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RESEARCH ARTICLE

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Existence of equilibria in matching models of money: a new technique

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Abstract This paper develops a new technique for proving the existence and indeterminacy of monetary equilibria in money search models with divisible money. Our technique is substantially simpler than standard constructive proofs in the literature.

Keywords Real indeterminacy · Matching model · Money · Existence of monetary equilibria

JEL Classification Numbers C78 · D51 · D83 · E40

1 Introduction

We develop a new technique to prove the existence of stationary equilibria with heterogeneous money holdings in random matching models with divisible money such as Camera and Corbae (1999), Green and Zhou (1998), and Zhou (1999). In the literature, the existence of equilibria has been proved using a constructive proof;

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namely, a candidate for equilibrium strategy is first presented, then it is verified that the candidate is indeed an equilibrium strategy by explicitly obtaining the value function.[1] Our technique is substantially simpler than the standard constructive proof.

Our technique relies on considering a limit point of money holdings distribution in which no one has money. This is not an equilibrium, since the amount of money held is zero while the money supply is positive. Then examining a point close to this limit point allows us to consider the existence of some price consistent with equilibrium trading behavior. That is, at this point, only a few agents have large amounts of money and the total amount is equal to the given money supply. Using the implicit function theorem, we demonstrate the existence of such a point in a neighborhood of the limit point.

More precisely, our technique proceeds by the following steps. First, we conjecture a trading pattern. Second, given this trading pattern, we determine laws of motion for money holdings that pin down conditions for the stationarity of the distribution of money. Third, we consider a distribution in which no one has money. This allows us to easily obtain discounted expected payoffs, since there is no buyer in the economy. Fourth, using the implicit function theorem, we obtain discounted expected payoffs corresponding to money holdings distributions in a neighborhood of this distribution. Finally, we check individual optimality conditions and demonstrate that a price exists that satisfies them.

The plan of this paper is as follows. In Section 2, we present a simple model to explain our technique. In Section 3, we present a general technique and apply it to a new model. In Section 4, we conclude the paper.

2 A simple example

We first investigate a simple example to develop intuition of our technique. This is a variant of Zhou (1999), which is based on Green and Zhou (1998). This model has only a two-point money holdings distribution, with agents either holding p money or none at all.

There is a continuum of agents with a mass of measure one. There are $k \geq 3$ types of agents with equal fractions, and the same number of types of goods. A type $i-1$ agent can produce just one unit of type i good and the production cost is $c > 0$. We assume that a type k agent produces type 1 good. A type i agent obtains utility $u > c$ only when she consumes one unit of type i good. There is perfectly divisible and durable fiat money of which constant nominal stock is $M > 0$. Time is continuous and pairwise random matchings take place according to Poisson process with parameter $\mu > 0$. For every matched pair, a seller posts a take-it-or-leave-it price offer without knowing the amount of a buyer's money holdings. Let $\gamma > 0$ be the discount rate.

The conditions for a stationary equilibrium are: (1) strategies are Markovian and symmetric with respect to types, (2) each agent maximizes the expected discounted

[1] It should be also emphasized that Green and Zhou (1998) and Zhou (1999) only prove the existence of equilibria with step value functions; i.e., an equilibrium with a continuous value function may exist but is not investigated. This is the same approach we take in this paper. Berentsen et al. (2004) presents a similar way by using lotteries.

payoff of lifetime utilities, (3) the money holdings distribution of the economy is stationary, i.e., time-invariant, and (4) the total amount of money the agents have is equal to M.

In what follows, we focus on a stationary distribution of money holdings of the agents with a support $\{0, p\}$ for some $p > 0$.[2] For simplicity, we assume that money holdings of the agents are in $[0, 2p)$.[3] Thus a money holdings distribution can be expressed by $h = (h_0, h_1)$, where h_n is a measure of the set of agents with np money. Of course, h should satisfy

$$h_0 + h_1 = 1, \tag{1}$$

$$h_n \geq 0, \quad n = 0, 1. \tag{2}$$

We focus on an equilibrium with the following candidate trading strategy:

- A seller with money holding $\eta \in [0, p)$ offers p.
- A seller with money holding $\eta \in [p, 2p)$ chooses no trade.
- A buyer with money holding η accepts offer prices less than or equal to η.

According to the strategy specified above, a type i agent without money makes a sale when she meets a type $i+1$ agent with money. The measure of agents without money is h_0 and the probability that they can make a sale is $(\mu/k)h_1$, and thus a set of agents with measure $(\mu/k)h_0h_1$ moves out from 0, i.e., it is an outflow at 0 as well as an inflow at p. On the other hand, a type i agent with p money makes a purchase when she meets a type $i - 1$ agent without money. The probability that they can make a purchase is $(\mu/k)h_0$, and thus a set of agents with measure $(\mu/k)h_1h_0$ moves out from p, i.e., it is an outflow at p as well as an inflow at 0. The stationary condition for $h = (h_0, h_1)$ requires that the time rate of inflow should be equal to the time rate of outflow at $n = 0$ and $n = 1$. Both conditions are the same and expressed as

