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A Contribution to the Pure Theory of Money*

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This article analyzes a general equilibrium model with search frictions and differentiated commodities. Because of the many differentiated commodities, barter is difficult in that it requires a double coincidence of wants, and this provides a medium of exchange role for fiat currency. We prove the existence of equilibrium with valued fiat money and show it is robust to certain changes in the environment. Rate of return dominance, liquidity, and the welfare implications of fiat money are discussed. *Journal of Economic Literature* Classification Numbers: 021, 311.

1. INTRODUCTION

One of the oldest questions in economics asks why trade so often takes place using money. Of particular interest is the phenomenon of *fiat currency* (an unbacked, intrinsically useless asset) circulating as a medium

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of exchange.[1] This paper adopts the venerable notion that agents accept fiat money in trade because they expect that others will do the same. We prove the existence of equilibria in which this is indeed the case and, further, show that such equilibria are *robust*. That is, even if fiat money has certain properties that reduce its desirability as an asset, it can maintain its value, due to its medium of exchange function.[2] We also use the model to address some classic issues in monetary economics, including liquidity, rate of return dominance, the welfare implications of monetary versus non-monetary equilibria, and the optimal quantity of money.

The approach here is similar in spirit to Kiyotaki and Wright [11]. Agents have idiosyncratic tastes for differentiated commodities and meet randomly over time in a way that implies trade must be bilateral and quid pro quo. However, our previous model assumed a great deal of asymmetry in goods and in agents, as we were interested in showing how certain commodities could endogenously become media of exchange, or commodity money, depending on their intrinsic properties as well as extrinsic beliefs. Even though we were also able to demonstrate the existence of equilibria with circulating fiat money, the analysis was rather complicated, and we never did check the robustness of those equilibria. In particular, we always assumed fiat currency had intrinsic properties making it the best available asset. Here, goods and agents will both display a certain symmetry, meaning there is no natural candidate for commodity money, but a potential role for fiat currency is highlighted and its analysis is simplified.

The basic environment in this paper is similar to, but also different in some important ways from, the standard general equilibrium search or matching framework, as popularized by Diamond [3], for example. The key difference is that Diamond's models have only one type of commodity, while we assume that different people have different preferences over a large number of differentiated goods. This is what makes pure barter difficult in our economy and, therefore, what allows for a genuine medium

[1] In Menger's [14] words, "It is obvious even to the most ordinary intelligence, that a commodity should be given up by its owner in exchange for another more useful to him. But that every economic unit in a nation should be ready to exchange his goods for little metal disks apparently useless as such, or for documents representing the latter, is a procedure so opposed to the ordinary course of things, that... [it is] downright 'mysterious.'"

[2] This is a robustness that does not appear, for example, in the overlapping generations model of fiat money—the presence of storage opportunities, or of other assets, with dominating rates of return necessarily drives money out of the system in that model, without auxiliary assumptions such as legal restrictions. Cash-in-advance or money in the utility function models, while clearly useful for some purposes, do not explain valued fiat currency at all; both appeal to implicit features of the environment in order to motivate the imposed role for money, and the problem is that once these features are made explicit, they may well have *other* implications that should not be ignored.

of exchange role for money. It is true that Diamond [4] also presents a "monetary" version of his economy, but in that model, money is used because barter is ruled out exogenously via a cash-in-advance constraint. The goal here is to capture monetary exchange as an equilibrium phenomenon and not to force it onto the system.[3]

We introduce the basic model without fiat currency in Section 2 and prove there exists a unique nondegenerate (symmetric steady state) equilibrium. In Section 3, we prove there also exist equilibria with valued fiat money. We note that in our monetary equilibria some barter always coexists alongside of monetary exchange. The liquidity of money and of real commodities is examined, and we show that an individual holding money acquires a desired commodity faster than one holding some other good, even though the latter could always barter directly and even though the former is more particular about the trades he is willing to make. In Section 4, we look at the robustness of monetary equilibria, by assuming fiat money is dominated in rate of return and by endowing it with a relatively high transaction cost. As long as the rate of return dominance and transaction cost are not too severe, the value of fiat money can survive. In Section 5 we study welfare and show that the use of fiat money may improve expected utility in the economy. Section 6 concludes the paper.

