Supernovas in Monetary Theory: Does the Ultimate Sunspot Rule Out Money?

By JON FAUST*

In recent years, economists have attempted to elucidate the micro-foundations of the demand for money by rigorously modeling monetary economies. Work toward this goal has been hampered by a perplexing problem: in standard models of *finite-duration* economies, there can be no demand for money. The explanation of this result starts with the fact that rational agents only accept fiat money with hopes of later spending it, and it is completed using backward induction from the final period. No agent will accept money in exchange during the final period since there will be no chance to spend it. Forward-looking agents will not accept money in the penultimate period since it is valueless in the final period, and so on. It seems that a necessary condition for money to have value at any time in an economy is that there be some prospect that it will have value at a later time.

Taken at face value, this "terminal problem" (Douglas Gale, 1982, p. 226) yields the questionable result that the death of our sun has implications for an area of monetary theory already plagued by sunspots (David Cass and Karl Shell, 1983). More importantly, the terminal problem has been an obstacle to building monetary models, requiring that theorists either work with infinite duration or rely on some artificial means to give money value, such as imposing a terminal condition on money balances (Frank Hahn, 1982; Gale, 1982) or putting money in the utility function.

Although each of these solutions is unappealing to those seeking a complete explanation of money's role in our finite world, the importance of the problem is unclear. Some seem to take the position that the problem reflects a fact of life, implying that the demand for money must rely on an infinite horizon (Cass and Shell, 1980; Ross Starr, 1980). The consensus, however, is probably that the terminal problem is a curious defect of current models with no substantive implications (Hahn, 1980). This view implicitly relies on the belief that there exist solutions to the puzzle that do not alter substantially the results that are obtained using *ad hoc* solutions.

In this paper a solution to the terminal problem is demonstrated using a model with variable holding periods in continuous time. It is verified that, at points in time far from the end of the world, the terminal point has almost no implications for the evolution of the economy. In Section I, the basic approach and model are described, and an example of a finite-duration monetary equilibrium is provided. Section II is dedicated to proving some general results about finite-duration equilibria, and the final section contains a discussion of the results and of some related literature.

I. A Model of Precautionary Money Holding

Modifying standard models by adding the realistic features of holding periods for money that are endogenously determined (Boyan Jovanovic, 1982; David Romer, 1986) and stochastic (Peter Diamond, 1984; Douglas Diamond and Philip Dybvig, 1983) has yielded many interesting results recently, and this approach also offers insight into the terminal problem. Suppose money is to become valueless at time τ. With variable holding periods in continuous time, no matter how imminent is time τ, any agent could accept money with the prospect of spending again before τ.

*Federal Reserve Bank of Kansas City, 925 Grand Ave., Kansas City, MO 64198. The views expressed herein are solely those of the author and do not necessarily reflect the views of the Federal Reserve Bank of Kansas City or of the Federal Reserve System. I am grateful for helpful comments from George Akerlof, Robert Anderson, and the anonymous referees.

In this world, there is no *last* period that money has value, and the necessary condition for money to function should be met. One might view this solution to the terminal problem as involving little more than the compression of an infinite number of trading periods into a finite length of time. While there is merit in this view, the transformation is not straightforward, as may become clear below.[1]

In the world described, money may be viewed as insurance against the costs of illiquidity from the current moment to time τ. As time τ approaches, the chance that the insurance will ever pay falls; as the value of the insurance falls, the money price of goods should rise without bound. During an era far from time τ, the insurance value of money should not be changing greatly and the economy should not behave much differently from an infinite-duration economy. These intuitions are verified in the context of the following model.

A. *Description of the Model*

The situation of the agents in this model is similar to that of agents in the models of William Baumol (1952) and James Tobin (1956), or that of Merton Miller and Daniel Orr (1966). Agents receive a weekly income and have weekly expenditures. The timing of some weekly consumption is known, and, for simplicity, it is assumed that the agents can time this consumption to occur on payday. Sometime during each week, agents are also confronted with a surprise desire for consumption, and consumers must hold money for the realistic purpose of making purchases when the desire for consumption arises but income does not. Since the pay periods overlap, surprised agents use money to purchase consumption from some agent who was just paid her (commodity) income.

