JOURNAL OF ECONOMIC THEORY **78**, 32–54 (1998)
ARTICLE NO. ET972353

A Dynamic Equilibrium Model of Search, Bargaining, and Money*

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Received January 19, 1996; revised August 21, 1997

This paper considers dynamic equilibria in a model with random matching, strategic bargaining, and money. Equilibrium in the bargaining game is characterized in terms of a simple differential equation. When we embed this characterization into the monetary economy, the model can generate outcomes such as limit cycles that never arise if one imposes a myopic Nash bargaining solution, as has been done in the past. *Journal of Economic Literature* Classification Numbers: C78, D83, E31. © 1998 Academic Press

1. INTRODUCTION

This paper considers dynamic equilibria in a model with random matching, strategic bargaining, and money. In particular, we extend work by Shi [26] and Trejos and Wright [30], who integrate explicit models of bilateral bargaining of the sort developed by Rubinstein [23] into search-theoretic models of fiat currency.[1] The analyses in those papers focus on steady states, or, when dynamics are discussed at all, simply impose Nash's [19] axiomatic bargaining solution. As we shall see, imposing the Nash solution out of steady state amounts to assuming that agents are myopic.

* We thank many of our friends and colleagues and participants in seminars and conferences (too many to mention by name) as well as the associate editor for helpful suggestions and comments. We also thank The National Science Foundation and La Comision Interministerial de Ciencia y Tecnologia for financial support. The usual disclaimer applies.

[1] Earlier search-based models of money along the lines Kiyotaki and Wright [12, 13] set prices exogenously.

0022-0531/98 $25.00

We solve the bargaining game here assuming that agents are forward looking and show how this makes a qualitative difference in the types of equilibria that can emerge.

It is well known that in stationary environments there is a close relationship between the models of Nash and Rubinstein. As shown by Binmore [2] and Binmore *et al.* [3], in the limit as the time between moves in the strategic model becomes small, the unique subgame perfect equilibrium outcome corresponds to the Nash solution with threatpoints and bargaining power that depend on details of the game. We generalize this result to environments that may be nonstationary. In particular, suppose there is no delay in bargaining and therefore immediate trade (in this paper we focus exclusively on such equilibria). If $q(t)$ denotes the terms of trade between two agents who meet at time t, then as the length of the period between moves in the bargaining game becomes small, the limiting path $[q(t)]_{t=0}^{\infty}$ will satisfy a simple differential equation. One can use this characterization in dynamic economic models in the same way that one uses the Nash solution in stationary models, as a "reduced form" for an explicit strategic bargaining game.[2]

To be more precise, suppose that the parameters of a bargaining game at time t are described by $\Theta(t)$, where $\Theta(t)$ may be endogenous in the market as a whole but is taken parametrically by the agents in their bilateral bargaining game (e.g., it may include the aggregate price level). We solve for the equilibrium value of $q(t)$ as a function of the expected path $[\Theta(s)]_{s=t}^{\infty}$. In comparison, the Nash solution depends only on current parameters. We show that if $\lim_{t\to\infty} \Theta(t)=\bar{\Theta}$ then $\lim_{t\to\infty} q(t)=\bar{q}$, where $\bar{q}$ is the Nash solution for parameters $\bar{\Theta}$; but, in general, imposing the same Nash solution with parameters $\Theta(t)$ for finite t amounts to assuming that agents are myopic. However, we also show that in the special case where all agents are risk neutral and have the same discount rate, $q(t)$ actually does satisfy the myopic Nash solution for all t and not just in the limit.[3]

[2] There are many prior analyses of nonstationary bargaining models. For example, going back to Stahl [29] people have considered finite horizons. Also, in some of the search and bargaining literature, as surveyed by Osborne and Rubinstein [20], the arrival rates for some agents change over time when others leave the market. More recently, Perry and Reny [21] analyze a continuous time bargaining model and Merlo and Wilson [15] analyze a stochastic discrete time model. None of these papers pursue the main objective here, however, which is to characterize equilibria in a way that is useful for applications in macro, labor and monetary economics.

[3] This is significant because several papers in the literature impose the Nash solution and then proceed to analyze non-steady state equilibria; see, for example, Pissarides [22], Drazen [8], Mortensen [17], or Mortensen and Pissarides [18]. As these papers assume risk neutral agents and common discount rates, our result suggests that the use of the Nash solution may in fact be justifiable as the limit of an explicit bargaining game when the time between moves becomes small. In related work, Coles and Hildreth [6] find that the Nash solution is also valid in a stochastic bargaining problem between a firm and a union when both agents are risk neutral.

In terms of results in monetary theory, we show that with forward looking bargaining there can be equilibria with limit cycles in nominal prices and real economic activity. Cycles are not possible in the same model with myopic bargaining. Hence the forward looking nature of the bargaining solution makes a qualitative difference. Moreover, cycles are not possible in a version of the model without fiat money, even with forward looking bargaining. Hence our results provide additional support for the long-standing notion that monetary economies are particularly susceptible to endogenous fluctuations induced by self-fulfilling prophecies, and show that search-theoretic models can generate equilibria that are similar to those generated by other, ostensibly quite different, models of fiat currency.[4]

The rest of the paper is organized as follows. In Section 2 we review the basic search-theoretic model of money. In Section 3 we analyze dynamic bargaining and derive our differential equation representation of equilibrium. In Section 4 we embed the solution to the dynamic bargaining game into the search model and describe market equilibria. We conclude in Section 5.

2. MONETARY THEORY

There is a $[0, 1]$ continuum of infinitely-lived individuals. There are $k > 2$ types of agents in equal numbers, and also k goods, with the property that type j only consumes good j and only produces good $j+1$ (modulo k). Agents meet in an anonymous random matching process where all exchange is bilateral and quid pro quo. Because of the way agents specialize in consumption and production there can be no direct barter. Also, we assume that the goods are nonstorable so as to rule out commodity money. Hence, if trade occurs at all in this economy it requires the use of fiat currency.[5]

At $t = 0$ a fraction $M \in [0, 1]$ of the population are each endowed with one unit of fiat currency, and the rest with production opportunities. For simplicity, we assume that when agents spend their money they spend all

[4] The notion that monetary economies are particularly susceptible to fluctuations can be found in the writings of Mill, Keynes, and Friedman, for example; see Smith [28] for a list of early references. This idea has been analyzed in overlapping generations models of money (see Shell [25] or Azariadis [1], e.g.), money-in-the-utility-function models (see Matsuyama [14] or Sims [27], e.g.), and cash-in-advance models (see Woodford [32], e.g.). Previous discussions in the context of search-based monetary models include Kehoe *et al.* [11], Wright [33], and Shi [26].