2. The Basic Nonmonetary Economy

Consider a model with a continuum of differentiated commodities, identified by points around a circle with circumference 2. There is also a continuum of infinite-lived agents with unit mass, identified by points on the same circle. Agents have diverse tastes for goods: the agent indexed by point i has as his most preferred commodity the good indexed by point i and derives utility $u(z)$ from consuming one unit of a good that is distance z from i, where $u'(z)<0$. For instance, each consumer has an ideal color and derives utility from consuming goods that decreases with the difference between their color and his ideal color. Note that the distance between a randomly selected good and a given agent's ideal is distributed uniformly on the unit interval. We do not need to assume that u is either concave or convex, although for certain purposes we do need to assume $u'(z)+zu''(z)\leqslant 0$ (which is automatically satisfied if u is concave).

[3] Jones [9], Oh [15], Iwai [7], and Ariga [1] have also analyzed money using models of bilateral trade. Somewhat similar are the spatial separation models studied by Townsend [17] or Townsend and Wallace [18]. Rather than attempt to review the many other relevant developments in monetary theory, we simply refer the reader to the recent survey by Ostroy and Starr [16].

Goods are technologically indivisible, come in units of size one, and may be stored one unit at a time. There is no direct cost or benefit to storage for now, but this is relaxed below. Goods are acquired in a production sector, where potential production opportunities arrive in continuous time according to a Poisson process with fixed arrival rate α. These opportunities, or projects, are characterized by a commodity type drawn randomly from the circle and a cost of production in terms of disutility c independently drawn from a cumulative distribution function $F(c)$. The greatest lower bound of $F(c)$ is set to zero. We assume that an agent cannot consume his own output; therefore, once a good is produced, he must proceed to a trading sector and attempt to exchange it for something else that he can consume.[4]

In the trading sector, agents are randomly matched in pairs over time according to a Poisson process with fixed arrival rate β, unless there are no other agents attempting to trade, in which case there are never any meetings. Letting N_1 be the measure of agents in the exchange process, we write the arrival rate of partners for a given individual as $B = B(N_1)$, where $B(0) = 0$ and $B(N_1) = \beta$ for $N_1 > 0$.[5] In a given meeting, trade occurs if and only if both partners agree and necessarily entails a one-for-one swap under our assumptions. However, there is a transaction cost in terms of disutility, ε, that must be incurred any time a good is accepted in trade. Letting $u_\varepsilon(z) = u(z) - \varepsilon$, we assume that $u_\varepsilon(0) > 0$ and that there exists $z_\varepsilon < 1$ such that $u_\varepsilon(z_\varepsilon) = 0$, where zero is also the utility of not consuming. Note that ε is strictly positive, although it can be arbitrarily small. If and when an exchange takes place, an individual can consume and proceed back to production whenever he wishes.

The agents in this economy choose strategies for determining when to produce, trade, and consume, in order to maximize their expected discounted utility from consumption net of production and transaction costs,

[4] The standard search models, such as Diamond [3, 4], also impose the restriction that individuals cannot consume their own output. In our model, because there are many differentiated commodities, there is an alternative story that leads to the same formal structure. Assume that production projects always yield *two* goods of opposite types (i and $i+1$) and that goods can only be enjoyed if consumed in pairs. Consuming two goods results in utility $u(z)$, where z is the distance between the two, and $u' < 0$ because it is preferable to combine things that are similar. Now even if we allow an agent to consume both units of his own output, he will typically prefer to trade one of them for something else so that he can enjoy a better combination. We prefer using the more standard story, however, where goods do not come in pairs but agents are simply not allowed to consume their own output, in order to facilitate comparison with existing search models.

[5] This is referred to as a *constant returns to scale* matching technology. To see why, let $\mu(N)$ be the number of meetings per unit of time when N agents are searching for a partner. Then $B(N) = \mu(N)/N$ is the rate at which a given agent finds a match. When $\mu(\cdot)$ displays constant returns, i.e., when $\mu(N) = \beta \cdot N$, we have $B(N) = \beta$ as long as $N > 0$.

given the strategies of others and the matching process. In this paper we will only consider outcomes where all agents and all goods play symmetric roles and where all agents are anonymous—i.e., the names of individuals do not matter (see Jovanovic and Rosenthal [10] for a general description of "anonymous sequential games" into which our model fits). We also concentrate exclusively here on steady states, where the strategies and distribution of agents are unchanging over time. We will describe the notion of a symmetric steady state equilibrium more precisely below, but before we do so, it is possible and useful to discuss certain obvious features of individual behavior.