The basic model is specified in the first seven assumptions below. The goal of proving the existence of finite-duration equilibria is simplified by using two further assumptions that limit our search: we will only consider equilibria in which prices are not random and never fall and in which agents are treated symmetrically. Given success in finding equilibria, the primary cost to limiting our search is that we may remain ignorant of still more finite-duration equilibria of the basic model.

ASSUMPTION A1: Each agent faces a weekly pay cycle of duration W that determines when he is paid a homogeneous commodity income, Q. The commodity perishes instantly if not consumed by someone. (The source of the commodity is ignored in the model).

ASSUMPTION A2: Each agent's marginal utility of consumption is zero at all times except as follows. A desire for consumption always arises on payday, called point g in the pay cycle. A stochastic need for consumption arises at a point labeled h in each pay cycle, and the agent derives utility from consumption at any time between h and the next payday. This will be referred to as *surprise consumption*. The agents all have identical utility functions that can be written,

$$(1) \qquad U(C_g, C_h) = u(C_g) + u(C_h),$$

where C is consumption at the indexed point in the pay cycle. For simplicity, it is assumed that agents do not discount consumption at h seen from g. The utility function is assumed to have some or all of the following properties as specified:

ASSUMPTION A3a: $u' > 0$, $u'' < 0$.

ASSUMPTION A3b: The coefficient of relative risk aversion for $u(x)$ is less than one and bounded away from zero for feasible levels of consumption:

$$(2) \qquad 0 < B < -xu''(x)/u'(x) \leq 1$$

for some B.

[1] Further insight into this issue can be gained from consideration of the more general case of continuous-time repeated games (Leo Simon and Maxwell Stinchcombe, 1987).

The timing of the surprise will be governed by one of two contrasting processes:

ASSUMPTION A4a: Point h always occurs halfway through the pay cycle.

ASSUMPTION A4b: Point h is drawn independently for each agent from a uniform distribution. For any agent at time t who has not been surprised since he was last paid and who will next be paid at time s, the prospect of being surprised in any interval can be derived from the density

$$(3)\quad \theta(t+\delta) = \begin{cases} 1/(s-t), & \delta \in [0, s-t] \\ 0, & \text{otherwise} \end{cases}$$

ASSUMPTION A5: There exist bits of paper, called money, that cannot be consumed and that are distributed initially in the quantity and manner described below.

ASSUMPTION A6: Only spot markets operate, and money balances must be nonnegative.

While this assumption may seem unnecessarily blunt, it can be justified in a number of ways. It is well known that unless something disturbs the smooth functioning of markets, agents could use full contingent claims markets to obviate any need for money. Rather than simply ruling out such markets, it might seem less arbitrary to specify any of a number of environments in which well-known problems upset the function of certain markets. Two drawbacks of that approach are the additional complexity of exposition entailed and the fact that the model would less clearly fulfill its essential role of illustrating the general point. In any case, one such environment—based on the possibility of hiding from one's creditors—is described in an earlier version of the paper and is available on request.

Finally, it should be noted that this model, including (A6), is no less palatable than an overlapping generations model. Indeed, it has essentially the same structure as an overlapping generations model with agents who live two periods (one pay cycle) and with one agent born to replace each that dies. Point g is the working phase, and point h is a stochastically timed retirement phase. The original interpretation is preferred, however, because it provides a more realistic interpretation for money than does the overlapping generations model—money is held between paychecks as opposed to between phases of life.