$$\frac{\mu}{k}h_0h_1 = \frac{\mu}{k}h_0h_1.$$

This is clearly an identity, and therefore any h satisfying (1) and (2) can be a stationary distribution. On the other hand, p is determined by the money market clearing condition

$$M = ph_1. \tag{3}$$

Let the expected discounted payoff function be denoted by $V : \mathbb{R}_+ \to \mathbb{R}$, and define $V_n = V(np)$. Then it satisfies

$$G_0 = V_0 - \frac{1}{\phi + 2}\left[h_1\left(V_1 - c\right) + h_0V_0 + V_0\right] = 0, \tag{4}$$

$$G_1 = V_1 - \frac{1}{\phi + 2}\left[V_1 + h_0\left(u + V_0\right) + h_1V_1\right] = 0, \tag{5}$$

where $\phi = k\gamma/\mu$.

[2] Although we look for an equilibrium in which the support of money is $\{0, p\}$, there may exist other types of equilibria.

[3] Without this assumption, Zhou (1999) shows that there is a region of parameters in which the existence of equilibria is assured. More precisely, in equilibrium, although agents can hold any amount of money, it is optimal for an agent with p money not to sell a good. Thus $\{0, p\}$ is the support of the stationary distribution.

The individual optimality conditions to play the candidate trading strategy in (4) and (5) are as follows:

$$-c + V_1 \geq V_0, \tag{6}$$

$$u + V_0 \geq V_1. \tag{7}$$

The first inequality is the condition that an agent without money will choose to sell her good. The second inequality is the condition that an agent with p money will choose to accept an offer price p. Note that defining $V(\eta) = V_{\lfloor \eta/p \rfloor}$, the optimality conditions at the other η follow from the above conditions, where $\lfloor \eta/p \rfloor$ is the largest integer less than or equal to η/p [See Zhou (1999)]. We should also check that agents will choose not to offer a non-integer multiple of p. This is clearly satisfied, since we defined $V(\eta) = V_{\lfloor \eta/p \rfloor}$ [See also Zhou (1999)]. Therefore a stationary monetary equilibrium, in which all agents choose the above strategy, is defined as (h_0, h_1, V_0, V_1, p) satisfying (1)–(7).

Now we are ready to proceed to our technique. The first step is to impose $(h_0, h_1) = (1, 0)$, and verify that conditions (1), (2), (4), (5), (6), and (7) hold. Note that (3) remains out, i.e., it is not an equilibrium, since the money supply does not equal to the money held in the economy. Equations (1) and (2) are clearly satisfied. Equations (4) and (5) are written as follows:

$$V_0 - \frac{1}{\phi + 2}\left[V_0 + V_0\right] = 0,$$

$$V_1 - \frac{1}{\phi + 2}\left[V_1 + (u + V_0)\right] = 0.$$

Since this system of equation is much simpler than (4) and (5), the solution is easily obtained as $V_0 = 0$, $V_1 = (1/(\phi + 1))u$. Clearly, (7) is satisfied with strict inequality for any u and ϕ. The necessary and sufficient condition for (6) is

$$\phi + 1 \leq \frac{u}{c}. \tag{8}$$

In what follows, we assume that (8) holds with strict inequality.

Clearly, this solution does not satisfy (3) for any $p > 0$. Then the last step of our technique is to slightly extend the point so that (3) is satisfied. To be more precise, we find $(h_0, h_1) = (1 - \epsilon, \epsilon)$ and corresponding (V_0, V_1) satisfying all conditions for stationary equilibrium for a sufficiently small $\epsilon > 0$. Clearly, (3) is satisfied for $p = (M/\epsilon) > 0$. To find such a point, we can simply apply the implicit function theorem. More precisely, the regularity of the system of Equations (4) and (5) at $h_0 = 1$ is satisfied as follows:

$$\det \begin{pmatrix} \dfrac{\partial G_0}{\partial V_0} & \dfrac{\partial G_0}{\partial V_1} \\ \dfrac{\partial G_1}{\partial V_0} & \dfrac{\partial G_1}{\partial V_1} \end{pmatrix}_{h_0=1} = \det \begin{pmatrix} 1 - \dfrac{1}{\phi+2}(1 + h_0) & -\dfrac{1}{\phi+2}(1 - h_0) \\ -\dfrac{1}{\phi+2} h_0 & 1 - \dfrac{1}{\phi+2}(2 - h_0) \end{pmatrix}_{h_0=1}$$

$$= \det \begin{pmatrix} \dfrac{\phi}{\phi+2} & 0 \\ -\dfrac{1}{\phi+2} & \dfrac{\phi+1}{\phi+2} \end{pmatrix} > 0.$$

Then, by the implicit function theorem, (V_0, V_1) satisfying (4) and (5) can be written as C^1 functions of ϵ, $(V_0(\epsilon), V_1(\epsilon))$. It remains to show that this solution satisfies the individual optimality conditions (6) and (7). Because (8) is satisfied with strict inequality, these conditions are still satisfied for a sufficiently small $\epsilon > 0$. This implies that

$$(h_0, h_1, V_0, V_1, p) = \left(1 - \epsilon, \epsilon, V_0(\epsilon), V_1(\epsilon), \frac{M}{\epsilon}\right)$$

is a stationary monetary equilibrium for a sufficiently small $\epsilon > 0$.