Consider an agent in the trading sector with good i, and let $\theta(i)$ denote the probability with which he believes that a randomly encountered agent is willing to accept i. Suppose he believes $\theta(i)=\theta_1$ is independent of i (i.e., all goods are equally acceptable). Then we claim that if he is offered good i' in exchange for good i, he will accept if and only if he is going to consume good i' immediately. If he does not consume it, the best he can do is to trade it again later. But since it is no easier to trade i' than it is to trade i, when $\theta(i')=\theta(i)=\theta_1$, he would not be willing to pay the transaction cost ε to make the exchange. Hence, trade only occurs here when both agents have something the other wants to consume, which is Jevons' [8] famous "double coincidence of wants" problem with pure barter. Furthermore, once an agent acquires a good that he is going to consume, he eats it and moves back to the production process immediately, simply because he discounts the future. This establishes the claim.[6]

Based on the above observations, we can formulate the problem faced by an individual as a relatively simple dynamic programming problem. Let j denote the agent's state: $j=0$ if he is in the production sector; $j=1$ if he is in the exchange sector. Let V_j be the optimal value function, which is independent of time and history, except as summarized by j, and is independent of the type of good an agent has in hand or the type of good he is trying to acquire (for the same reason that θ_1 is). Then, if r is the constant rate of time preference, the value functions in steady state will satisfy the following versions of Bellman's equations of dynamic programming:

$$V_0=\int_0^\infty \alpha e^{-(r+\alpha)t}\int_0^\infty \max(V_0,\, V_1-c)\, dF(c)\, dt \tag{2.1}$$

$$V_1=\int_0^\infty B\theta_1\, e^{-(r+B\theta_1)t}\int_0^1 \max[V_1,\, u_\varepsilon(z)+V_0]\, dz\, dt. \tag{2.2}$$

[6] Below we show that the assumption $\theta(i)=\theta_1$ for all i is consistent with rational beliefs in equilibrium. We do not claim that other types of equilibria could not exist, by the way. Equilibria where commodities play asymmetric roles could lead one to identify some of them as commodity monies, as in Kiyotaki and Wright [11], for example, but the purpose here is to concentrate exclusively on fiat money.

Equation (2.1) describes the expected discounted value of the problem for an agent in the production sector, who we call a *producer*. Since production opportunities are located according to a Poisson process with arrival rate α, the waiting time until the next arrival is distributed exponentially with parameter α (i.e., its density function is $\alpha e^{-\alpha t}$). Once a project arrives, the producer has a choice of whether to reject it and continue in production, which yields value V_0, or to switch to exchange at cost c, which yields $V_1 - c$. Equation (2.2) describes the value of the problem for an agent in the trading sector, who we call a *trader*. Other traders are located and are willing to accept his commodity according to a Poisson process with arrival rate B times θ_1, and once a willing partner arrives, the agent has a choice of whether to reject his offer, which implies continuing with value V_1, or to accept, pay the transaction cost ε, consume, and return to production, which implies the value $u_\varepsilon(z) + V_0$.

The maximization problems in (2.1) and (2.2) are over decision rules mapping the supports of c and z into {Accept, Reject} and are solved by *reservation strategies*: (i) accept a production project iff $V_1 > V_0 + c$, or equivalently iff $c < k$ where $k \equiv V_1 - V_0$; (ii) accept a trade iff $u_\varepsilon(z) + V_0 > V_1$, or equivalently iff $z < x$ where $u_\varepsilon(x) \equiv V_1 - V_0$. These equations can therefore be simplified to

$$rV_0 = \alpha \int_0^k (V_1 - V_0 - c)\, dF(c) = \alpha \int_0^k (k - c)\, dF(c)$$

$$rV_1 = B\theta_1 \int_0^x [u_\varepsilon(z) + V_0 - V_1]\, dz = B\theta_1 \int_0^x [u(z) - u(x)]\, dz.$$