[5] It is interesting and feasible to include direct barter, commodity money, credit, and many other things in addition to fiat money in the model, but we ignore these complications here in order to focus on bargaining and dynamics.

of it (say, because it comes in indivisible units), and that except for those initially endowed with production opportunities no agent can produce until after he consumes. So in this economy every exchange involves exactly one unit of currency going from one agent to another, for some quantity of output q to be determined below, and therefore individual money holdings are always in $\{0, 1\}$. Call the M agents with money *buyers*, and the $1-M$ agents with no money but with production opportunities *sellers*.[6]

Consumption of q units of one's consumption good generates utility $U(q)$, while production of one's production good generates disutility $c(q)$. Assume $U'(q)>0$, $U''\leqslant 0$, $0<c'(0)<1$, and $c''(q)>0$ for all $q\geqslant 0$, and $c(q)>U(q)$ for large q. Also assume for now that $U(0)=c(0)=0$ (although later we will also consider the model with a fixed cost). When it is convenient below we will sometimes normalize $U(q)=q$, which one always can do without loss in generality as long as one renormalizes the cost function $c(q)$, since we can always let agents bargain over utility rather than physical units of output.

Suppose time is considered as a sequence of discrete periods of length $\Delta>0$. Agents meet according to a Poisson process with arrival rate α, which means that the probability that a seller meets a buyer in a period is approximately $\alpha\Delta M$ and the probability that a buyer meets a seller is approximately $\alpha\Delta(1-M)$. When a buyer and seller meet the probability is $1/k$ that the latter can produce the former's desired good. In that event, if they can settle on a quantity q, the buyer hands over his money, enjoys utility $U(q)$, and becomes a seller, while the seller takes the money, suffers disutility $-c(q)$, and becomes a buyer.

Assuming that at each date t agents who can trade complete negotiations immediately at some quantity $q(t)$, which could possibly be random, let V_b and V_s denote the value functions for buyers and sellers. Then the standard dynamic programming equations are

$$V_b(t)=\frac{1}{1+r\Delta}\left\{\alpha\Delta(1-M)\frac{1}{k}\left[E_t U(q(t+\Delta))+V_s(t+\Delta)\right]\right.$$

$$\left.+\left[1-\alpha\Delta(1-M)\frac{1}{k}\right]V_b(t+\Delta)+o(\Delta)\right\} \quad (1)$$

[6] Recent papers by Molico [16], Green and Zhou [9], Zhou [34], and Camera and Corbae [5] all study steady state versions of this model where agents hold quantities of money in some set other than $\{0, 1\}$. We make assumptions to guarantee individual money holdings are in $\{0, 1\}$ because it then is trivial to solve for the aggregate distribution of money holdings (which in general is very complicated) in equilibrium, and this allows us to focus attention on the dynamics of bargaining.

$$V_s(t)=\frac{1}{1+r\Delta}\left\{\alpha\Delta M\frac{1}{k}[-E_t c(q(t+\Delta))+V_b(t+\Delta)]\right.$$

$$\left.+\left[1-\alpha\Delta M\frac{1}{k}\right]V_s(t+\Delta)+o(\Delta)\right\}, \tag{2}$$

where r is the rate of time preference and the term $o(\Delta)$, which appears because of the Poisson approximation, satisfies $o(\Delta)/\Delta\to 0$ as $\Delta\to 0$.[7] One can rearrange these and take the limit as $\Delta\to 0$ to arrive at the standard continuous time equations,

$$rV_b(t)=\alpha(1-M)\frac{1}{k}[E_t U(q(t))+V_s(t)-V_b(t)]+\dot V_b(t) \tag{3}$$

$$rV_s(t)=\alpha M\frac{1}{k}[-E_t c(q(t))+V_b(t)-V_s(t)]+\dot V_s(t). \tag{4}$$

It remains to determine $q(t)$. In the next section we analyze explicit strategic bargaining games. However, for the sake of illustration, let us first consider what happens if one adopts the Nash bargaining solution: $q(t)=q^n(t)$, where

$$q^n(t)=\arg\max_q\,[U(q)+V_s(t)-T_b(t)]^{\theta}\,[-c(q)+V_b(t)-T_s(t)]^{1-\theta} \tag{5}$$

and $T_i(t)$ is the threatpoint of agent i and θ is the bargaining power of the buyer. Also, we need to impose that the maximization is subject to constraints that guarantee trade is voluntary:

$$U(q)+V_s(t)\geqslant V_b(t) \tag{6}$$

$$-c(q)+V_b(t)\geqslant V_s(t). \tag{7}$$

Given threatpoints and bargaining power, an equilibrium can be defined as a list of nonnegative and bounded paths $[V_s(t), V_b(t), q^n(t)]_{t=0}^{\infty}$ satisfying for all t the dynamic programming equations (in either discrete or continuous time) and the maximization problem in (5) subject to (6) and (7).

Shi [26] and Trejos and Wright [30] study steady states of this model. Those papers also discuss how the Nash solution can be interpreted as the outcome of an explicit bargaining game in the limit as the time between moves goes to 0. Of course, this has also been shown by Binmore [2] and

[7] For example, (1) says that between t and $t+\Delta$ a buyer meets a seller that can produce his desired good with probability $\alpha\Delta(1-M)\frac{1}{k}$, which yields payoff $E_t U(q(t+\Delta))+V_s(t+\Delta)$, and with probability $1-\alpha\Delta(1-M)\frac{1}{k}$ he does not, which yields payoff $V_b(t+\Delta)$. Note that the value functions are not indexed by agent type j because we only consider symmetric equilibria, in which all types get the same payoff and use the same strategies.

Binmore *et al.* [3] in a general context for models that are stationary, as is the above model as long as one only looks at steady states. Trejos and Wright [30] also discuss non-steady state solutions; but since it has never been claimed that the Nash solution can be derived from an explicit strategic bargaining model when the payoffs are changing over time, that specification is very much *ad hoc*. In the next section we analyze bargaining games without imposing stationarity so that in the section after that we can define equilibrium in a more satisfactory way.

3. BARGAINING THEORY

Suppose there are two agents, labeled $i=1, 2$, and 1 is interested in obtaining some q from 2 in exchange for a fixed amount of something else (like a unit of labor services, a given amount of money, or whatever). If agent 2 gives q to agent 1 at time t their instantaneous payoffs at t are given by $u_1(q, t)$ and $u_2(q, t)$, where u_1 is increasing and u_2 decreasing in q, and both depend explicitly on time. Agent i discounts the future at rate $r_i>0$, so the payoff for i from trading at t discounted back to date 0 is $e^{-r_i t}u_i(q, t)$. Assume $u_i \in C^2$, u_i concave in q for all t, $u_i(q, t)$ bounded in t, and $\partial u_i(q, t)/\partial t$ bounded for all (q, t). Agents derive some utility from not trading at all, normalized to 0. Define $\mathscr{A}(t)=\{q: u_i(q, t) \geqslant 0, i=1, 2\}$, and assume that $\mathscr{A}(t)$ is non-empty for all t, and uniformly bounded in t.[8]

The monetary exchange model in the previous section is a special case of the above specification, where agent 1 is the buyer and agent 2 is the seller, and the payoffs are $u_1(q, t)=U(q)+V_s(t)$ and $u_2(q, t)=-c(q)+V_b(t)$. These payoffs include the instantaneous utility from trading plus the value functions because in the monetary model the agents separate and return to the market once a trade is concluded. However, given the anonymous random matching process, once a bargain is completed the two agents will never meet again, and so their interaction is over once they part. In a bilateral bargaining game the agents may have arbitrary beliefs about the expected value of returning to the market, although of course the equilibrium we describe in the next section will require that these beliefs are consistent with market outcomes.