The final assumption of the basic model is specified to create aggregate certainty with individual uncertainty. Since the only stochastic feature in the model is the timing of the consumption surprise, this goal can be achieved if, loosely speaking, the number of surprised agents at any time can be known with certainty. The intuition is to model a large population in which the stochastic element of the timing of individual surprises disappears due to some sort of law of large numbers. This simplifying move has been used frequently in macroeconomics, but recent work of Kenneth Judd (1985) and Mark Feldman and Christian Gilles (1985) demonstrates that approaches implicit in much of such work are flawed. While there is agreement that solutions exist, there is little agreement on the best way to proceed. Because the mathematics of aggregate certainty is not the main topic here, the required result is simply assumed.[2]

ASSUMPTION A7: There is an infinite number of agents whose initial paydays are arranged uniformly over time $[t_0, t_0+W]$ (each agent is paid with certainty every W units of time since last payment). The aggregate outcome of the independent and uni-

[2]Feldman and Gilles (1985) suggest using a countable set of agents and a finitely additive measure space: $(A, P(A), \nu)$, with $A = 1,2,3,\ldots$, $P(A)$ the power set of A, and ν defined uniquely to satisfy Feldman and Gilles Proposition 3 (1985, p. 31). Feldman and Gilles show that the required law of large numbers result is satisfied in this construction, and the primary substantive change this brings to the analysis here is that the aggregate resource constraint involves integration with respect to a charge. In favor of this idealization of a large economy, the authors cite Thomas Armstrong and Marcel Richter (1984) and Ernst-August Weiss (1981).

formly distributed surprises for each individual is such that the density of agents being surprised is with probability one uniform and equal to the uniform density of agents being paid.

Next are the assumptions that limit our search. First, we assume that all agents in similar circumstances are treated similarly in equilibrium. Given the equal quantity of buyers and sellers in any interval of time, the assumption of equal treatment allows us to model the economy as if agents pair off to trade.

ASSUMPTION A8: Agents are treated symmetrically in equilibrium so that the consumption foregone by one agent paid at t is precisely the consumption at t of one surprised agent.

ASSUMPTION A9: The price of the commodity in terms of money is known with certainty and never falls as time passes.

Given aggregate certainty, it is natural to look for equilibria with nonstochastic prices. Limiting consideration to equilibria in which prices never fall simplifies the problem in two respects. First, it makes trivial the agent's decision about when to purchase consumption after being surprised: the purchase should occur as soon as possible. Second and more important, in combination with the (A3) assumptions regarding the utility function, (A9) guarantees that nobody would want to carry money over from one pay period into the next (which is to say, the marginal utility of income in equilibrium is nonincreasing as time passes). In conjunction with ruling out borrowing, this lack of saving allows us to treat each pay period as an independent optimization problem.

Given the setup of the model, the role of money is obvious. The commodity perishes instantly if not consumed by someone; thus, if trade is not possible, the utility gained by each agent over any pay cycle is simply $u(Q)$. Introducing fiat money might allow Pareto-improving trades, in which an agent who has just been paid sells the commodity to an agent who has just been surprised. To prove this is possible in equilibrium, we will look for an equilibrium in which 1) each agent has maximized utility subject to individual budget constraints, 2) with a positive, finite price level, the money stock is voluntarily held, and 3) the aggregate resource constraint holds. In this model, the aggregate resource constraint requires that the integral of consumption in any time interval equals the integral of commodity income in that interval. In constructing an equilibrium, (A8) is imposed, requiring that the consumption foregone by one paid agent is the consumption of one surprised agent and guaranteeing that the aggregate budget constraint will be satisfied.[3]

To establish a basis for evaluating the importance of stochastic consumption and finite duration, we begin by looking at the infinite-duration version of the model in which the timing of the surprise is fixed at the midpoint of the cycle. Consider the pay cycle beginning at a time t when some agent is paid. The agent will wish to maximize expected utility, (1), subject to the following constraints:

$$P_gQ = P_gC_g + M, \tag{4}$$

$$M = P_hC_h, \tag{5}$$

where P is the price of Q at the indexed instant, and M is nominal money balances.[4]

Assume that at time t, all agents who have not purchased surprise consumption since they were last paid are holding \$$M$ in cash. If price expectations are given by

$$P_g = P_h = M/(Q/2) \tag{6}$$

for all t, then it is easy to see that the economy is on a rational expectations equilibrium path. The agents at (or after) point h in their cycles will want to spend all of their cash. Once the constraint (5) is substituted into (1), the problem of those at payday becomes to select C_g and M to satisfy (4)

[3]This result will hold for any construction of the measure space of agents and stochastic process that provide with probability one that the measure of agents paid and surprised in any interval is equal.

