different bargaining powers. This also works, but things turn out to be a lot cleaner if we assume agents are identical in an ex ante sense and have different bargaining powers in different meetings.

on is to think about the model in the following way. Agents believe that there is some economy-wide probability λ with which they can get goods for a unit of money. Then, when a particular buyer-seller pair meet, they bargain over the probability $\check{\lambda}$ that they will use to trade, taking λ as given. This generates a version of a best response function, of which an equilibrium is a fixed point. For the commodity money model, the best response function can easily be shown to be $\check{\lambda} = \min\{A(\lambda), 1\}$, where $A(\lambda)$ is linear:

$$A(\lambda) = \frac{\theta U + (1 - \theta) C}{CU(1 + r)} \{ \gamma + [(1 - M) U + MC] \lambda \}.$$

If $\gamma = 0$ (fiat money), the intercept of $A(\lambda)$ is zero. When θ is small, the slope of A is less than 1, and given any economy-wide λ our particular buyer-seller pair will bargain to $\check{\lambda} < \lambda$, so the only fixed point is $\lambda = 0$. When θ is big, the slope of $A(\lambda)$ is greater than 1, and given $\lambda > 0$ our pair will bargain to $\check{\lambda} > \lambda$, so $\lambda = 1$ is fixed point, as well as $\lambda = 0$. With $\gamma > 0$, however, the intercept of $A(\lambda)$ is strictly positive, and so there is a unique fixed point, it is always positive, and it less than 1 iff θ is small. Intuitively, the idea is that even if other agents are giving goods with probability $\lambda = 0$ in exchange for money, as long as $\gamma > 0$ you would be willing to trade your good with some positive probability to get a unit of money. This is why the model with $\gamma > 0$ always has $\lambda > 0$, and can have $\lambda < 1$. The idea is essentially the same when $\omega > 0$. Any value to money outside of the Nash bargain that determines λ could play the same role.

To close this section we address one final concern: does the nontrivial role for lotteries arise when we solve a social planner's problem, or only when we impose an arbitrary bargaining solution, in the sense of an arbitrary value of θ ? Consider a world with an exogenous supply of M units of commodity money with return $\gamma > 0$. Goods as well as money are indivisible and there is a unit storage capacity. The social planner gets to choose a lottery (τ, λ) . Welfare is given by

$$W = MV_1 + (1 - M) V_0 = M(1 - M) \lambda(U - C) + M\gamma,$$

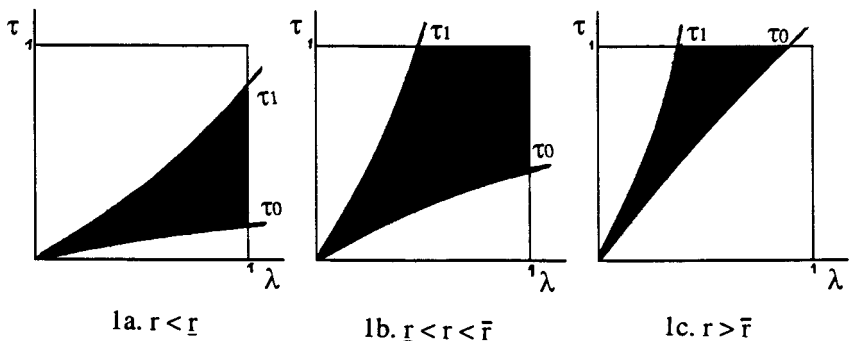
which is increasing in λ , and independent of τ (since it does not matter in the aggregate who holds the asset and therefore it cannot matter if money changes hands).

The planner is faced with the following incentive constraints:

$$\tau(V_1 - V_0) - \lambda C \geq 0 \quad \text{which holds iff} \quad r \leq \left[\frac{\gamma + (1 - M) \lambda(U - C)}{\lambda C} \right] \tau$$

$$\lambda U + \tau(V_0 - V_1) \geq 0 \quad \text{which holds iff} \quad r \geq \left[\frac{\gamma - M\lambda(U - C)}{\lambda U} \right] \tau.$$

Case 1: $\gamma / M(U-C) > 1$



Case 2: $\gamma / M(U-C) < 1$

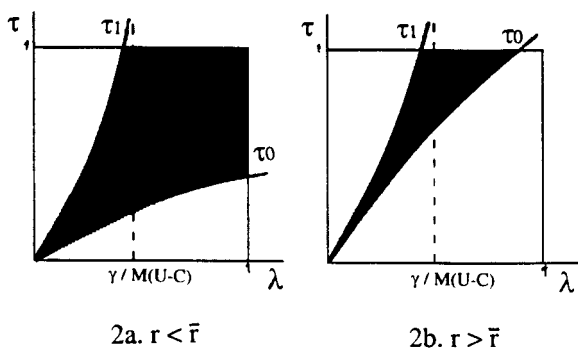


FIG. 3. Incentive feasible allocations in lottery space.