Note that simple systems of inequalities as considered above can be solved directly, but in the cases where the models are complex, the system may not be solved directly. However, our technique is also applicable to such models as we show in Section 3.

3 A general model

In this section, we extend our technique to more general environment and apply it to a rather complicated model, which can be considered as a modified version of Camera and Corbae (1999)'s model. Note that the general model is a simplified version of Kamiya and Shimizu (2006)'s model.

3.1 A general environment

Throughout this section, we adopt mostly the same environment as in the previous section. The only differences are that in this section we do not specify the bargaining procedure and that we make no assumption on the divisibility of goods. Furthermore, we allow for general discrete money holdings distributions. We confine our attention to the case that, for some $p > 0$, all trades occur with integer multiple amounts of money p. In what follows, we focus on a stationary distribution of money holdings on $\{0, p, \ldots, Np\}$ expressed by $h = (h_0, \ldots, h_N)$, where h_n is a measure of agents with np money, and $N < \infty$ is the upper bound of the distribution. More precisely, we seek for an equilibrium in which the finite support $\{0, p, \ldots, Np\}$ holds true in equilibrium. Our model includes the case of exogenously determined N as well as the case of endogenously determined N.

Of course, $h_n \geq 0$ and $\sum_{n=0}^{N} h_n = 1$ must hold. We assume that fiat money is perfectly divisible.[4] Let $M > 0$ be a given nominal stock of money. Since p is uniquely determined by $\sum_{n=0}^{N} pnh_n = M$ for a given h with $h_0 \neq 1$, then, deleting p from $\{0, p, \ldots, Np\}$, the set $\{0, \ldots, N\}$ can be considered as the state space.

3.2 Example: a model with divisible goods

In this subsection, we introduce a modified version of Camera and Corbae (1999)'s model of which environment is a special case of the general model in Subsection 3.1. We make the following assumptions.

[4] For the case of indivisible money, see footnote 8.

(a) Money holdings of the agents are in $[0, (N+1)p)$ for some positive integer N, where N is exogenously given.
(b) Goods are perfectly divisible. Let $C(q) = q$ be the cost function and $U(q) = q^{\frac{1}{2}}$ be the utility function.
(c) Agents can observe the type and the current money holdings of a matched agent at the beginning of the bargaining. In a buyer–seller match, the seller and the buyer are equally likely to make an offer. If the offer is rejected, both obtain zero gain.

The bargaining procedure is a unilateral offer scheme with a random proposer. Note that it mimics the take-it-or-leave-it bargaining scheme adopted by Camera and Corbae (1999), although a seller does not necessarily receive zero gain.

The conditions for a stationary equilibrium are similar to that in the previous section. For the definition, see the next subsection.

We search for a monetary equilibrium in which there exists a price $p > 0$ supporting the following candidate trading strategy:

- A seller with money holding $\eta < Np$ always offers (p, q_s) for some $q_s > 0$. The offer is accepted by any buyer with money holdings more than or equal to p.
- A seller with money holding $\eta \geq Np$ offers no trade.
- A buyer with money holding more than or equal to p always offers (p, q_b) for some $q_b > 0$. The offer is accepted by any seller with money holdings less than Np.

Although the off-equilibrium strategy is not completely specified in the above, it will be determined by the value function. Note that the above is sufficient for finding the equilibrium value function.

Given this trading strategy, the stationary condition for $h = (h_0, h_1, \ldots, h_N)$ is

$$\sum_{n=0}^{N} h_n - 1 = 0, \tag{9}$$

$$\frac{\mu}{k}\,[h_1(1-h_N) - h_0(1-h_0)] = 0, \tag{10}$$

$$\frac{\mu}{k}\left[\{h_{n-1}(1-h_0) + h_{n+1}(1-h_N)\} - h_n\{(1-h_0) + (1-h_N)\}\right] = 0, \quad 1 \leq n \leq N-1, \tag{11}$$

$$\frac{\mu}{k}\left[h_{N-1}(1-h_0) - h_N(1-h_N)\right] = 0. \tag{12}$$

Note that (10)–(12) correspond to the stationarity at $n = 0$, $n = 1, \ldots, N-1$, and $n = N$, respectively. As in the previous section, it is easily verified that two equations among them are redundant. Thus, in what follows we focus on (9) and (11). Note also that the above stationary condition is exactly the same as that of Camera and Corbae (1999). Let

$$F_0 = \sum_{n=0}^{N} h_n - 1 = 0,$$
$$F_n = \{h_{n-1}(1-h_0) + h_{n+1}(1-h_N)\} - h_n \{(1-h_0) + (1-h_N)\} = 0,$$
$$1 \le n \le N-1.$$

Then we obtain the following stationary distribution from the stationary condition:

$$h_n = h_0 \left(\frac{1-h_0}{1-h_N} \right)^n, \quad n = 1, \ldots, N, \tag{13}$$

where h_N is determined so that

$$h_N(1-h_N)^N = h_0(1-h_0)^N. \tag{14}$$

It is verified that, for any $h_0 \in [0,1]$, there uniquely exist $h_1, \ldots, h_N \in [0,1]$ satisfying (9), (13), and (14).[5] In other words, for any $h_0 \in [0,1]$, there is a corresponding distribution h satisfying the stationary condition.