The bargaining procedure assumes random alternating offers. Suppose at time t the agents have not yet reached agreement. With probability π_1 ($\pi_2=1-\pi_1$) nature chooses player 1 (player 2) to propose a value of q. Given that offer, the other agent decides either to accept or reject. If he accepts, exchange takes place, payoffs are realized, and the agents part

[8] This ensures that q is bounded. In principle we could also impose a constraint such as $q \in [0, \hat{q}]$, but we simply assume that such contraints are not binding in most of what follows.

company. If he rejects, they realize no instantaneous utility that period and the game moves ahead one period of length Δ. At $t+\Delta$ nature chooses the next proposer with the same probabilities, and this continues until an offer is accepted. Notice that if no agreement is reached at t we assume that the players continue bargaining (rather than, say, dropping out of the game). In the monetary exchange model, constraints (6) and (7) imply that this is consistent with individual optimizing behavior.[9]

Our goal is to characterize subgame-perfect equilibria in strategies that are history independent, although typically nonstationary because payoffs depend on t. Also, we focus here on equilibria in which there is no delay: upon meeting, the agents always reach immediate agreement. Much of the literature has focussed on situations where delay may occur in equilibrium (see, e.g., Binmore [2] and Merlo and Wilson [15]) but this is not our concern here—indeed, the dynamic programming equations in the previous section were derived under the assumption that there is no delay in equilibrium. We refer to an equilibrium of this class as an *Immediate Trade Equilibrium* (ITE).

Given history independent strategies, if 1 is willing to accept q at t then he must also be willing to accept $q' > q$ at t. Similarly, if 2 is willing to accept q at t then he must also be willing to accept $q' < q$ at t. Hence, there exist *reservation values* $q_1(t)$ and $q_2(t)$ such that at time t agent 1 accepts any $q \geqslant q_1(t)$ and agent 2 accepts any $q \leqslant q_2(t)$. Moreover, the best proposal is always the reservation value of the other agent. This implies that we can identify an ITE strategy profile with $[q_1(t), q_2(t)]_{t=0}^{\infty}$, where at time t agent 1 proposes $q_2(t)$ if it is his turn to make an offer and accepts any $q \geqslant q_1(t)$ if it is his turn to respond, while agent 2 proposes $q_1(t)$ if it is his turn to make an offer and accepts any $q \leqslant q_2(t)$ if it is his turn to respond.

The reservation values satisfy the following recursive relations:

$$u_1[q_1(t), t] = \frac{1}{1+r_1\Delta}\{\pi_2 u_1[q_1(t+\Delta), t+\Delta] + \pi_1 u_1[q_2(t+\Delta), t+\Delta]\} \quad (8)$$

$$u_2[q_2(t), t] = \frac{1}{1+r_2\Delta}\{\pi_2 u_2[q_1(t+\Delta), t+\Delta] + \pi_1 u_2[q_2(t+\Delta), t+\Delta]\}. \quad (9)$$

For example, (8) says that agent 1 is indifferent between accepting his reservation value at t, or delaying until $t+\Delta$ when a new proposer will be

[9] That is to say, agents will never voluntarily walk away from a potential trading partner in equilibrium. At the end of this section we generalize the model to allow for exogenous breakdowns in the negotiations.

determined. These equations are *forward looking* in the sense that reservation values at t are defined in terms of reservation values at $t+\Delta$. If nothing is changing over time then (8) and (9) determine a pair of numbers (q_1, q_2). Here they constitute a dynamical system that determines paths $[q_1(t), q_2(t)]_{t=0}^{\infty}$. Such paths constitute an ITE if and only if $q_1(t) \leqslant q_2(t)$ for all t, since this ensures that at each stage there exists a mutually acceptable agreement.

We begin the analysis with some preliminary technical results, the proofs of which are in the Appendix. In what follows, $q_i(t)$ is actually a function of both t and Δ, but to conserve on notation we sometimes suppress the Δ argument when there is no risk of ambiguity.

LEMMA 1. *In ITE,* $u_j[q_i(t), t] \geqslant 0$ *and* $q_i(t)$ *is bounded in* t.

LEMMA 2. *In ITE, for all* t, *let* $\varepsilon(t) = q_1(t) - q_2(t)$; *then* $\varepsilon(t)$ *converges to zero at rate* Δ *as* $\Delta \to 0$.

Let $q(t)$ denote the expected terms of trade,

$$q(t) = \pi_2 q_1(t) + \pi_1 q_2(t).$$

Lemma 2 implies that when Δ becomes small $q_1(t)$ and $q_2(t)$ both converge to $q(t)$. The goal now is to characterize the behavior of $q(t)$.

THEOREM 1. *In ITE, in the limit as* $\Delta \to 0$, $q(t)$ *is a differentiable function of* t *and satisfies*

$$\dot{q} = \pi_2 \left[\frac{r_1 u_1(q, t) - \partial u_1(q, t)/\partial t}{\partial u_1(q, t)/\partial q} \right] + \pi_1 \left[\frac{r_2 u_2(q, t) - \partial u_2(q, t)/\partial t}{\partial u_2(q, t)/\partial q} \right]. \quad (10)$$

Proof. Given $\Delta > 0$, let $\varepsilon(t, \Delta) = q_1(t, \Delta) - q_2(t, \Delta)$, so that $q_1 - q = \pi_1 \varepsilon$ and $q_2 - q = -\pi_2 \varepsilon$. Given u_i is continuously differentiable and $\varepsilon(t, \Delta)$ is $O(\Delta)$ for all t by Lemma 2 (i.e., ε converges to 0 at rate Δ as $\Delta \to 0$), first order Taylor expansions of (8) and (9) about $q(t, \Delta)$ on the LHS and $q(t+\Delta, \Delta)$ on the RHS yield

$$u_1[q(t, \Delta), t] + \pi_1 \varepsilon(t, \Delta) \frac{\partial u_1[q(t, \Delta), t]}{\partial q} = \frac{u_1[q(t+\Delta, \Delta), t+\Delta]}{1 + r_1 \Delta} + o(\Delta) \quad (11)$$

$$u_2[q(t, \Delta), t] - \pi_2 \varepsilon(t, \Delta) \frac{\partial u_2[q(t, \Delta), t]}{\partial q} = \frac{u_2[q(t+\Delta, \Delta), t+\Delta]}{1 + r_2 \Delta} + o(\Delta) \quad (12)$$

where $o(\Delta)/\Delta \to 0$ as $\Delta \to 0$. If we multiply (11) by $\pi_2 \partial u_2[q(t, \Delta), t]/\partial q$ and (12) by $\pi_1 \partial u_1[q(t, \Delta), t]/\partial q$, add the equations, and simplify, we get

$$\pi_2 \left\{ u_1[q(t, \Delta), t] - \frac{u_1[q(t+\Delta, \Delta), t+\Delta]}{1+r_1\Delta} \right\} \frac{\partial u_2[q(t, \Delta), t]}{\partial q} + \pi_1 \left\{ u_2[q(t, \Delta), t] - \frac{u_2[q(t+\Delta, \Delta), t+\Delta]}{1+r_2\Delta} \right\} \frac{\partial u_1[q(t, \Delta), t]}{\partial q} = o(\Delta).$$