[4]Equation (5) exploits the assumptions of known and monotonically increasing prices as discussed in (A9).

and

$$u'(M/P_h)/P_h = u'(C_g)/P_g. \tag{7}$$

Together with the price in (6), this implies that consumption at the two points in the cycle is the same, $(Q/2)$. With consumption evenly spread between periods, this is obviously a Pareto optimal equilibrium.

Now introduce uncertainty over the timing of h by using (A4b), the uniformly distributed surprise.[5] To find the equilibrium in the stochastic case, time indices must be added to the variables. Consumption at t by those paid at t is $C_g(t)$; $C_h(t)$ is analogously defined. Using W for the time span $(s-t)$ in (3), the first-order condition (7) becomes

$$\int_t^{t+W} \frac{u'(M/P(s))}{P(s)}\theta(s)\,ds = \frac{u'(C_g(t))}{P(t)}. \tag{8}$$

While (8) explicitly shows the roles of the various variables, the equation is greatly simplified by some redefinitions and normalizations. Set Q and W to one, and define $\phi(t)$ to be the proportion of real income sold by agents paid at t. The consumption by one surprised agent is the consumption foregone by one agent at payday; thus,

$$\phi(t) \equiv C_h(t) = 1 - C_g(t)$$

and substituting ϕ in (5) yields $P(t) = M/\phi(t)$. Equation (8) then reduces to

$$\int_t^{t+1} \phi(s)u'(\phi(s))\,ds = \phi(t)u'(1-\phi(t)). \tag{9}$$

[5]One complication must be considered: there is zero probability that an agent will be surprised at the exact moment some agent is being paid, implying surprised agents must wait for someone with whom to trade. Under the assumptions specified, however, buyers and sellers will both be uniform and dense in any interval. They can be paired off by moving surprised agents an arbitrarily small distance and without changing their distribution.

An infinite-horizon, rational expectations equilibrium of this model occurs when, for some function ϕ, equation (9) holds and is expected to hold for all t. Notice that $\phi(t) = 1/2$ for all t is an equilibrium path for consumption: uncertainty does not eliminate Pareto optimality of the equilibrium. This is because, given the aggregate certainty in the model, the addition of uncertainty may change the name of the agent that happens to be buyer at any time, t, but whichever agent happens to be a buyer at time t has the same demand function as before.

B. *Finite-Duration Monetary Equilibria*

Now consider an economy that lasts one pay period starting at time zero and ending at time $\tau = 1$. Obviously, the economy reverts to autarchy at time one, and $\phi(t)$ equals zero from time one onward. The question is whether there is some equilibrium path such that $P(t)$ is positive and finite over the interval from time zero up to up to point τ. To verify that this is the case, one must find some function $\phi(t)$ that satisfies,

$$\int_t^{1} \phi(s)u'(\phi(s))\,ds = \phi(t)u'(1-\phi(t))t \in [0,1]. \tag{10}$$

Any closed form solution to this problem obviously depends on the specific utility function that is chosen. A simple case arises when $u(x)$ can be represented by the natural log of x. The equilibrium path for ϕ is then,

$$\phi(t) = (1-t)/(2-t), \tag{11}$$

which can be verified by substituting (11) into (10).

This is the solution for a one-period economy beginning at time zero. At the beginning of the economy, $\phi(t)$ is equal to $1/2$, its infinite-horizon equilibrium value, and at the end, the value of money and $\phi(t)$ are zero. If the economy lasted $K+1$ pay periods, then the equilibrium path would coincide with the Pareto optimal, infinite-horizon equilibrium for the first K periods and then switch to the path given by (11). This can be verified by substituting the proposed solution into (15) below.