Notice that if $\gamma = 0$, then the second constraint is not binding and the first holds iff $r \leq \tau(1 - M)(U - C)/C$, which does not depend on λ . Thus, with $\gamma = 0$ there is no role for lotteries, since: (i) λ does not affect the incentive conditions and $\tau < 1$ only makes them less likely to hold; and (ii) the objective function W is unaffected by τ and increasing in λ . This means that either $\lambda = 1$ is feasible, in which case we can always set $\tau = 1$, or $\lambda = 1$ is not feasible, in which case we cannot do better than autarchy. So if there is a role for lotteries here it must be that $\gamma > 0$.

Figure 3 shows the set of points in lottery space satisfying the incentive conditions for various parameter configurations.¹³ The cases are similar

¹³ The figure is drawn by solving for the $\tau = \tau_0(\lambda)$ and $\tau = \tau_1(\lambda)$ that solve the two incentive conditions with equality. The values for \bar{r} and $\bar{\bar{r}}$ in the figure are given by $\bar{r} = [\gamma - M(U - C)]/U$ and $\bar{\bar{r}} = [\gamma + (1 - M)(U - C)]/C$.

except for the location of $(1, 1)$ relative to this set. In case 1a, for instance, $(1, 1)$ is not feasible; it is feasible to choose $\lambda = 1$ only if $\tau < 1$. This case corresponds to a large value of γ , which means money holders will not give up money with probability 1 to get the good, but they would give it up with some probability $\tau < 1$. In case 1b, $\lambda = 1$ is feasible for τ in some range that includes 1, which means we do not need lotteries. In case 1c, we cannot have $\lambda = 1$; the best λ we can achieve is attained by setting $\tau = 1$, but this entails $\lambda < 1$. In this case, γ is small, so again we need lotteries but now it is the good that trades with probability less than 1. The bottom line is that if $\gamma > 0$ there can be a nontrivial role for lotteries in the planner's problem, either because we need to set $\tau < 1$ to achieve $\lambda = 1$, or because the best λ we can achieve is less than 1.¹⁴

6. CONCLUSION

This paper has introduced lotteries into search-theoretic models of monetary exchange. It has been shown that, in general, private agents (or a planner) may want to use randomized trading in this environment. In the model with indivisible goods, we discussed how lotteries give us a way to analyze prices, and also how lotteries eliminate mixed strategy equilibria. When goods are divisible, we found that the quantity produced is never more than the efficient quantity—something that is not generally true without lotteries. Also, we found that as a commodity money gets more and more valuable, it drops out of circulation probabilistically but the outcome is still efficient—again something that is not true without lotteries. A general conclusion is that future work should take into account the fact that lotteries can have a nontrivial role to play in monetary economics.

APPENDIX

1. *Proof of Proposition 6.* The argument is similar to the proof of Proposition 1, although here we use the fact that $\lambda_0 = \lambda_1 = \lambda$. First order conditions for the bargaining problem are:

¹⁴ Note that there is no discontinuity at $\gamma = 0$: as $\gamma \rightarrow 0$, $\tau_1(\lambda)$ approaches the vertical axis and $\tau_0(\lambda)$ approaches a line which coincides with the vertical axis up to $\tau = rC/(1 - N)(U - C)$ and then becomes horizontal, which gives exactly the set of incentive feasible lotteries with $\gamma = 0$.

$$\begin{aligned}
& -\theta[\tau(V_1 - V_0) - \lambda C](V_1 - V_0) \\
& + (1 - \theta)[\tau(V_0 - V_1) + \lambda U](V_1 - V_0) - \eta_\tau \leq 0, \quad = \text{if } \tau > 0 \\
& \hspace{15em} (16)
\end{aligned}$$

$$\theta[\tau(V_1 - V_0) - \lambda C]U - (1 - \theta)[\tau(V_0 - V_1) + \lambda U]C - \eta_\lambda \leq 0, \quad = \text{if } \lambda > 0.$$

Given $\tau > 0$, the first condition in (16) holds with equality. Consider the case $\tau < 1$, which implies $\lambda = 1$, as in Proposition 1. Substituting the V_j 's into first condition in (16), we can solve for $\tau = \tilde{\tau}$, where $\tilde{\tau}$ is defined above, and show $\tilde{\tau} \in (0, 1)$ iff

$$\theta > \bar{\theta} = \frac{(r + M)U - MC - \gamma}{(1 + r)(U - C)}.$$

Notice $\bar{\theta} > 0$ iff $(r + M)U - MC > \gamma$. The incentive conditions are satisfied at $\tau = \tilde{\tau}$ and $\lambda = 1$. Hence, there exists an equilibrium with $\lambda = 1$ and $\tau = \tilde{\tau} \in (0, 1)$ iff $\theta > \bar{\theta}$.