Given $\varepsilon(t, \Delta)$ is $O(\Delta)$ and $\partial u_1/\partial t$ is bounded, (11) implies that $q(t, \Delta) - q(t+\Delta, \Delta)$ is $O(\Delta)$. Hence, given $q(t+\Delta, \Delta)$, a first order Taylor expansion of the previous equation about $q(t+\Delta, \Delta)$ implies $q(t, \Delta)$ must satisfy

$$\frac{q(t+\Delta, \Delta) - q(t, \Delta)}{\Delta} = \pi_2 \left\{ \frac{r_1 u_1[q(t+\Delta, \Delta), t+\Delta] - \partial u_1[q(t+\Delta, \Delta), t+\Delta]/\partial t}{\partial u_1[q(t+\Delta, \Delta), t+\Delta]/\partial q} \right\} + \pi_1 \left\{ \frac{r_2 u_2[q(t+\Delta, \Delta), t+\Delta] - \partial u_2[q(t+\Delta, \Delta), t+\Delta]/\partial t}{\partial u_2[q(t+\Delta, \Delta), t+\Delta]/\partial q} \right\} + \frac{o(\Delta)}{\Delta}.$$

If an ITE exists in the limit as $\Delta \to 0$, then taking the limit as $\Delta \to 0$ in the previous equation implies that the limiting solution $[\lim_{\Delta \to 0} q(t, \Delta)]_{t=0}^{\infty}$ must satisfy the differential equation (10) for all t. This completes the proof. ∎

Based on Theorem 1, if we know $q(t) = \hat{q}$ at some point $\hat{t}$, say, then the entire path $[q(t)]_{t=0}^{\infty}$ can be found by iterating on (10). The remaining problem is to determine such a condition and so identify an ITE. The next result considers the case where the payoff functions u_i settle down over time.

THEOREM 2. *Suppose* $u_i(q, t) \to \bar{u}_i(q)$ *as* $t \to \infty$, $i = 1, 2$, *where* $\bar{u}_i$ *satisfies all of the assumptions on* u_i. *Then, in the limit as* $\Delta \to 0$, *if an ITE exists it is unique, and* $q(t) \to \bar{q}$ *as* $t \to \infty$ *where*

$$\bar{q} = \arg\max \bar{u}_1(q)^{\theta}\, \bar{u}_2(q)^{1-\theta}$$

with $\theta = \pi_1 r_2/(\pi_1 r_2 + \pi_2 r_1)$.

Proof. If $u_i(q, t) = \bar{u}_i(q)$ then (10) becomes

$$\dot{q} = \left(\frac{\pi_2 r_1 \bar{u}_1}{\bar{u}_1'} + \frac{\pi_1 r_2 \bar{u}_2}{\bar{u}_2'} \right) \equiv \Upsilon(q).$$

The solution to $\Upsilon(q)=0$ is $\bar{q}$. Moreover, $\Upsilon'(q)>0$, which implies that if $q(t)>\bar{q}$ in the limit then $q(t)$ increases without bound, and if $q(t)<\bar{q}$ in the limit then $q(t)$ decreases without bound. But Lemma 1 says that q is bounded, and so it must converge to $\bar{q}$. The solution to (10) given this boundary condition is unique, and therefore there is a unique ITE. ∎

Consider again the Nash solution

$$q^n(t)=\arg\max_q \,[u_1(q,t)-T_1(t)]^{\theta}\,[u_2(q,t)-T_2(t)]^{1-\theta} \qquad (13)$$

with $T_i(t)=0$ and $\theta=\pi_1 r_2/(\pi_1 r_2+\pi_2 r_1)$. The previous result says that if the payoff functions settle down over time, then $\lim_{t\to\infty} q(t)=\lim_{t\to\infty} q^n(t)$; but $q(t)$ does *not* generally coincide with $q^n(t)$ for $t<\infty$.[10]

Assuming $q^n(t)$ describes the outcome of the bargaining game typically requires assuming that agents are myopic: they believe the payoff functions will stay at their current values for the entire future even as they are changing. There is a special case, however, in which $q(t)$ and $q^n(t)$ do coincide along the entire path, and not merely in steady state.

THEOREM 3. *Suppose* $u_i(q,t)=\eta_i q+\varphi_i(t)$, *where* $\eta_1>0>\eta_2$, *and* $r_1=r_2=r>0$. *Then if an ITE exists it is unique and* $q(t)=q^n(t)$ *with* $T_i(t)=0$ *and* $\theta=\pi_1$.

[10] For example, let $u_1(q,t)=q^{\beta}$, with $0<\beta<1$, and $u_2(q,t)=e^{-\delta t}-q$, so that the surplus to be divided is depreciating at rate δ (or, if $\delta<0$, appreciating). Assume $r_i=r$ and $\pi_i=\frac{1}{2}$, in which case (10) becomes

$$\dot{q}=\frac{r(1+\beta)\,q-\beta(r+\delta)\,e^{-\delta t}}{2\beta}.$$

In this example Theorem 2 implies $q(t)\to 0$, and the solution to the differential equation subject to this boundary condition is

$$q^*=\frac{\beta(r+\delta)\,e^{-\delta t}}{r(1+\beta)+2\delta\beta}.$$

One can check that immediate trade is an equilibrium as long as $r+\delta>0$. By way of comparison, the Nash solution with the threatpoints and θ that apply in steady state implies that

$$q^n=\frac{\beta e^{-\delta t}}{1+\beta}$$

for all t. While q^* and q^n converge to the same limit as $t\to\infty$, for finite t we have $q^*>q^n$ if $\delta>0$ and $q^*<q^n$ if $\delta<0$.

Proof. Without loss in generality normalize $\eta_1 = 1$ and $\eta_2 = -1$. Then (10) reduces to

$$\dot{q} = \pi_2(r_1 q + r_1 \varphi_1 - \dot{\varphi}_1) - \pi_1(-r_2 q + r_2 \varphi_2 - \dot{\varphi}_2),$$

which implies

$$\dot{q} - (\pi_2 r_1 + \pi_1 r_2)\, q = \pi_1(\dot{\varphi}_2 - r_2 \varphi_2) - \pi_2(\dot{\varphi}_1 - r_1 \varphi_1).$$

As these terms only have a common integrating factor if $r_1 = r_2$, a simple solution only exists for such r. Given $r_1 = r_2$, integration implies

$$q = \pi_1 \varphi_2 - \pi_2 \varphi_1 + \eta_0 e^{rt},$$

where η_0 is a constant. Since q is bounded by Lemma 1, we have $\eta_0 = 0$ and $q(t) = \pi_1 \varphi_2(t) - \pi_2 \varphi_1(t)$. For these functional forms, this is identical to $q^n(t)$. ∎

In other words, $q^n(t)$ equals $q(t)$ for all t when payoffs are linear in q, separable between q and t, and $r_1 = r_2$.[11] This is significant because some authors, including Pissarides [22], Drazen [8], Mortensen [17], and Mortensen and Pissarides [18], have effectively imposed the solution $q^n(t)$ in dynamic models. As those models assume linear and separable payoffs as well as equal discount rates, one may conjecture that the use of the Nash solution may be justifiable in the sense that the outcome would be the same if they used a forward looking bargaining game and took the limit as the time between moves went to 0; of course, this needs to be checked carefully, as there are some differences in the environments used by those authors (including the fact that in those models agents who reach an agreement enter into an ongoing relationship, rather than simply trading and parting). In any case, from the perspective of the monetary model in the previous section, Theorem 3 is not of much help: it requires that both $U(q)$ and $c(q)$ are linear in q, and it is easy to show that in this case there does not exist a steady state with valued fiat money. To the extent that one is interested in a model which has a monetary steady state one must assume that at least one of the functions $U(q)$ or $c(q)$ is nonlinear, and then use the forward looking solution (10) to analyze dynamics.