In the subgame in which agents are matched and a seller is chosen as a proposer, subgame perfection requires that the seller's offer to the buyer with np money, (p, q_s^n), satisfies

$$U(q_s^n) = V_n - V_{n-1}. \tag{15}$$

Similarly, the buyer's offer to a seller with np money, (p, q_b^n), satisfies

$$C(q_b^n) = V_{n+1} - V_n. \tag{16}$$

Then $(V_0, V_1, \ldots, V_N)$ should satisfy the following equations:

$$V_0 = \frac{1}{\phi+2} \left\{ \frac{1}{2} \sum_{i=1}^{N} h_i (V_1 - C(q_s^i)) + \frac{1}{2} h_0 V_0 + \frac{1}{2} V_0 + V_0 \right\}, \tag{17}$$

$$V_n = \frac{1}{\phi+2} \left\{ \frac{1}{2} \sum_{i=1}^{N} h_i (V_{n+1} - C(q_s^i)) + \frac{1}{2} h_0 V_n + \frac{1}{2} V_n + \frac{1}{2} \sum_{i=0}^{N-1} h_i (V_{n-1} + U(q_b^i)) + \frac{1}{2} h_N V_n + \frac{1}{2} V_n \right\}, \quad n = 1, \ldots, N-1, \tag{18}$$

[5] More precisely, there are four cases: (1) $h_0 = 0$, (2) $h_0 = 1$, (3) $h_0 \in (0,1) \setminus \{1/(N+1)\}$, and (4) $h_0 = 1/(N+1)$. Let $\xi(h_N) \equiv h_N(1-h_N)^N - h_0(1-h_0)^N$. ξ is strictly increasing for $h_N \in (0, 1/(N+1))$ and strictly decreasing for $h_N \in (0, 1/(N+1))$, and has the maximum value $N^N/(N+1)^{N+1} - h_0(1-h_0)^N$ at $h_N = 1/(N+1)$. In cases (1)–(3), the maximum value is positive, and hence $\xi(h_N) = 0$ has two solutions. One solution is clearly $h_N = h_0$. In this case, by (13), $h_n = h_0, n = 1, \ldots, N-1$ holds and h does not satisfy (9). It is verified that the other solution satisfies all conditions. In case of (4), the maximum value is equal to zero, and hence $\xi(h_N) = 0$ has the unique solution $h_N = 1/(N+1)$, and thus we obtain $h_n = 1/(N+1), n = 1, \ldots, N-1$. They clearly satisfy all conditions.

$$V_N = \frac{1}{\phi+2}\left\{V_N + \frac{1}{2}\sum_{i=0}^{N-1} h_i (V_{N-1} + U(q_b^i)) + \frac{1}{2}h_N V_N + \frac{1}{2}V_N\right\}. \quad (19)$$

3.3 The definition of stationary equilibrium

In this subsection, we formally define the concept of (symmetric) stationary equilibria of the general model in Subsection 3.1. When a type i agent meets a type $i+1$ agent, the former agent becomes a seller and the latter becomes a buyer. When a seller's money holding is np, she chooses an action in her finite action space A_n^S. For example, an action is an offer price. Similarly, when a buyer's money holding is np, he chooses an action in his finite action space A_n^B. Let $A_n = A_n^S \times A_n^B$ and $A = \Pi_{n=0}^N A_n$. Since A_n is a finite set, it can be written as $A_n = \{a_{n1}, \ldots, a_{ns(n)}\}$, where $s(n)$ is the number of elements in A_n, i.e., the product of the number of elements in A_n^S and that in A_n^B. We confine our attention to stationary equilibria in which all agents choose a pure strategy. As for mixed strategy equilibria, see Kamiya and Shimizu (2006). Given a (pure) strategy $a = (a_0, \ldots, a_N)$ of an agent, where $a_n \in A_n$ is the action taken at np, define $\alpha(a) = \{(n, j) \mid a_n = a_{nj}\}$. In other words, $(n, j) \in \alpha(a)$ means that an agent with np money chooses an action pair $a_{nj} = (a_{nj}^S, a_{nj}^B)$. In what follows, we focus on symmetric equilibria in which all gents choose a common strategy a.