This equilibrium illustrates the two intuitions discussed above: there is an arbitrarily long phase at the beginning of the economy in which the existence of a terminal point in the distant future does not affect the functioning of money. As the end becomes near, an unbounded inflation begins that ultimately leaves money valueless. The next section is dedicated exclusively to showing that these properties carry over to a broader class of utility functions.

II. Some General Results

The following results are demonstrated here: a monetary equilibrium of this model exists in general (for utility functions satisfying (A3)); the equilibrium always involves the price level becoming unbounded as time τ approaches; and, if the entire duration of the economy is "long enough," equilibrium values will be arbitrarily close to the infinite-duration equilibrium values for an arbitrarily long interval of time.

The first result is that any equilibrium ends in an unbounded inflation.

PROPOSITION 1: *If $u(\cdot)$ satisfies (A3a), then for any $\phi(t)$ that solves (10), $\lim_{t\to 1} \phi(t)=0$.*

PROOF:

From (10) it is clear that

$$\lim_{t\to 1} \phi(t)u'(1-\phi(t))=0, \tag{12}$$

since the integral on the left side of (10) goes to zero. If $\phi(t)$ does not go to zero, then $u'(1-\phi(t))$ must go to zero. By (A3a), this marginal utility is strictly positive in the relevant range, however. □

The $K+1$ period equilibrium is constructed backward from time one beginning with the final pay period.

PROPOSITION 2: *Using preferences satisfying (A3a) and (A3b), a solution exists for ϕ in (10) in which ϕ is monotonically declining, strictly positive and finite for $t\in[0,1]$, and with $\phi(1)=0$.*

PROOF:

Differentiating (10) with respect to t and rearranging yields,

$$\phi'(t)=\frac{-\phi u'(\phi)}{u'(1-\phi)-\phi u''(1-\phi)} \equiv f(\phi(t)). \tag{13}$$

This is a first-order ordinary differential equation; and, since $\phi(t)$ and t are both in the unit interval, we are interested in the equation's behavior in the unit square. Strict concavity of $u(\cdot)$ guarantees that $f(x)\le 0$ and that ϕ falls monotonically. An upper bound on the absolute value of $f(x)$ is given by the ratio of the maximum (over x) of the numerator divided by the minimum of the denominator (this bound is attained for the natural log utility function). The numerator of f is proportional to the marginal utility of a dollar at the point of surprise. Call this $\mu(x)$. Using (A3b), the restriction on risk aversion, one can see that the first derivative of μ with respect to x is positive:

$$\frac{\partial\mu(x)}{\partial x}=xu''(x)+u'(x)\ge 0. \tag{14}$$

Thus, the maximum of the numerator is $u'(1)$. The minimum of the denominator occurs at $x=0$. At this point, the first term is at its minimum of $u'(1)$ and the second is at its minimum (absolute) value of zero.

Together, the results imply that $f(x)\in[-1,0]$ and is continuous for x and t in their domain. These facts are sufficient to show that the ordinary differential equation in (13) has a continuous solution through any point, (t,x), in the closed unit square and, specifically, that there is a solution satisfying the Proposition 2 requirement that $\phi(1)=0$ (Earl Coddington and Norman Levinson, 1955, Theorem 1.2).

Unfortunately, we must deal with the possibility that the solution we have found has $\phi(t)=0$ for all time. Such nonmonetary equilibria—equilibria in which money never has value—are common phenomena in models (Hahn, 1982). Inspection of (10) reveals that $\lim_{x \to 0} xu'(x) = 0$ is necessary and sufficient for the nonmonetary equilibrium to exist, and the goal is to show that a second, monetary, equilibrium exists whenever this condition is satisfied. This final step in proving Proposition 2 is demonstrated in the Appendix. □

PROPOSITION 3: *Under preferences satisfying (A3a) and (A3b), the model has a monetary equilibrium of arbitrary duration with ϕ declining monotonically.*

PROOF:
The proof, sketched here, is in the Appendix.