Notice that until now we have been assuming immediate trade. As remarked above, it must be checked that this is consistent with equilibrium behavior. Let $\Pi_i(t) = e^{-r_i t} u_i[q(t), t]$ be the equilibrium discounted payoff

[11] In the simple example in the previous footnote, if we assume that $\beta = 1$, then the assumptions of Theorem 3 are satisfied and one can easily check that $q^n = q^*$ for all t.

to i if agreement is made at time t, given q solves (10). Then immediate trade for all t constitutes an equilibrium if

$$\Pi_i(t) > 0 \qquad \text{and} \qquad \Pi_i'(t) < 0 \tag{14}$$

for all t and $i = 1, 2$. The first inequality says the agents always want to trade, while the second says they always want to trade sooner rather than later.[12]

Rather than verifying (14) directly in each application, note that $\Pi_i'(t) < 0$ for both agents if and only if

$$\left(r_1 u_1 - \frac{\partial u_1}{\partial t}\right)\frac{\partial u_2}{\partial q} - \left(r_2 u_2 - \frac{\partial u_2}{\partial t}\right)\frac{\partial u_1}{\partial q} < 0. \tag{15}$$

A simple sufficient condition which guarantees (15) is that $e^{-r_i t} u_i(q, t)$ is decreasing in t for all $q \in A(t)$ for all t, and is strictly decreasing for one agent (see Binmore [2]). However, as these payoff functions may be endogenously determined as part of a bigger model (as in the previous section), it may not be possible to establish whether this condition holds before solving the entire model. In that case, it is necessary to generate a candidate ITE and then check that (15) holds along the equilibrium path.

The last thing we do in this section is to briefly consider the case where we allow exogenous breakdowns in the bargaining game. Let λ_i be the Poisson arrival rate with which i believes an exogenous breakdown will occur, and b_i his payoff in this event.[13] In this case, a straightforward generalization of Theorem 1 yields

$$\dot{q} = \pi_2 \left[\frac{(r_1 + \lambda_1)\, u_1 - \lambda_1 b_1 - \partial u_1/\partial t}{\partial u_1/\partial q}\right] + \pi_1 \left[\frac{(r_2 + \lambda_2)\, u_2 - \lambda_2 b_2 - \partial u_2/\partial t}{\partial u_2/\partial q}\right].$$

Following Theorem 2, one can show that when $u_i(q, t)$ converges over time to $\bar{u}_i(q)$ then $\lim_{t \to \infty} q(t) = \bar{q}$, where $\bar{q}$ is the Nash solution with

[12] By inspection of (11), a sufficient condition that $\varepsilon(t) \leqslant 0$ for all t in the limit as $\Delta \to 0$, and hence $q_1(t) \leqslant q_2(t)$ for all t, is that $u_1[q(t), t] - u_1[q(t+\Delta), t+\Delta]/(1 + r\Delta)$ is strictly positive and is $O(\Delta)$ for all t. From the definition of $\Pi_1(t)$, a sufficient condition for existence of an ITE in the limit as $\Delta \to 0$ is that $\Pi_1'(t) < 0$ for all t. Similarly for $i = 2$.

[13] We do not necessarily impose $\lambda_1 = \lambda_2$, since in applications in the literature a breakdown occurs for one agent but not the other if the former is replaced by an identical new agent (see, e.g., Wolinsky [31]). In such a model, b_i typically represents the value of searching for another trading partner.

$$T_i = \frac{\lambda_i b_i}{r_i + \lambda_i};$$

$$\theta = \frac{\pi_1(r_2+\lambda_2)}{\pi_1(r_2+\lambda_2)+\pi_2(r_1+\lambda_1)}.$$

Finally, following Theorem 3, one can show that if the payoffs are linear in q and $r_i + \lambda_i$ is the same for both agents, then $q(t)$ is the same as the myopic Nash solution along the entire path, not just in steady state. Also note that $\theta = \pi_1$ in this case.

4. MARKET EQUILIBRIA

We are now in a position to characterize equilibrium in the monetary model with forward looking strategic bargaining. Because agents are of measure zero, the probability that two particular traders meet again once they have separated is zero. Also, any delay to trade between two agents does not affect the aggregate market outcome. Hence, while bargaining each agent takes the expected value of returning to the market as given. For simplicity, we assume in this section that in the bargaining game each agent has an equal probability of making the next offer ($\pi_1 = \frac{1}{2}$), they have common discount rates ($r_i = r$), and there are no exogenous breakdowns ($\lambda_i = 0$). If delay occurs, (6) and (7) imply that both agents prefer to continue bargaining than separate.

In discrete time, to describe an ITE with history independent bargaining strategies we need to determine $[V_b(t), V_s(t), q_b(t), q_s(t)]_{t=0}^{\infty}$. All agents believe that $V_i(t)$ is the value of search at time t in the market and $q_i(t)$ is the amount traded when agent i receives an offer at time t, $i = b, s$. If trade q occurs at t then the expected payoffs to the buyer and seller are $u_1(q, t) = U(q) + V_s(t)$ and $u_2(q, t) = -c(q) + V_b(t)$. Of course, equilibrium requires that beliefs are consistent with market outcomes; i.e., taking $[q_b(t), q_s(t)]_{t=0}^{\infty}$ as given $[V_b(t), V_s(t)]_{t=0}^{\infty}$ must satisfy (1) and (2); and taking $[V_b(t), V_s(t)]_{t=0}^{\infty}$ as given $[q_b(t), q_s(t)]_{t=0}^{\infty}$ must be consistent with equilibrium in the bargaining game.

Now consider the limiting case as $\Delta \to 0$. Then we need to determine $[V_b(t), V_s(t), q(t)]_{t=0}^{\infty}$ where the $V_i(t)$ solve the continuous time dynamic programming equations (3) and (4), and $q(t)$ is the limiting value of both $q_b(t)$ and $q_s(t)$ as $\Delta \to 0$. Theorem 1 says that if an ITE exists then q satisfies (10), which in this model simplifies to

$$\dot{q} = \frac{rU(q) + rV_s - \dot{V}_s}{2U'(q)} + \frac{rc(q) - rV_b + \dot{V}_b}{2c'(q)}. \tag{16}$$

Thus, equilibrium is given by paths for $[V_b(t), V_s(t), q(t)]_{t=0}^{\infty}$ satisfying (3), (4) and (16), subject to constraints (6) and (7), plus the immediate trade condition (14) for all t.