The monetary transition resulting from transaction among a matched pair is described by a function f. When a seller with np money and action a_{nj}^S meets a buyer with $n'p$ money and $a_{n'j'}^B$, their states, i.e., money holdings, become $(n + f(n, j; n', j'))p$ and $(n' - f(n, j; n', j'))p$, respectively. That is f maps an ordered pair $(n, j; n', j')$ to a non-negative integer $f(n, j; n', j')$. When N is exogenously determined, we assume $N \geq n + f(n, j; n', j')$ and $n' - f(n, j; n', j') \geq 0$. When N is endogenously determined, we assume the latter condition while the former one should be satisfied on the equilibrium path. Let $\theta \in \mathbb{R}^L$ be the parameters of the model.

Let V_n be the expected discounted payoff of lifetime utilities at state n, $n = 0, \ldots, N$, and $V = (V_0, V_1, \ldots, V_n)$. The variables in the model are denoted by $x = (h, V, a)$. Let $W_{nj}(x; \theta)$ be the the expected discounted payoff of lifetime utilities of one-shot deviation by action j at state n. Thus, in equilibrium, $W_{nj}(x; \theta) = V_n$ holds for $(n, j) \in \alpha(a)$. Note that $W_{nj}(x; \theta)$ includes the utility and/or the production cost of perishable goods.

We define $h_{nj} = h_n$ if $a_{nj} = a_n$ and $h_{nj} = 0$ if $a_{nj} \neq a_n$. Then by the random matching assumption and the definition of f, the inflow I_n into state n and the outflow O_n from state n are defined as follows:

$$I_n(h, a; \theta) = \frac{\mu}{k}\left[\sum_{(i,j,i',j')\in X_n} h_{ij}h_{i'j'} + \sum_{(i,j,i',j')\in X_n'} h_{ij}h_{i'j'}\right],$$

$$O_n(h, a; \theta) = \frac{\mu}{k}\left[\sum_{(j,i',j')\in Y_n} h_{nj}h_{i'j'} + \sum_{(j,i',j')\in Y_n'} h_{nj}h_{i'j'}\right],$$

where

$$X_n = \{(i, j, i', j') \mid f(i, j; i', j') > 0,\ i + f(i, j; i', j') = n\},$$
$$X'_n = \{(i, j, i', j') \mid f(i, j, i', j') > 0,\ i' - f(i, j; i', j') = n\},$$
$$Y_n = \{(j, i', j') \mid f(n, j; i', j') > 0\},$$
$$Y'_n = \{(j, i', j') \mid f(i', j'; n, j) > 0\}.$$

We denote $I_n - O_n$ by D_n. Then the condition for stationarity is $D_n = 0$ for $n = 0, \ldots, N$ and $\sum_{n=0}^{N} h_n = 1$. Clearly, $\sum_{n=0}^{N} D_n = 0$ holds as an identity, and thus at least one equation is redundant. The following theorem shows that one more equation is always redundant.

Theorem 1[6] *For any a,*

$$\sum_{n=0}^{N} n O_n(h, a; \theta) = \sum_{n=0}^{N} n I_n(h, a; \theta), \tag{20}$$

is an identity, i.e., $\sum_{n=0}^{N} n D_n(h, a; \theta)$ *is identically zero.*

Proof Consider a pair of pairs (n, j) and (n', j'). Given the matchings between them, the proportion $(\mu/k)h_{nj}h_{n'j'}$ of agents move from n to $n + f((n, j), (n', j'))$, and the same proportion of agents move from n' to $n' - f((n, j), (n', j'))$. Corresponding to the moves, the following terms appear in the RHS and in the LHS of (20):

the LHS	the RHS
$n\frac{\mu}{k}h_{nj}h_{n'j'}$	$(n + f((n, j), (n', j')))\frac{\mu}{k}h_{nj}h_{n'j'}$
$n'\frac{\mu}{k}h_{nj}h_{n'j'}$	$(n' - f((n, j), (n', j')))\frac{\mu}{k}h_{nj}h_{n'j'}$

Clearly, the sum of the terms in the LHS is equal to that in the RHS. Since this holds for any pair of pairs (n, j) and (n', j'), (20) holds. □

Together with the identity $\sum_{n=0}^{N} D_n(h, a; \theta) = 0$, the above theorem implies that h is a stationary distribution if and only if $D_n(h, a; \theta) = 0, n = 1, \ldots, N-1$, and $\sum_{n=0}^{N} h_n = 1$ hold. Thus the equilibrium condition is expressed as follows:

Definition 1 *Given* θ, $x = (h, V, a) \in \mathbb{R}_+^{N+1} \times \mathbb{R}_+^{N+1} \times A$ *is a* (*pure strategy*) *stationary equilibrium if it satisfies the following*:

$$h_0 \neq 1, \tag{21}$$

$$\sum_{n=0}^{N} h_n - 1 = 0, \tag{22}$$

$$D_n(h, a; \theta) = 0, \quad n = 1, \ldots, N-1 \tag{23}$$

$$V_n - W_{nj}(x; \theta) = 0, \quad (n, j) \in \alpha(a) \tag{24}$$

$$V_n - W_{nj}(x; \theta) \geq 0, \quad (n, j) \notin \alpha(a). \tag{25}$$

[6] Kamiya and Shimizu (2002) first presents this theorem. See also Kamiya and Shimizu (2006), a revised version of the first half of the paper.