The equilibrium of arbitrary duration is constructed backward in time from the final period equilibrium just established. The first step involves taking a solution for time t onward and finding an equilibrium for all pay periods that overlap time t. This proof closely follows the proof of Proposition 2. The remaining step is to show that $\phi(t)$ does not go to its upper or lower bounds, so that this process can be continued indefinitely back in time. This step is easily completed once monotonicity of prices is established.

The final result to be proven is that the equilibrium of an economy of long duration will involve a long period nearly indistinguishable from the Pareto optimal, infinite-horizon equilibrium.

PROPOSITION 4: *For any $k > 0$ and any $\delta > 0$, there exists a duration for the economy, K, such that for an interval of time k pay periods in length, $\phi(t)$ is within δ of $1/2$.*

PROOF:
The proof follows directly from two facts demonstrated in proving Proposition 3: $\phi(t)$ is monotonically increasing as time recedes and between zero and one-half. This implies that if the economy lasts long enough, $\phi(t)$ must be essentially unchanged for a long period of time. When $\phi(t)$ is flat, prices are, by definition, constant, and the infinite-horizon equilibrium path is the only possible solution. Slightly more formally, it is straightforward to show that the limit of $\phi(t)$ as time recedes is $1/2$. Thus, we can select K large enough that $\phi(t)$ is within δ of $1/2$ for any number of periods. □

III. Discussion

The model of Sections I and II provides equilibria with intuitively appealing characteristics. At a point distant in time from the end of the world, the fact that the world will indeed end has almost no implications for how agents carry out their day-to-day transactions, and the infinite-horizon equilibrium very nearly prevails. As Armageddon draws near, the value of money goes to zero, as one would expect. Overall, the results of this simple model are intended to illustrate why using *ad hoc* terminal conditions or infinite-horizon approximations to the actual finite-duration economy may not introduce undesired results—so long as one is interested in studying the "stable" period early on in the life of the economy.

The results here also have application to the problem of bubbles in asset prices. As Jean Tirole (1985) has noted, monetary equilibria involve bubbles since there is a positive price for money, an asset that pays no dividends and cannot be consumed. These results, then, suggest that finite-duration bubbles are possible in continuous time. Using a different structure from that in this paper, this point was recently established by Franklin Allen and Gary Gorton (1988).

It must be said that one should be dubious of the practical importance of results that hinge on whether time comes in small but fixed lumps or flows continuously. Both constructions of time may have unpalatable aspects, and either might be appropriate in specific cases. If one appeals to the sensibleness of the results implied by the model, however, I believe that the justification for continuous time is persuasive in this case, for it is difficult to believe that our concerns about the end of the universe have any current role in the function of money.

There remains a question as to whether there are other solutions to the terminal problem. Three comments are warranted.

First, we know that a monetary model with discounting can generally be viewed as a model of a finite-duration world in which the probability that the world will exist tomorrow, given existence today, equals the discount factor. This simple solution has drawbacks, however. It requires that we have a particular belief about the nature of the process selecting the end of the world—there must be a positive probability that the world will last beyond any point in finite time.[6] Further, it leaves intact the peculiar result that use of money is irrational today if the world will certainly end in several centuries.

The second comment is that loosening some standard assumptions clearly would give rise to other solutions. For instance, Darrell Duffie (1986) presents a static model in which "money" has value. It seems, however, that a totally satisfactory model of money must admit some dynamic notion of monetary exchange (money is accepted for goods in hopes of trading it for other goods later). There appears to be no dynamic account lurking behind this static model,[7] and so this solution seems to rest on a less sensible notion of monetary exchange.