As is often true in search-theoretic models, we can reduce the dimensionality of the system by defining $x = V_b - V_s$. Subtracting (3) and (4) yields

$$\dot{x} = rx - \alpha(1-M)\frac{1}{k}[U(q) - x] - \alpha M\frac{1}{k}[c(q) - x],$$

and inserting (3) and (4) into (16) yields

$$\dot{q} = \frac{rU(q) + \alpha M\frac{1}{k}[x - c(q)]}{2U'(q)} + \frac{rc(q) + \alpha(1-M)\frac{1}{k}[x - U(q)]}{2c'(q)}.$$

Moreover, constraints (6) and (7) can be rewritten as $(q, x) \in \mathscr{B}$, where $\mathscr{B} = \{(q, x) : c(q) \leqslant x \leqslant U(q)\}$. Thus, the model reduces to a system in (q, x),

$$\begin{bmatrix} \dot{q} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} \dfrac{rq + M[x - c(q)]}{2} + \dfrac{rc(q) + (1-M)(x-q)}{2c'(q)} \\ (1+r)\,x - Mc(q) - (1-M)\,q \end{bmatrix}, \tag{17}$$

where in order to reduce the notation we have, with no loss in generality, normalized time by setting $\alpha/k = 1$ and normalized $U(q) = q$. Then an (immediate trade) equilibrium is any solution to (17) that stays in $\mathscr{B}$, and also satisfies $\Pi_b = e^{-rt}(q + V_s)$ and $\Pi_s = e^{-rt}[-c(q) + V_b]$ decreasing in t.

A special case is a steady state, which is an equilibrium where q and x are constant (there are no initial conditions here since both q and x can take on any values at date 0). It is easy to see that $(q, x) = (0, 0)$ is a steady state. It is also easy to show that whenever there exists a monetary steady state (i.e., one with $q > 0$) it is unique, and it exists if and only if $c'(0)$ is below some threshold $\tilde{c}$.[14] For the remainder of the analysis we assume a monetary steady state exists and proceed to consider dynamic equilibria.

[14] From (17), $\dot{q} = \dot{x} = 0$ is equivalent to $\Psi(q) = 0$, where $\Psi(q)$ will be defined in equation (18) in the text below. Moreover, a necessary and sufficient for $(q, x) \in \mathscr{B}$ is that $q \leqslant \hat{q}$, where $\hat{q}$ is defined by $(1-M)\,\hat{q} = (r+1-M)\,c(\hat{q})$. Note that $\hat{q} > 0$ as long as $c'(0) < \tilde{c}$, where $\tilde{c}$ is the smaller root of the quadratic

$$\tilde{c}^2 - \frac{2(r+M)}{M}\tilde{c} + \frac{(r+M)(1-M)}{M(r+1-M)} = 0.$$

One can show $\Psi(0) = 0$, $\Psi(\hat{q}) < 0$, and $\Psi'(q) < 0$ at any $q \in (0, \hat{q}]$ such that $\Psi(q) = 0$. One can also show that $\Psi'(0) > 0$ if and only if $c'(0) < \tilde{c}$. Hence, if $c'(0) < \tilde{c}$ then there is a unique $q \in (0, \hat{q})$ such that $\Psi(q) = 0$, and therefore a unique monetary steady state; otherwise, there is no such q and no monetary steady state.

The Jacobian of (17) is

$$J=\begin{bmatrix} r-\frac{Mc'}{2}-\frac{1-M}{2c'}-\frac{[rc+(1-M)(x-q)]\,c''}{2(c')^2} & \frac{M}{2}+\frac{1-M}{2c'} \\ -Mc'-(1-M) & 1+r \end{bmatrix}.$$

We compute $\det(J)=-\Psi'(q)/2c'(q)$, where

$$\Psi(q)=(r+M)[(1-M)\,q-(r+1-M)\,c(q)] \\ -(r+1-M)[(r+M)\,q-Mc(q)]\,c'(q). \tag{16}$$

One can show $\Psi'(0)>0$, and so $\det(J)<0$, at the nonmonetary steady state, and it is a saddle point. At the monetary steady state, $\Psi'(q)<0$, and so $\det(J)>0$ and the monetary steady state is either a sink or a source. As

$$\text{trace}(J)=1+r+\frac{(1-M-Mc')^2-\Psi'}{2(1+r)\,c'},$$

we see that $\Psi'(q)<0$ implies $\text{trace}(J)>0$ at the monetary steady state, and so it is a source.

One can also show that along the boundary of the set $\mathscr{B}$ the flow is outward, and so orbits never enter from outside of $\mathscr{B}$. Since $(0,0)\in\mathscr{B}$, the saddle path leading to the nonmonetary steady state lies entirely in $\mathscr{B}$, and so any orbit beginning on the saddle path is an equilibrium. Furthermore, since it cannot come from outside of $\mathscr{B}$, the saddle path must either emanate from the monetary steady state or from a cycle surrounding the monetary steady state. We found in examples that we studied that the saddle path always emerged from the monetary steady state, although sometimes $[q(t), x(t)]$ converged monotonically to the nonmonetary steady state and sometimes it first spiraled around the monetary steady state before converging to $(0,0)$. In these examples, the complete set of equilibria consists of the monetary steady state, the nonmonetary steady state, and a continuum of dynamic equilibria indexed by any initial beliefs $[q(0), x(0)]$ on the saddle path and converging to the $(0,0)$ (any initial beliefs not on the saddle path sets us off on an orbit that eventually leaves the set $\mathscr{B}$, which although it cannot be seen in the figure, is compact because $c(q)>U(q)$ for large q). A typical case is shown in Figure 1.

Although we could not rule out the existence of stable limit cycles, we did not find any in simple examples. To construct an explicit example with a stable limit cycle, therefore, we complicate things slightly by considering

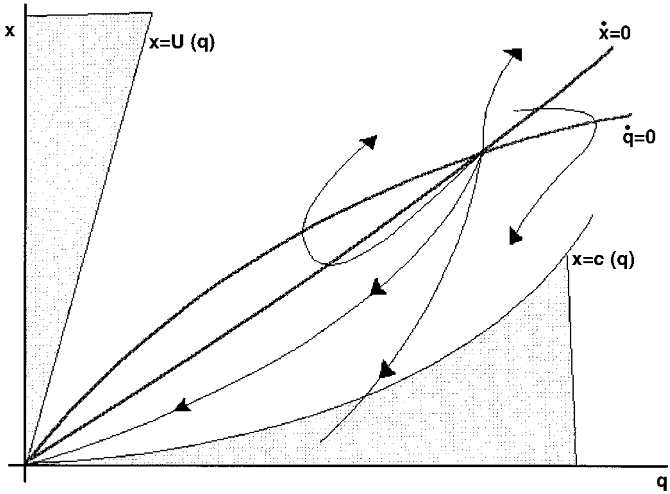

FIGURE 1

the case with a fixed cost, $c(0)=\bar{c}>0$.[15] This shifts the $\dot{q}=0$ and $\dot{x}=0$ curves so that they now intersect twice in the positive quadrant, as shown in Figure 2. Let (q^0, x^0) and (q^*, x^*) denote the lower and higher steady states; these both constitute monetary equilibria as long as they are in $\mathscr{B}$, which will be the case as long as the fixed cost is not too big. And, of course, as always there is a nonmonetary steady state at the origin. The low level monetary steady state (q^0, x^0) inherits its stability properties from the nonmonetary steady state that we analyzed when $\bar{c}=0$; i.e., it is a saddle point. As shown in the figure, there are dynamic equilibria leading to (q^0, x^0) starting both from $q(0)<q^0$ and $q(0)>q^0$.