To reduce notation, we write these as

$$rV_0 = \alpha s_0(k) \qquad \text{and} \qquad rV_1 = B\theta_1 s_1(x) \tag{2.3}$$

where $s_0(k) \equiv \int_0^k (k - c)\, dF(c)$ and $s_1(x) \equiv \int_0^x [u(z) - u(x)]\, dz$. Note that $s_0(k)$ is increasing for $k \in [0, \infty)$, while $s_1(x)$ is strictly increasing and weakly convex, due to the assumption $u'(z) + zu''(z) \leqslant 0$.[7]

Now suppose all agents use symmetric reservation trading strategies, and let z be the distance along the circle between a fixed good i and the ideal commodity type of a randomly sampled agent. Then z is distributed uniformly on [0, 1], and so the probability that a randomly encountered trader is willing to accept good i is $\mathrm{pr}(z < x) = x$, which is independent of i. Therefore, it is indeed rational for individuals to believe $\theta(i) = \theta_1 = x$

[7] One can interpret $s_0(k)$ as the expected gain from searching in the production sector, or the expected producer surplus, per unit of time, and $s_1(x)$ as the expected gain from trade, or the expected consumer surplus, per unit of time.

will not depend on i. Furthermore, given x and k, the steady state measure of agents in the exchange sector N_1 is a solution to

$$\dot{N}_1 = \alpha F(k)\, N_0 - Bx^2 N_1 = 0, \tag{2.4}$$

where $N_0 = 1 - N_1$.

We now define a symmetric steady state equilibrium—or, for short, simply an equilibrium—for the nonmonetary model, as follows.

DEFINITION 1. An *equilibrium* for the nonmonetary economy is a list $(k, x, V_0, V_1, N_0, N_1)$ satisfying

(a) Eq. (2.4) and $\theta_1 = x$ (rational expectations in steady state);

(b) the equations in (2.3), with $u_\varepsilon(x) = k = V_1 - V_0$ (maximization).

Note that $N_1 = 0$ (along with $V_0 = V_1 = k = 0$ and $x = z_\varepsilon$) will always constitute a degenerate equilibrium, because when there is no one with whom to trade, the best response is to not produce. We also have the following, more interesting, result.

PROPOSITION 1. *There exists a unique nondegenerate equilibrium for the nonmonetary economy.*

Proof. A nondegenerate equilibrium satisfies (2.3) with $\theta_1 = x$, $B = \beta$, and $k = u_\varepsilon(x) = V_1 - V_0$. These conditions reduce to one equation in x,

$$T(x) = ru_\varepsilon(x) + \alpha s_0[u_\varepsilon(x)] - \beta x s_1(x) = 0.$$

Any solution to $T(x) = 0$ (along with the implied values for the other variables) constitutes an equilibrium, which involves production, or is nondegenerate, iff $k = u_\varepsilon(x) > 0$, which means iff $x < z_\varepsilon$. It is easy to verify that $T(0) > 0 > T(z_\varepsilon)$ and $T'(x) < 0$. Hence, there exists a unique solution $x \in (0, z_\varepsilon)$ to $T(x) = 0$. ∎

Activity in the nondegenerate equilibrium can be summarized as follows. Agents in the production sector look for projects and accept them iff they cost less than k. When production occurs, an agent moves to the exchange sector to look for other traders. When two traders meet, if $z < x$ for both they trade, and otherwise they part company. When a trade occurs, the agent consumes and enjoys utility $u(z)$ minus transaction cost ε and then moves back to production. The key aspect of the model, and that which departs from the standard search equilibrium framework, is that the agents have heterogeneous tastes over the differentiated goods. This is what makes pure barter difficult: the probability of trade in a given meeting is the probability that $z < x$ for both traders, which is x^2, where the fact x is squared represents the double coincidence. In the next section we ask if this might give rise to a role for fiat currency.

3. The Monetary Economy

We now hypothesize the existence of a new object—one that does not provide utility to any agent and does not aid in production (intrinsic uselessness) and is not a convertible claim to something else that does. These are the defining characteristics of fiat money (see Wallace [19]). The new object could be thought of as a particular type of paper, or perhaps a shell, with its supply fixed at $M \leqslant 1$ units. In order to maintain the same basic framework used above, assume that this shell is indivisible and that each agent can carry no more than one unit of either shells or "real" commodities at a time; hence, if they are held, the M units must be held by a fraction M of the agents at any point in time.[8] We generally assume there is a transaction cost η to accepting a shell, analogous to the cost ε of accepting goods. However, it facilitates the presentation considerably to set $\eta = 0$ for now and to return to the general case in the next section.