Finally, it is interesting to note the parallels between the issues here and the literature on equilibria of finitely repeated games. Because of a regress starting with lack of cooperation in the final period, such games often have no cooperative solutions. Cooperation has been shown to emerge, however, in continuous time (Simon and Stinchcombe, 1987) and with discounting (Drew Fudenberg and Eric Maskin, 1986). Cooperation can also emerge if one injects even a tiny amount of irrationality (David Kreps and Robert Wilson, 1982). This latter approach could certainly be used in the monetary case, so long as one is willing to suppose that money's existence rests on irrationality.

In the game theory literature, there is only one other relevant solution to the terminal problem of which I am aware. If there are multiple equilibria of the one-shot game, cooperation can help agents reach the best equilibrium in the last period (Jean-Pierre Benoit and Vijay Krishna, 1985). It is difficult to imagine what second equilibrium could exist in the final period of the monetary model besides one in which the value of money is zero. Thus, if results from this well-studied problem in the game theory literature provide a clue, we may have exhausted the solutions to the terminal problem that maintain most conventional assumptions.

[6] Hahn (1982), for example, contends that this is not satisfactory because there is an absolute outer bound on the life span of the universe.

[7] In this one-period model, buyers and sellers face different prices, and in equilibrium the total price paid for goods exceeds the price received for goods by an amount equal to the money stock. Without relying on some *ad hoc* mechansism such as taxation to account for this difference in prices, a dynamic exchange story consistent with this model would seem to involve agents simply voluntarily leaving money on the ground at the site of each transaction.

APPENDIX

AI. *Existence of Nontrivial Solutions*

The goal here is to show that, when there is a solution to (13) in which $\phi(t)=0$ for all t, there is also a solution in which $\phi(1)=0$ but $\phi(t)$ is positive and finite until time one. It has already been proven that there is a solution to (13) through any point in the unit square that can be continued forward and backward in time to borders of the square. This leaves open the possibility that all of the solutions for which ϕ is positive at some point might never cross $\phi(t)=0$. The strategy here is to rule out this possibility by finding a differential equation, $v'(t)$, for which the solution is known and for which 1) $v'(t)<0$, 2) $v(1)=0$, and 3) $v(t)$ is flatter than $\phi(t)$ at any point of intersection of the two functions occurring where $\phi(t)<q$ for some small $q>0$. Existence of this function, v, is important, because if v is flatter than $\phi(t)$ when $\phi(t)<q, \phi$ cannot cut through v from below in this region. Our desired solution for (13) can be found by noting that the solution through any point (t_1, x) with $x=v(t_1)<q$, must be bounded above by $v(t)$ for any $t>t_1$ and therefore must attain $\phi(t_0)=0$ for some $t_0 \leq 1$. Because (13) is autonomous, we are free to re-label time so that the first t for which $\phi(t)=0$ is labeled time one. Thus, finding an appropriate $v'(t)$ would complete the proof.

One function satisfying the requirements for $v'(t)$ is,

$$v'(t) = -Av(t)^{1-b}/b \quad A>0 \quad 0<b<1,$$

which has the solution,

$$v(t) = (A(1-t))^{1/b}$$

As required, $v'(t)<0, v(1)=0$. It must be shown that we can choose A, q, and b such that,

$$v'(t) - \phi'(t) \geq 0 \quad \text{whenever } v(t) = \phi(t) \in (0, q].$$

Substituting for the definitions of $v'(t)$ and $\phi'(t)$ using $x \equiv v(t) = \phi(t)$ gives,

$$\frac{x\left(u'(x) - Ax^{-b}/b\left[u'(1-x) - xu''(1-x)\right]\right)}{u'(1-x) - xu''(1-x)} \geq 0$$

for all $x \in (0, q]$. The denominator and the leading x term may be disregarded, as they are nonnegative. It is easy to verify that for $q < 1/2$ the term in square brackets in the numerator is bounded above by $c(q) \equiv 2u'(1-q)$. As the second term in the numerator is negative, if we replace the term in brackets by $c(q)$ and find A, q, and b such that the resulting inequality is satisfied, then the same parameters must satisfy the original inequality. These changes give

$$u'(x) - c(q)Ax^{-b}/b \geq 0 \quad x \in (0, q].$$

Of course, the second term is simply the marginal utility from the constant relative risk aversion utility function $w(x) \equiv [c(q)A/(b(1-b))]x^{1-b}$, for which b is the coefficient of relative risk aversion. We have assumed in (A3b) that the relative risk aversion of $u(\cdot)$ is bounded below by some constant B. This provides sufficient information to choose our parameters.