Even with $\bar{c}>0$, if we impose the myopic Nash bargaining solution, then one can still show that the saddle path converges monotonically from (q^*, x^*) to (q^0, x^0). Therefore, we will now construct a cycle around (q^*, x^*) and be able to argue that this depends on the forward looking nature of our bargaining solution. Our strategy is as follows: First, we fix $M=\bar{M}$ and $r=\bar{r}$. Then let $c(q)=a_0+a_1 q+a_2 q^2$ in the neighborhood of (q^*, x^*), and choose the coefficients a_j so that $(q^*, x^*)\in \text{int}(\text{B})$ and

[15] An alternative that we suspect would lead to similar results is to allow a positive probability of direct barter for agents who are not holding money. The effect of either a fixed cost or allowing some barter is to prevent q from converging to 0 in equilibrium, since no one would incur the fixed cost to get money which can only be exchanged for a very small q, nor would any one be willing to give up the option of barter for a very small q. A very similar effect arises if we impose a storage cost on money.

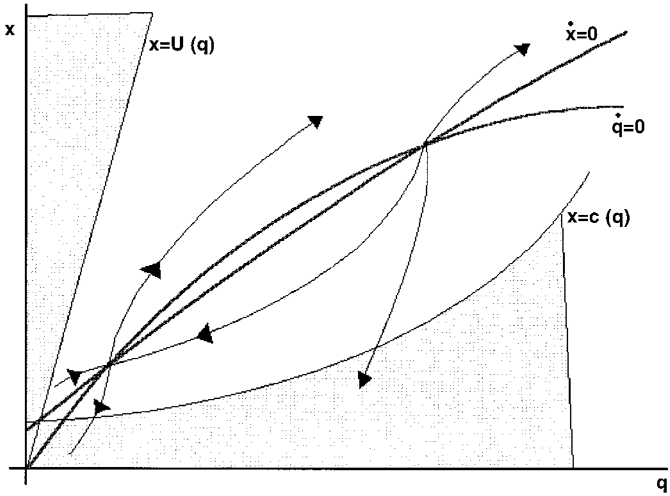

FIGURE 2

trace$(J)=0$ at (q^*, x^*).[16] Then one should expect cycles in the neighborhood of these parameters. We verify this by studying the system numerically (we used the software *Phaseplane*).

For small enough $r<\bar{r}$, one finds that the branch of the unstable manifold of (q^0, x^0) denoted by W^u in Figure 3 lies outside the branch of the stable manifold of (q^0, x^0) denoted by W^s in Figure 3. As we increase r, W^u and W^s get closer together until, at some $r=\hat{r}<\bar{r}$, they coalesce to form a homoclinic orbit that starts at (q^0, x^0), loops around (q^*, x^*), and returns to (q^0, x^0). For $r\in(\hat{r}, \bar{r})$, W^u lies inside of W^s, as shown in Figure 4, and so there is a region around (q^*, x^*) within which orbits cannot escape. But for $r<\bar{r}$, we have trace$(J)>0$ and so (q^*, x^*) is a source, which means that orbits in this region cannot converge to (q^*, x^*). Applying the Poincare–Bendixson Theorem (see Guckhenheimer and Holmes [10]), they must converge to a limit cycle around (q^*, x^*) for all $r\in(\hat{r}, \bar{r})$. The size of the cycle is decreasing in r, and for $r>\bar{r}$ it collapses into (q^*, x^*), since for $r>\bar{r}$ we have trace$(J)<0$ and so (q^*, x^*) is a sink.

The key result is that for all $r\in(\hat{r}, \bar{r})$ there is a region such that any orbit that starts in this region converges to a limit cycle around (q^*, x^*). To argue that such a path is an equilibrium, we need to verify two more things: that it stays within $\mathscr{B}$, and that it satisfies the immediate trade

[16] Note that this is impossible when $a_0=0$, as we argued earlier that trace$(J)>0$ at the monetary steady state under the assumption $c(0)=0$; but it is possible if $a_0>0$ (this is where the fixed cost comes in). With $\bar{M}=0.5$ and $\bar{r}=0.01$, we have $a_0=13.57451$, $a_1=-27.04902$, and $a_2=13.57451$. For these parameter values, $c(q)$ is increasing and convex in the relevant range.

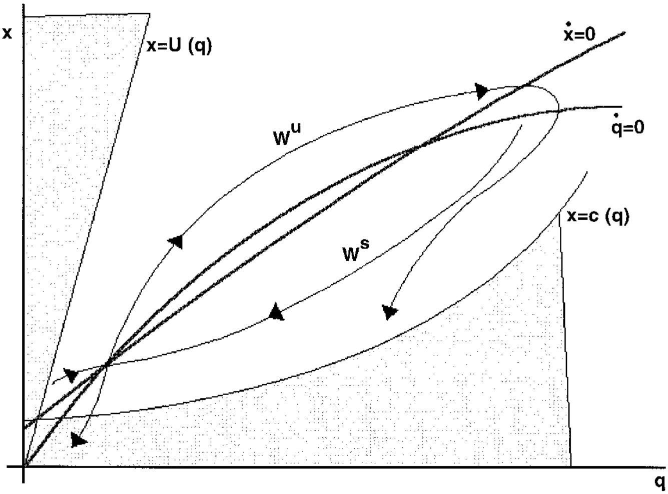

FIGURE 3

condition, $\Pi_i'(t) < 0$ for all t. Since $(q^*, x^*) \in \text{int}(\mathscr{B})$, at least for r near $\bar{r}$ the cycles are sufficiently small they must stay in $\mathscr{B}$. We then verified numerically in our examples that $\Pi_i'(t) < 0$ along the cycle. Hence, there exist paths converging to limit cycles that satisfy all of the conditions for monetary equilibria. We conclude that cycles are possible in the model with forward looking bargaining, but not with the myopic Nash solution.