(h, V) is called a stationary equilibrium for a and θ if (h, V, a) is a stationary equilibrium for θ. A stationary equilibrium is called a monetary equilibrium if $V_n - V_0 > 0$ for some $n > 0$.

Condition (21) is required for the existence of $p > 0$ satisfying $\sum_{n=0}^{N} pnh_n = M$. Equations (22) and (23) are the stationary conditions. Note that, because of Theorem 1, the stationary conditions at $n = 0$ and N are dropped. Equation (24) is the condition that the equilibrium strategy indeed realizes the value. Condition (25) is the relevant optimality condition. We define

$$
\begin{aligned}
F_0 &= \sum_{n=0}^{N} h_n - 1, \\
F_n &= D_n(h, a; \theta), \quad n = 1, \ldots, N-1, \\
G_n &= V_n - W_{nj}(x; \theta), \quad (n, j) \in \alpha(a).
\end{aligned}
$$

In addition to the above equilibrium conditions, the following conditions are typically required to be an "equilibrium" in most of matching models with money: (1) agents will not choose an action out of our action space,[7] and (2) agents will choose the equilibrium strategy at state $\eta \notin \{0, p, \ldots, Np\}$. However, they are not very restrictive. For a sufficient condition for (1) and (2), see Kamiya and Shimizu (2006). Moreover, it is satisfied in all of the matching models with divisible money known so far, such as Zhou (1999)'s model, a divisible money version of Camera and Corbae (1999)'s model, and a divisible money version of Trejos and Wright (1995)'s model. As for the case of the model in Subsection 3.2, see Subsection 3.5.

It is worthwhile noting that Kamiya and Shimizu (2006) shows there is a continuum of stationary equilibria in a broad class of search models with divisible money. More precisely, under a regularity condition, the stationarity condition of money holdings distribution always leads to indeterminacy.

3.4 The technique

Given a and θ, let $\Psi : \mathbb{R}^{N+1} \times \mathbb{R}^{N+1} \to \mathbb{R}^N \times \mathbb{R}^{N+1}$ be defined as

$$
\begin{aligned}
&\Psi\left((h_0, \ldots, h_N), (V_0, \ldots, V_N)\right) \\
&\quad = (F_0, \ldots, F_{N-1}, G_0, \ldots, G_N)(h_0, \ldots, h_N, V_0, \ldots, V_N, a; \theta).
\end{aligned}
$$

Denote by $\det D\Psi(h, V)$ the determinant of the following $(2N+1) \times (2N+1)$ matrix:

$$
\begin{pmatrix}
\frac{\partial F_0}{\partial h_1}(h, V) & \ldots & \frac{\partial F_0}{\partial h_N}(h, V) & \frac{\partial F_0}{\partial V_0}(h, V) & \ldots & \frac{\partial F_0}{\partial V_N}(h, V) \\
\vdots & & \vdots & \vdots & & \vdots \\
\frac{\partial F_{N-1}}{\partial h_1}(h, V) & \ldots & \frac{\partial F_{N-1}}{\partial h_N}(h, V) & \frac{\partial F_{N-1}}{\partial V_0}(h, V) & \ldots & \frac{\partial F_{N-1}}{\partial V_N}(h, V) \\
\frac{\partial G_0}{\partial h_1}(h, V) & \ldots & \frac{\partial G_0}{\partial h_N}(h, V) & \frac{\partial G_0}{\partial V_0}(h, V) & \ldots & \frac{\partial G_0}{\partial V_N}(h, V) \\
\vdots & & \vdots & \vdots & & \vdots \\
\frac{\partial G_N}{\partial h_1}(h, V) & \ldots & \frac{\partial G_N}{\partial h_N}(h, V) & \frac{\partial G_N}{\partial V_0}(h, V) & \ldots & \frac{\partial G_N}{\partial V_N}(h, V)
\end{pmatrix}.
$$

[7] For example in Section 3, a seller may offer a price which is not an integer multiple of p.

Then, our technique proceeds by the following steps:

(1) Find a candidate strategy a for a stationary monetary equilibrium.
(2) Setting $(h_0, h_1, \ldots, h_N) = (1, 0, \ldots, 0)$, obtain V satisfying (24). Denote $(h, V) = ((1, 0, \ldots, 0), V)$ by y^*.
(3) Verify that det $D\Psi(y^*)$ is nonzero. Then, by the implicit function theorem, there are C^1 functions $(h_1(\epsilon), \ldots, h_N(\epsilon), V_0(\epsilon), \ldots, V_N(\epsilon))$ which, together with $h_0 = 1 - \epsilon$, a, and θ, satisfies (21)–(24) for a sufficiently small $\epsilon > 0$. Denote $y^*(\epsilon) = (1 - \epsilon, h_1(\epsilon), \ldots, h_N(\epsilon), V_0(\epsilon), \ldots, V_N(\epsilon))$.
(4) Verify that $h_n(\epsilon) \geq 0$ for $n = 1, 2, \ldots, N$ for a sufficiently small $\epsilon > 0$.
(5) Verify the optimality condition (25) for a sufficiently small $\epsilon > 0$.