Set $b^* = B$; select an arbitrary $q^* < 1/2$; and A^* so that the inequality above is satisfied with equality at consumption q^*. By assumption,

$$-xu''(x)/u'(x) > b^* = -xw''(x)/w'(x)$$

for all feasible x, implying that at any level of consumption for which the marginal utilities for the two functions are equal (such as consumption q^*), $w''(x) > u''(x)$. As w'' is negative, this proves the required result: at any point of intersection of the two marginal utility functions, $u'(x)$ must be rising faster than $w'(x)$ as consumption falls, and $u'(x)$ must therefore exceed $w'(x)$ at any consumption level below q^*. This establishes that $v(\cdot)$ has the desired properties, and the proof is complete. □

AII. *An Equilibrium of Arbitrary Duration*

A $K+1$ period equilibrium is built up from the final period equilibrium in Proposition 2. Call the pay period beginning at $t = -j$, for $j = 0, \ldots, K$, the jth period, and define ϕ_j as a solution for ϕ in the jth pay period, given a solution for ϕ_{j-1}. This implies that ϕ_j is a function satisfying (9) for all times between $-j$ and $1-j$. Substituting for ϕ in (9) yields, for $j = 1, \ldots, K$, and for all $t \in [0,1]$,

$$\text{(A1)} \quad \int_r^{r+t} \phi_j(s)u'(\phi_j(s))\,ds + \int_{r+t}^{r+1} \phi_{j-1}(s)u'(\phi_{j-1}(s))\,ds = \phi_j(r)u'(1-\phi_j(r)),$$

where $r \equiv 1-(j+t)$. Differentiating with respect to t gives,

$$\text{(A2)} \quad f_j(t, \phi_j(r)) \equiv \phi'_j(r) = \frac{\phi_{j-1}(1+r)u'(\phi_{j-1}(1+r)) - \phi_j(r)u'(\phi_j(r))}{u'(1-\phi_j(r)) - \phi_j(r)u''(1-\phi_j(r))}.$$

For completeness, call f in (13) f_0. The denominator of f_j is the same as for f_0 and therefore is positive with a lower bound of $u'(1)$. The numerator is the change in μ between the beginning and end of the pay period beginning at r. Since μ is bounded below by zero and above by $u'(1)$, we know $f_j(t, x) \in [-1, 1]$ for t and x in their domain.

It is possible, however, to rule out positive values for f_j. If $\phi(t)$ falls for an entire pay period beginning at time t then, from (14), the change in μ over that pay period must be negative and, from (16), using the continuity of $f_j, \phi(t)$ must be falling in some neighborhood of time preceding t. Because $\phi(t)$ falls throughout the final pay period, the obvious application and continuation of this argument establishes that $f_j(\cdot)$ must be negative for all $j = 1, 2, \ldots, K$ and for all t in the unit interval.

Establishing that $\phi(t)$ is strictly declining allows one to establish an upper bound of $1/2$ for $\phi(t)$, implying that our solutions can be continued to the left (back in time) without worrying about $\phi(t)$ going to the top of its range. Simply notice that the term to be integrated in equilibrium condition (9) is $\mu(s)$. If $\phi(t) = 1/2$, then $\mu(s)$ equals the right-hand side of (9) at $(s = t)$ and must decline with ϕ thereafter. Since the integration is over a span of one, (9) could not be satisfied.

Overall, each f_j, premised on the solution for $\phi_i, i = 1, \ldots, j-1$, has the same properties as those shown for f_0, which means that the conditions for a continuous solution to (16) in which $\phi(t)$ falls monotonically are satisfied. □

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