Now consider the case where $\tau = 1$. Inserting the V_j 's into the second equation in (16) at equality and rearranging, we have

$$\begin{aligned}
& \lambda\{\theta U[(1 - M)U - (r + 1 - M)C + \gamma] \\
& - (1 - \theta)C[(r + M)U - MC - \gamma/\lambda]\} = (1 + r)\eta_\lambda.
\end{aligned}$$

Consider the case $\lambda < 1$, which implies $\eta_\lambda = 0$ and $\tau = 1$. Given this, we can substitute the V_j 's into second condition in (16) at equality and solve for $\lambda = \tilde{\lambda}$, where $\tilde{\lambda}$ is defined above. Notice $\tilde{\lambda} \in (0, 1)$ iff $\theta < \underline{\theta}$, where

$$\underline{\theta} = \frac{C[rU + M(U - C) - \gamma]}{(U - C)[U(1 - M) + CM + \gamma]}.$$

The incentive conditions are satisfied at $\lambda = \tilde{\lambda}$. Hence, there exists an equilibrium with $\tau = 1$ and $\lambda = \tilde{\lambda}$ iff $\theta < \underline{\theta}$.

Finally, consider $\lambda = \tau = 1$. Note that $\tau = 1$ satisfies the first condition in (16) iff $\theta \leq \bar{\theta}$ and $\lambda = 1$ satisfies the second condition iff $\theta \geq \underline{\theta}$. Also, the incentive compatibility constraints are satisfied at $\tau = \lambda = 1$. Hence, there exists an equilibrium with $\lambda = 1$ and $\tau = 1$ iff $\underline{\theta} \leq \theta \leq \bar{\theta}$. Also, it is easy to see that $\gamma < \bar{\gamma}$ implies $0 < \underline{\theta} < \bar{\theta} < 1$ and $\gamma > \bar{\gamma}$ implies $\bar{\theta} < 0$. ■

2. *Proof that we cannot have $\hat{\tau} = 1$ and $\hat{\lambda} = (V_1 - V_0)/C$.* Assuming $\hat{\tau} = 1$ and $\hat{\lambda} = (V_1 - V_0)/C$, we will derive a contradiction from the first order conditions from the Nash bargaining problem for (τ, λ) (these are given

explicitly in the proof of Proposition 7). There are several cases. First, we cannot have $\tau = \lambda = 0$ in a monetary equilibrium, and we cannot have $\lambda, \tau < 1$ because this generates the same contradiction as Proposition 1. Now suppose $\lambda < 1$ and $\tau = 1$; then the first order conditions, which are again given by (16), imply

$$CU = \frac{[\theta U + (1 - \theta) C](1 - \omega)[(1 - M) U + MC]}{r + (1 - M) \omega(1 - U/C) + 1 - \omega},$$

after substituting the value functions; this can hold only for degenerate parameter values. Now suppose $\lambda = 1$ and $\tau \leq 1$; then the first order conditions imply

$$\theta[\tau(V_1 - V_0) - C] = (1 - \theta)[U - \tau(V_1 - V_0)],$$

which implies $\tau = [(1 - \theta) U + \theta C]/(V_1 - V_0) > 1$ because $V_1 - V_0 \leq C$ (otherwise $\hat{\lambda} > 1$). In each case we have a contradiction. ■

Proof of Proposition 7. We are looking for equilibria with $\hat{\tau} = C/(V_1 - V_0)$, $\hat{\lambda} = 1$, and (τ, λ) satisfying the first order conditions, which are again given by (16). Again we emulate the proof of Proposition 1 and consider each combination of (τ, λ) in turn. As above, we cannot have $\lambda, \tau < 1$.

Now consider $\lambda < 1$ and $\tau = 1$. The second equation in (16) implies, after inserting the value functions,

$$\lambda = \frac{[\theta U + (1 - \theta) C](1 - M) \omega(U - C)}{CU(r + 1 - \omega) - (1 - \omega)[\theta U + (1 - \theta) C][(1 - M) U + MC]}.$$

One can check that the first condition in (16) is satisfied, and that $\lambda < 1$ iff $\theta < \underline{\theta}$ where

$$\underline{\theta} = \frac{C[Ur + (M - \omega)(U - C)]}{(U - C)\{(1 - M) \omega(U - C) + (1 - \omega)[(1 - M) U + MC]\}}.$$

One can also solve for

$$\hat{\tau} = \frac{CU(r + 1 - \omega) - [\theta U + (1 - \theta) C](1 - \omega)[(1 - M) U + MC]}{U(1 - M) \omega(U - C)},$$

and show that $\hat{\tau} < 1$ under the assumption $\omega > \underline{\omega}$. Hence, equilibrium exists with $\hat{\tau} < 1$, $\lambda < 1$ and $\tau = 1$ for all $\theta \in [0, \underline{\theta})$.