The question we ask here is simple: Can this worthless shell or paper take on value, in the sense that agents willingly accept it in trade for their commodities, merely because they believe that others will do the same for them in the future?

Again, V_j is the value function, but now j denotes one of three states: $j = 0$, 1, or m indicates an agent is a producer, a *commodity trader* in the exchange sector with one of the consumption goods, or a *money trader* in the exchange sector with fiat currency. The proportions of the population in the three states are (N_0, N_1, N_m). Let the probability of a good being accepted by a random commodity trader be θ_1, let the probability of a good being accepted by a random money trader be θ_2, and let the probability of fiat money being accepted by a random commodity trader be θ_3. We only consider symmetric steady state outcomes, which means that all real commodities are equally acceptable, and therefore the V_j and θ_j do

[8] This assumption is, of course, "unrealistic" in the sense that it seems feasible in the real world to hold both money and other objects simultaneously. But the puzzle in monetary economics is to explain why people voluntarily surrender goods for worthless objects like paper or shells, which would be a puzzle whether or not goods and money could be held simultaneously, whether or not they were divisible, etc. Our restriction implies that agents must give up *all* of their goods in order to acquire and store money, which makes valued fiat currency seem all the more puzzling. Also, we want to emphasize that M is the quantity of real balances in the economy. If we were to let S be the total physical number of shells and P the number of shells that exchange for one real commodity in any given transaction, then changing S and P proportionally has no effect on the model.

not depend on the good one has in hand or the good one is trying to acquire. As a final piece of notation, let $m = N_m/(N_m + N_1)$ be the fraction of agents in the trading sector with money, or the probability that a randomly sampled trader is a money trader as opposed to a commodity trader.

By an argument analogous to the one in the preceding section, the value functions can now be shown to satisfy the following conditions:

$$
\begin{aligned}
rV_0 &= \alpha s_0(k) \\
rV_1 &= B(1-m)\,\theta_1 s_1(x) + Bm\theta_2 \max_{\pi}\,[\pi(V_m - V_1)] \qquad (3.1)\\
rV_m &= \max\{B(1-m)\,\theta_3 s_1(y), rV_0\}.
\end{aligned}
$$

The first equation describes the return to search for a producer, as in the preceding section. The second describes the return to search in the exchange sector for a commodity trader: the first term on the right side is the expected gain from meeting a willing commodity trader, while the second is the gain from meeting a willing money trader, whereupon the commodity trader can choose a *probability* $\pi \in [0, 1]$ of accepting the money (i.e., we allow mixed strategies). He will always accept money ($\pi = 1$) if $V_m > V_1$, he will never accept money ($\pi = 0$) if $V_m < V_1$, and he may randomize ($0 < \pi < 1$) if $V_m = V_1$. Finally, the third equation describes the return for a money trader, which is equal to one of two things. If $V_m > V_0$, then $rV_m = B(1-m)\,\theta_3 s_1(y)$, where $u_\varepsilon(y) = V_m - V_0$ defines the reservation commodity for a money trader; otherwise, we assume that money is freely disposed of and the agent switches to production, so that $V_m = V_0$.

The symmetric use of the strategies described above implies $\theta_1 = x$, $\theta_2 = y$, and $\theta_3 = \pi$. The steady state conditions for the distribution of agents are given by

$$N_m = M \qquad \text{if } \theta_3 > 0, \qquad N_m = 0 \qquad \text{if} \quad \theta_3 = 0 \qquad (3.2)$$

$$\dot{N}_1 = \alpha F(k)\,N_0 - B[(1-m)\,x^2 + my\theta_3]\,N_1 = 0. \qquad (3.3)$$

Equation (3.2) says that if money is accepted the population of money traders is equal to the quantity of real money balances (because of our indivisibility assumption), while if money is not accepted and is instead thrown away, there are no money traders in equilibrium. Equation (3.3) equates the flow into the population of commodity traders to the flow out, where now the latter consists of commodity traders who meet other commodity traders and barter, consume, and return to production plus commodity traders who meet money traders and accept their currency. See Fig. 1.