The advantage of our technique is that it is applicable to various models, since the Bellman equations are typically simple at $h_0 = 1$.[8]

3.5 The example with $N = 2$

In this subsection, we apply the technique in the previous subsection to the example in Subsection 3.2 with $N = 2$. We begin with Step (2), since Step (1) has been done in Subsection 3.2. We investigate a solution at $h_0 = 1$. Using (16), (V_0, V_1, V_2) must satisfy the following equations:

$$V_0 = \frac{1}{\phi + 2}\left\{\frac{1}{2}V_0 + \frac{1}{2}V_0 + V_0\right\},$$
$$V_1 = \frac{1}{\phi + 2}\left\{\frac{1}{2}V_1 + \frac{1}{2}V_1 + \frac{1}{2}[V_0 + U(V_1 - V_0)] + \frac{1}{2}V_1\right\},$$
$$V_2 = \frac{1}{\phi + 2}\left\{V_2 + \frac{1}{2}[V_1 + U(V_1 - V_0)] + \frac{1}{2}V_2\right\}.$$

We obtain $q_b^0 = A^2$, $V_0 = 0$, $V_1 = A^2$, and $V_2 = A^2(1 + A)$, where $A = (1/(2\phi + 1))$.[9] Below, we extend the solution to the case of $h_0 < 1$. Note that the nice feature is that, by using the implicit function theorem, there is no need to solve explicitly for the equation in a neighborhood of the degenerate distribution. This stands in contrast with the previous approach to proving the existence for a specific trading strategy.

We proceed to Step (3). Let G_0 be the difference between V_0 and the RHS of (17). G_1 and G_2 are similarly defined. Let $y = (h_0, h_1, h_2, V_0, V_1, V_2)$ and

$$y^* = \left(1, 0, 0, 0, A^2, A^2(1 + A)\right).$$

Let $\Psi : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^2 \times \mathbb{R}^3$ be defined as

$$\Psi(y) = (F_0(y), F_1(y), G_0(y), G_1(y), G_2(y)).$$

[8] This argument can be easily applied to models with indivisible money. Suppose Δ is a minimum unit of fiat money. The set of admissible prices is $\{0, \Delta, 2\Delta, \ldots\}$. From $\sum_{n=0}^{N} pnh_n = M$, it follows that among a continuum of stationary money holdings distributions only a finite number of them, if any, are in the set. Therefore, for a sufficiently small Δ, we can find a stationary equilibrium in the neighborhood of $h_0 = 1$.

[9] The other solution is $q_b^0 = 0$ and $V_n = 0$, $n = 0, 1, 2$.

Denote by $\det D\Psi(y)$ the Jacobian of Ψ at y with respect to $(h_1, h_2, V_0, V_1, V_2)$. Then Step (3) is to verify that $\det D\Psi(y^*) \neq 0$ at y^*. Indeed, $\det D\Psi(y^*)$ is calculated as follows:

$$\det D\Psi(y^*) = \det \begin{pmatrix} \Upsilon_1 & 0 \\ \Upsilon_2 & \Upsilon_3 \end{pmatrix},$$

where

$$\det \Upsilon_1 = \det \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = 2,$$

and

$$\det \Upsilon_3 = \det \begin{pmatrix} -\dfrac{\phi}{\phi+2} & 0 & 0 \\ -\dfrac{1}{\phi+2} - \dfrac{1}{4A} & \dfrac{\phi+1}{\phi+2} + \dfrac{1}{4A} & 0 \\ -\dfrac{1}{2(\phi+2)} - \dfrac{1}{4A} & -\dfrac{1}{2(\phi+2)} + \dfrac{1}{4A} & -\dfrac{\phi+1}{\phi+2} \end{pmatrix}$$
$$= \frac{\phi(\phi+1)}{(\phi+2)^2} \left(\frac{\phi+1}{\phi+2} + \frac{1}{4A} \right) > 0.$$

Then $\det D\Psi(y^*) \neq 0$. By the implicit function theorem, a solution to $\Psi(y) = 0$ can be written as a C^1 function of $\epsilon > 0$, where $h_0 = 1 - \epsilon$.

Step (4) can be easily done. Indeed, it follows from (13) that, for a sufficiently small $\epsilon > 0$, the corresponding $y(\epsilon) = ((1-\epsilon, h_1(\epsilon), h_2(\epsilon)), (V_0(\epsilon), V_1(\epsilon), V_2(\epsilon)))$ satisfies $h_1(\epsilon) > 0$ and $h_2(\epsilon) > 0$.