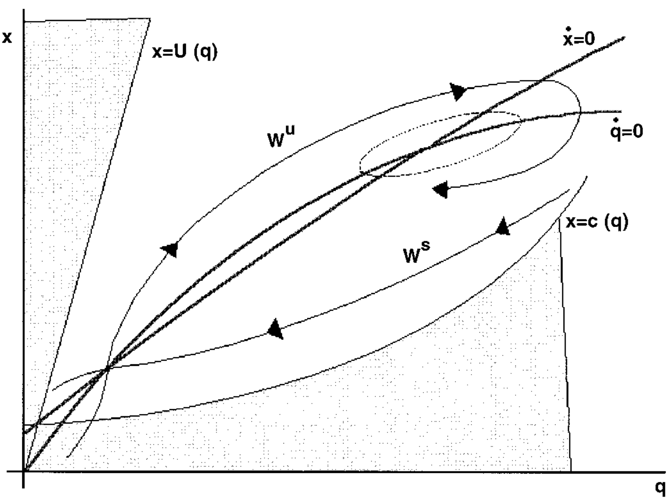

FIGURE 4

One can get cycles in some closely related models, like Diamond and Fudenberg [7] or Boldrin *et al.* [4], but those models work in a very different way. In particular, they do not include fiat currency and there is no bargaining (all trades are one-for-one swaps), and the possibility of cycles depends on increasing returns to scale in the exogenously specified matching technology. We do not need increasing returns in the matching technology, although we obviously did assume a fixed cost in the production technology. Also note that a version of this model without fiat currency but extended to allow direct barter would look very much like a standard (nonmonetary) search and bargaining model, along the lines of Rubinstein and Wolinksy [24]. That model would not have multiple equilibria and would not have cyclical equilibria. It seems to be the combination of monetary exchange and forward looking bargaining that is the key to the results here.

5. CONCLUSION

This paper has analyzed a model of search and bargaining with fiat money. The solution to the bargaining problem was characterized by a simple dynamical system. It is possible that this characterization will be useful in dynamic analyses beyond the application to monetary economics considered here, in much the same way the Nash solution is useful in static or steady state analyses. In the context of the fiat currency model, our characterization of the bargaining solution gives the same answer as the myopic Nash solution only in steady state. In particular, we constructed an example with limit cycles, something that cannot happen if one imposes the myopic bargaining solution.

APPENDIX

Proof of Lemma 1. By always rejecting offers and making offers $q_i(t) \in \mathscr{A}(t)$, agent i can always guarantee a non-negative payoff. Hence an equilibrium offer must lie in $\mathscr{A}(t)$. Then $q_i(t)$ must be bounded because $\mathscr{A}(t)$ is uniformly bounded. ∎

Proof of Lemma 2. We must show that for all t, for small Δ, $q_2(t) - q_1(t) = O(\Delta^a)$ where $a \geqslant 1$. By way of contradiction, suppose that at some t we have $q_2(t) - q_1(t) = O(\Delta^a)$ with $a < 1$. Notice that ITE requires $q_2(t) > q_1(t)$, while Lemma 1 requires $a \geqslant 0$. Now let $h = h_0 \Delta^b$, where $h_0 > 0$

and $a<b<1$, and consider the time interval $T_h=[t, t+h]$. By construction, $h \to 0$ as $\Delta \to 0$. Also, if N denotes the number of Δ time periods in T_h then $N \to \infty$ as $\Delta \to 0$.

The following result sets up the required contradiction.

Claim. Fix $\Delta>0$ and $k>0$. Let $n=1, 2, ...,$ and let M be the number of time periods in an ITE where $t+n\Delta \in T_h$ and

$$\frac{u_1[q_2(t+n\Delta), t+n\Delta]}{(1+r_1\Delta)^n} > u_1[q_1(t), t]+k\Delta. \tag{19}$$

Then, as $\Delta \to 0$, $M/N \to 0$.

Proof. Let $P_1(t)$ be the expected payoff to player 1 at t if agreement is not reached at t. Player 1 can always use the following strategy in the subgame:

1. Always reject player 2's offer;
2. In period $t+n\Delta$, propose $q>q_2(t+n\Delta)$ if (19) does not hold;
3. In period $t+n\Delta$, propose $q=q_2(t+n\Delta)$ if (19) holds.

Given player 2's strategy in ITE, this strategy implies

$$P_1(t) \geqslant \{u_1[q_1(t), t]+k\Delta\}(1-\pi_2^M). \tag{20}$$

Settlement occurs in the third contingency in the above list; the probability that this never occurs is $\pi_2{}^M$, in which case $u_1 \geqslant 0$. Now ITE requires $P_1(t) \leqslant u_1[q_1(t), t]$. This and (20) imply

$$\pi_2{}^M \geqslant \frac{k\Delta}{u_1[q_1(t), t]+k\Delta},$$

or, equivalently,

$$M \leqslant \frac{\log(u_1+k\Delta)-\log(k\Delta)}{-\log(\pi_2)}.$$

Now consider the limit as $\Delta \to 0$. If $u_1=0$ then $M=0$. If $u_1>0$ (but bounded) then, noting that $1/N=O(\Delta^{1-b})$, we have

$$\frac{M}{N} \leqslant O(-\Delta^{1-b}\log \Delta).$$

Hence, $M/N \to 0$. This proves the claim.

By symmetry, the same result holds for player 2. Hence, as $\Delta \to 0$, most time periods $t + n\Delta \in T_h$ are characterized by

$$\frac{u_1[q_2(t+n\Delta), t+n\Delta]}{(1+r_1\Delta)^n} \leqslant u_1[q_1(t), t] + k\Delta \tag{21}$$

$$\frac{u_2[q_1(t+n\Delta), t+n\Delta]}{(1+r_2\Delta)^n} \leqslant u_2[q_2(t), t] + k\Delta. \tag{22}$$

By concavity, (21) implies

$$\frac{u_1[q_2(t+n\Delta), t+n\Delta]}{(1+r_1\Delta)^n} \leqslant u_1[q_2(t+n\Delta), t] + [q_1(t) - q_2(t+n\Delta)] \frac{\partial u_1[q_2(t+n\Delta), t]}{\partial q} + k\Delta.$$

This can be rewritten as

$$q_2(t+n\Delta) \leqslant q_1(t) + R_1(t, t+n\Delta, \Delta),$$

where $R_1(t, t+n\Delta, \Delta)$ is defined to make the statements equivalent.

We know $q_2(t+n\Delta)$ is bounded and u_1 is continuous with a bounded time derivative. As $n\Delta < h$, it follows that $|R_1(t, t+n\Delta, \Delta)| = O(\Delta^b)$. Similarly,

$$q_1(t+n\Delta) \geqslant q_2(t) + R_2(t, t+n\Delta, \Delta),$$

where $|R_2(t, t+n\Delta, \Delta)| = O(\Delta^b)$ (note that the inequality is reversed because $\partial u_2/\partial q < 0 < \partial u_1/\partial q$). Subtracting,

$$q_2(t+n\Delta) - q_1(t+n\Delta) \leqslant -[q_2(t) - q_1(t)] + R_1(t, t+n\Delta, \Delta) - R_2(t, t+n\Delta, \Delta).$$

But $q_2(t) - q_1(t) > 0$ and is $O(\Delta^a)$, where $a < b$. Hence, as $\Delta \to 0$, there must exist many time periods $t + n\Delta \in T_h$ where $q_2(t+n\Delta) - q_1(t+n\Delta) < 0$, which contradicts an ITE. This completes the proof. ∎

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