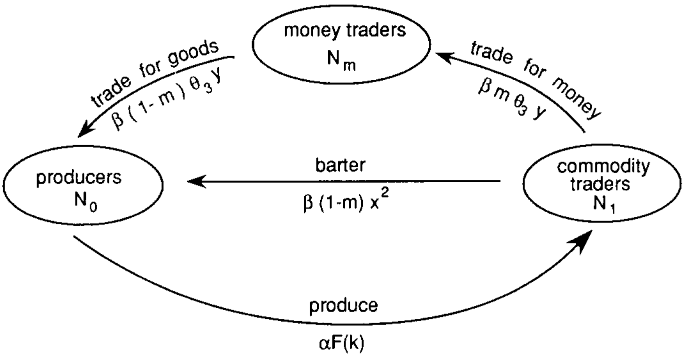

FIGURE 1

We now define a symmetric steady state equilibrium—or simply an equilibrium, for short—for this economy with M fixed exogenously, where M is the supply of real money balances.

DEFINITION 2. Given M, an *equilibrium* for the monetary economy is a list $(k, x, y, V_0, V_1, V_m, N_0, N_1, N_m)$ satisfying

(a) Eqs. (3.2) and (3.3) plus $\theta_1 = x$, $\theta_2 = y$, and $\theta_3 = \pi$ (rational expectations in steady state);

(b) the equations in (3.1), with $k = u_\varepsilon(x) = V_1 - V_0$ and $u_\varepsilon(y) = V_m - V_0$ (maximization).

We say that an equilibrium is *nondegenerate* if production takes place and $N_1 > 0$. We say that it is *monetary* if money is accepted, $\pi > 0$, and *nonmonetary* otherwise. It is a *pure* monetary equilibrium if $\pi = 1$, in which case money is universally accepted, and a *mixed* monetary equilibrium if $0 < \pi < 1$, in which case money is only partially accepted.

The first thing to note is that when $\theta_3 = 0$, the system describing the monetary economy reduces exactly to that of the nonmonetary economy. When agents believe fiat money will be valueless, the monetary economy will have only nonmonetary equilibria (those described in Section 2). A minimal requirement for money to circulate is that agents believe in it. We now show there exist pure monetary equilibria where such beliefs are rational.

PROPOSITION 2. *For any $M \in (0, 1)$ there exists a nondegenerate pure monetary equilibrium.*

In order to prove this proposition, we first prove the following lemma:

LEMMA 1. *For any* $m \in (0, 1)$, *there exists a unique solution* (k, x, y, V_0, V_1, V_m) *to the conditions of Definition* 2 *excluding the steady state Eqs.* (3.2) *and* (3.3), *with* $\pi = \theta_3 = 1$ *and* $k > 0$.

Proof. We seek a solution to (3.1) with $(\theta_1, \theta_2, \theta_3) = (x, y, 1)$ and $B = \beta$,

$$rV_0 = \alpha s_0(k) \tag{3.4}$$

$$rV_1 = \beta(1-m)\, xs_1(x) + \beta my(V_m - V_1) \tag{3.5}$$

$$rV_m = \beta(1-m)\, s_1(y) \tag{3.6}$$

plus $k = u_\varepsilon(x) = V_1 - V_0 > 0$ and $u_\varepsilon(y) = V_m - V_0$, for a given m. From (3.4) and (3.6), we have

$$\beta(1-m)\, s_1(y) - ru_\varepsilon(y) = \alpha s_0[u_\varepsilon(x)]. \tag{3.7}$$

Since the left hand side of (3.7) is increasing in y and the right hand side is decreasing in x, this implicitly defines a decreasing continuous function $x = \varphi(y)$, where $\varphi\colon [y_1, \bar{y}] \to [0, z_\varepsilon]$. Here, y_1 is the unique solution to $\beta(1-m)\, s_1(y_1) = ru_\varepsilon(y_1)$, and $\bar{y}$ is either z_ε or the value of y that satisfies (3.7) with $x = 0$, if such a y exists and is less than z_ε.