Finally, we proceed to Step (5), i.e., we verify that the individual optimality conditions are satisfied at $h_0 = 1$ with strict inequalities. It follows that, for a sufficiently small $\epsilon > 0$, the corresponding $y(\epsilon)$ satisfies the optimality conditions. The relevant optimality conditions are as follows:

(a) When a buyer with ip money and a seller with jp money meet and the former proposes an offer, she prefers offering p to no trade, for $i = 1, 2$ and $j = 0, 1$.

(b) When a buyer with $2p$ money and a seller without money meet and the former proposes an offer, she prefers offering p to $2p$.

(c) When a seller with ip money and a buyer with jp money meet and the former proposes an offer, she prefers offering p to no trade, for $i = 0, 1$ and $j = 1, 2$.

(d) When a seller without money and a buyer with $2p$ money meet and the former proposes an offer, she prefers offering p to $2p$.

In the above, we restricted our attention to actions with integer multiples of p and to money holdings that are realized in equilibrium. Since agents can offer a noninteger multiple of p, we need to check that they will not make such offers. Moreover, we need to check the optimality of the strategy in the off-equilibrium path, i.e., money holdings $\eta \in (0, p) \cup (p, 2p) \cup (2p, 3p)$. However, defining

$V_\eta = V_{\lfloor \eta/p \rfloor}$, where $\lfloor \eta/p \rfloor$ is the integer part of η/p, it is verified that conditions (a)–(d) are sufficient.

For example, consider an agent with $1.5p$ money. Suppose she is a buyer and makes an offer to a seller with η money. Her optimal offer is determined by

$$\begin{aligned}&\max_{(d,q)}\ U(q) + V_{1.5p-d}\\&\text{s.t. } C(q) = V_{\eta+d} - V_\eta\\&\eta + d < 3p,\ 0 \le d \le 1.5p.\end{aligned}$$

Note that we only need to investigate the cases of $\eta = 0$, p, and $2p$, since on the equilibrium path sellers have $\eta = 0$, p, or $2p$ with probability one. We first investigate the optimality of offering p instead of $d \in (0, p)$. Let $q_{(d,k)}$ be the quantity of the good such that a seller with kp money, $k = 0, 1, 2$ is indifferent between accepting $(d, q_{(d,k)})$ and rejecting it, i.e., $q_{(d,k)} = V_{kp+d} - V_{kp}$.

Since V is a step function, $V_{kp+d} - V_{kp} = V_k - V_k = 0$ holds and thus $q_{(d,k)} = 0$. Thus offering $d \in (0, p)$ is not better than no trade, since the amount she obtains is zero. Similarly, an offer price $d \in (p, 1.5p)$ is not better than p. Therefore, we need to show that she prefers offering p to no trade, i.e.,

$$U\left(q_{(p,k)}\right) + V_0 \ge V_1. \tag{26}$$

Condition (26) is the same as the condition that a buyer with p money prefers offering p to no trade. Clearly, (26) is a special case of (a). Similar arguments apply to the other cases.

Below, we check the optimality conditions (a)–(d). As for (a), the strict optimality condition for choosing the strategy is

$$U\left(V_{j+1} - V_j\right) + V_{i-1} - V_i > 0.$$

Since

$$\begin{aligned}U\left(V_{j+1} - V_j\right) + V_{i-1} - V_i &= A^{1+.5j}\left(1 - A^{1-.5j}\right)\\&\ge A^{1+.5j}\left(1 - A^{.5}\right),\end{aligned}$$

the strict optimality condition is always satisfied. As for (b), the strict optimality condition is

$$U\left(V_1 - V_0\right) + V_1 - U\left(V_2 - V_0\right) + V_0 > 0.$$

Thus by (a), a sufficient condition is

$$V_2 - U\left(V_2 - V_0\right) + V_0 > 0.$$

Clearly,

$$V_2 - U\left(V_2 - V_0\right) + V_0 = A^2(1 + A)\left(1 - \frac{1}{A(1 + A)^{.5}}\right)$$

is strictly positive when A is sufficiently close to 1, i.e., ϕ is sufficiently small. Then the strict optimality condition is satisfied for a sufficiently small ϕ. Similarly, we can verify that (c) and (d) hold for a sufficiently small ϕ. In other words, for a sufficiently small ϕ, $y(\epsilon)$ is a monetary equilibrium when ϵ is sufficiently small.

For $N > 2$, similar arguments can be applied. However, the optimality condition at $h_0 = 1$ is not strict in some cases. Thus we need to choose strategies which the agents prefer even for $h_0 = 1 - \epsilon$.

4 Concluding remarks

In this paper, we develop a new technique for proving the existence of monetary equilibria in search models with divisible money. The nice feature is that, by using the implicit function theorem, there is no need to solve explicitly for the Bellman equation in a neighborhood of the degenerate distribution. This stands in contrast with the previous approach to proving the existence for a specific trading strategy. A shortcoming of our approach is that we cannot obtain closed-form solutions of Bellman equations, while we obtain equilibrium strategies. Thus if all we need are the existence of an equilibrium and a corresponding strategy, our technique is easily applied. Finally, we note that a more extensive and general treatment of our technique and indeterminacy of equilibria are found in Kamiya and Shimizu (2002).

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