Consider $\lambda = 1$ and $\tau = 1$. Then

$$\hat{\tau} = \frac{C(r + 1 - \omega)}{(1 - M) U + MC - \omega C}.$$

The first order conditions can be shown to hold iff $\underline{\theta} \leq \theta \leq \bar{\theta}$, where

$$\bar{\theta} = \frac{rU + (M - \omega)(U - C)}{(1 - \omega + r)(U - C)}.$$

Hence, equilibrium exists with $\hat{\tau} < 1$ and $\tau = \lambda = 1$ for all $\theta \in [\underline{\theta}, \bar{\theta}]$.

Now consider $\lambda = 1$ and $\tau < 1$. The first equation in (16) implies

$$\tau = \frac{r[(1 - \theta)U + \theta C]}{[\theta - M + \omega(1 - \theta)](U - C)}.$$

The second condition in (16) is satisfied iff $\theta > \underline{\theta}$, and $\tau < 1$ iff $\theta > \bar{\theta}$. Also,

$$\hat{\tau} = \frac{rC[U(\theta - M) + C(M - \omega\theta)]}{[(1 - M)U + (M - \omega)C][\theta - M + \omega(1 - \theta)](U - C)}.$$

One can show that $\hat{\tau} < 1$. Hence, equilibrium exists with $\hat{\tau} < 1$, $\tau < 1$ and $\lambda = 1$ for all $\theta > \bar{\theta}$. ■

REFERENCES

1. A. Berentsen, Money inventories in search equilibrium, *J. Money, Credit, Banking* **32** (2000), 168–178.
2. K. Binmore, A. Rubinstein, and A. Wolinsky, The Nash bargaining solution in economic modeling, *Rand J. Econ.* **17** (1986), 176–188.
3. G. Camera and D. Corbae, Money and price dispersion, *Int. Econ. Rev.* **40** (1999), 985–1008.
4. S. Chatterjee and D. Corbae, Valuation equilibria with transactions costs, *Amer. Econ. Rev.* **85** (1995), 287–290.
5. M. Coles and R. Wright, A dynamic equilibrium model of search, bargaining, and money, *J. Econ. Theory* **78** (1998), 32–54.
6. P. Diamond, Pairwise credit in search equilibrium, *Quart. J. Econ.* **105** (1990), 285–319.
7. E. Green and R. Zhou, A rudimentary model of search with divisible money and prices, *J. Econ. Theory* **81** (1998), 252–271.
8. N. Kiyotaki and R. Wright, On money as a medium of exchange, *J. Polit. Econ.* **97** (1989), 927–954.
9. N. Kiyotaki and R. Wright, A contribution to the pure theory of money, *J. Econ. Theory* **53** (1991), 215–235.
10. N. Kiyotaki and R. Wright, A search-theoretic approach to monetary economics, *Amer. Econ. Rev.* **83** (1993), 63–77.
11. M. Molico, The distribution of money and prices in search equilibrium, manuscript (1996).
12. M. J. Osborne and A. Rubinstein, “Bargaining and Markets,” Academic Press, San Diego, 1990.
13. E. C. Prescott and R. M. Townsend, Pareto optima and competitive equilibrium with adverse selection and moral hazard, *Econometrica* **52** (1984), 21–45.

14. E. C. Prescott and R. M. Townsend, General competitive analysis in an economy with private information, *Int. Econ. Rev.* **25** (1984b), 1–20.
15. R. D. Rogerson, Indivisible labor, lotteries and equilibrium, *J. Monet. Econ.* **21** (1988), 3–16.
16. A. Rubinstein, Perfect equilibrium in bargaining model, *Econometrica* **50** (1982), 97–109.
17. A. Rubinstein and A. Wolinsky, Equilibrium in a market with sequential bargaining, *Econometrica* **53** (1985), 1133–1150.
18. K. Shell and R. Wright, Indivisibilities, lotteries, and sunspot equilibria, *Econ. Theory* **3** (1993), 1–17.
19. S. Shi, Money and prices: A model of search and bargaining, *J. Econ. Theory* **67** (1995), 467–496.
20. A. Taber and N. Wallace, A matching model with bounded holdings of indivisible money, *Int. Econ. Rev.* **40** (1999), 961–984.
21. A. Trejos and R. Wright, Search, bargaining, money and prices, *J. Polit. Econ.* **103** (1995), 118–141.
22. F. Velde, W. Weber, and R. Wright, A model of commodity money, with applications to Gresham's law and the debasement puzzle, *Rev. Econ. Dynam.* **2** (1999), 291–323.
23. R. Zhou, Individual and aggregate real balances in a random matching model, *Int. Econ. Rev.* **40** (1999), 1009–1038.