From (3.5) and (3.6), we now get

$$\Phi(y, x) = (r + \beta my)[u(y) - u(x)] - \beta(1-m)[s_1(y) - xs_1(x)] = 0. \tag{3.8}$$

We now seek a solution to $T(y) \equiv \Phi(y, \varphi(y)) = 0$. Observe that $y > y_1$ iff $x < z_\varepsilon$; that is, $y > y_1$ iff $k = u_\varepsilon(x) > 0$. Also define y_2 by $y_2 = \varphi(y_2)$, where $y_1 < y_2 < \bar{y}$ from (3.7) and the properties of φ, and observe $V_m - V_1 = u[\varphi(y)] - u(y) > 0$ iff $y < y_2$. In other words, a necessary and sufficient condition for $k > 0$ and $V_m > V_1$ is that the solution to $T(y) = 0$ is strictly between y_1 and y_2. As we have

$$T(y_1) = \Phi(y_1, z_\varepsilon) = \beta my_1 u_\varepsilon(y_1) + \beta(1-m)\, z_\varepsilon s_1(z_\varepsilon) > 0,$$

$$T(y_2) = \Phi(y_2, y_2) = -\beta(1-m)\, s_1(y_2)(1 - y_2) < 0,$$

and $T(y)$ is continuous, there exists $y \in (y_1, y_2)$ such that $T(y) = 0$. Since it is straightforward to show that $T'(y) < 0$ when $T(y) = 0$, with the assumption $u'(z) + zu''(z) \leqslant 0$, the solution y is unique. See Fig. 2. Given y, we have $x = \varphi(y)$ and $k = u_\varepsilon(x)$, while the V_j uniquely solve (3.4)–(3.6). This completes the proof of the lemma. ▌

Proof of Proposition 2. Lemma 1 yields only part of a nondegenerate pure monetary equilibrium, in the sense that it determines the values of y, x, k, and the V_j for a given m, without regard to the steady state

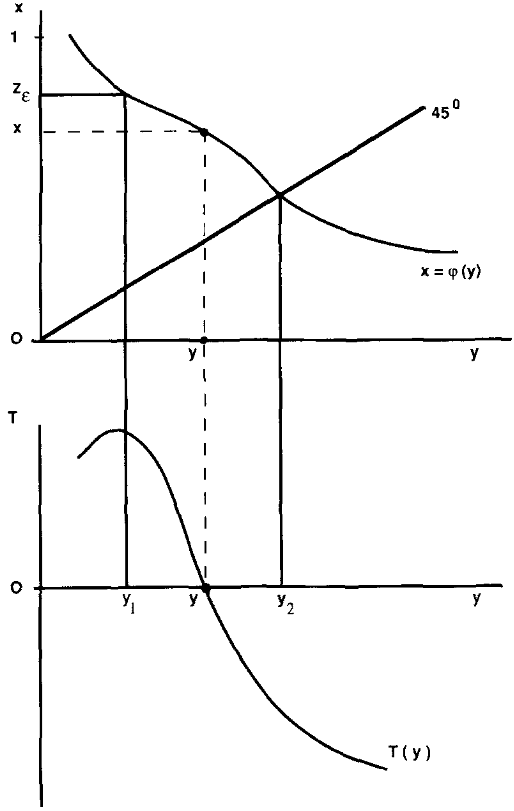

FIGURE 2

conditions. However, we can rearrange these conditions, (3.2) and (3.3), to yield

$$M = M(m) = \frac{m\alpha F[u_\varepsilon(x)]}{\alpha F[u_\varepsilon(x)] + \beta x^2(1-m)^2 + \beta m(1-m)y}. \tag{3.9}$$

Note that $M(m)$ is continuous on $(0, 1)$, $M(m) \to 0$ as $m \to 0$, and $M(m) \to 1$ as $m \to 1$. Hence, for any $M \in (0, 1)$, there exists at least one $m \in (0, 1)$ satisfying $M = M(m)$. This means that, starting with M, we can always find a value of m consistent with a pure monetary steady state, and for this m Lemma 1 yields the values of the other endogenous variables that constitute the desired equilibrium. ∎

Remark. Although Lemma 1 demonstrates that the system has a unique solution for a given m, we have not ruled out the possibility that there might be multiple equilibria for a given value of M. Uniqueness for a given M would follow if we could show that $M = M(m)$ is invertible; this is not obvious, because we know little about how x and y depend on m and therefore little about the function defined by (3.9). Nevertheless, we have no example of multiplicity, and $M(m)$ was certainly invertible in any example we were able to construct.

Proposition 2 demonstrates that there exists a pure monetary equilibrium, for any given $M \in (0, 1)$, in which commodity traders universally surrender goods for fiat currency. The reason agents are always willing to trade goods for money is that $V_m > V_1$. The reason $V_m > V_1$ is that agents believe others are always willing to trade goods for money. Through this circularity the value of fiat money supports itself. Note, however, that some barter always co-exists with monetary exchange in equilibrium—if two commodity traders meet who happen to have a double coincidence, the model predicts that they will trade. There is no *constraint* that says agents have to use cash in this model, as there is in some models, and so we think that it is reasonable to claim that the use of money is endogenous here.

Of course, the double coincidence problem makes monetary trade easier than barter—the chance of a commodity trader's partner being willing to trade is only $\theta_1 = x < 1$, while the chance of a monetary trader's partner being willing to trade is $\theta_3 = 1$. When two commodity traders meet, they exchange with probability $\theta_1 \theta_1 = x^2$, while when commodity and money traders meet, they exchange with probability $\theta_2 \theta_3 = y$. The next proposition shows that $y > x^2$, so the probability of trade is greater in the latter case. This is true even though $y < x$, so that money traders are more demanding in the sense that their reservation good is closer to their ideal good, which means they make better trades on average.

PROPOSITION 3. *In a nondegenerate pure monetary equilibrium,* $x^2 < y < x$.

Proof. See Appendix 1.

The above results are closely related to the concept of *liquidity*. Let D_j be the random duration of time it takes to acquire a consumption good for an agent in state j, $j = 1$ or m. To capture the notion that objects are "less liquid" the longer it takes to exchange them for something that you want to consume, define the liquidity of real commodities or money by $\lambda_j = 1/ED_j$, $j = 1$ or m.[9] For a money trader, D_m is exponentially distributed with $ED_m = 1/[\beta(1-m)y]$, so the liquidity of money is $\lambda_m = \beta(1-m)y$. The liquidity of a commodity is more difficult to compute (because commodity

[9] For a general discussion of "first passage times" like D_j, see, e.g., Heyman and Sobel [5]. See Lippman and McCall [13] for a discussion of some related notions of liquidity.

traders can acquire consumption goods directly via barter or indirectly by trading goods for money for goods); in Appendix 2 we derive $\lambda_1 = \beta(1-m)[(1-m)x^2 + my]$. Hence, $\lambda_m - \lambda_1 = \beta(1-m)^2(y - x^2)$, which is positive, since $y > x^2$ by Proposition 3. A higher acceptance probability for money is therefore reflected in a greater liquidity of money.

We now turn to a discussion of mixed monetary equilibria—that is, equilibria with θ_3 strictly between 0 and 1. This can occur only if $V_1 = V_m$, and so the conditions $u_\varepsilon(x) = V_1 - V_0$ and $u_\varepsilon(y) = V_m - V_0$ immediately imply $x = y$. Then from (3.1) we also have $\pi = \theta_3 = x$, and combined with (3.3), this reduces the conditions for the existence of equilibrium to

$$T(x) = ru_\varepsilon(x) + \alpha s_0[u_\varepsilon(x)] - \beta(1-m)\, xs_1(x) = 0.$$

This equation is the same as the nondegenerate equilibrium condition for the nonmonetary economy in the proof of Proposition 1, except that the effective arrival rate of trading partners is reduced from β to $\beta(1-m)$. Thus, it has a unique solution $x \in (0, z_\varepsilon)$. Furthermore, the steady state conditions in this case reduce to

$$M = \hat{M}(m) = \frac{m\alpha F[u_\varepsilon(x)]}{\alpha F[u_\varepsilon(x)] + \beta(1-m)\, x^2}. \tag{3.10}$$

As with pure monetary equilibria, for any $M \in (0, 1)$ there is at least one m satisfying (3.10), and hence there exists an equilibrium for that M in which money is only partially acceptable.[10]

Of course, rather than agents using mixed strategies where they accept money with probability π, it is equivalent for a proportion π of the population to always accept fiat money while the rest always reject it. In this case, some but not all agents use fiat currency. Under either interpretation, in monetary equilibria where money is not universally accepted, it does not stimulate trade—the probability of randomly sampled commodity and money traders exchanging is the same as the double coincidence probability, x^2. We also note that, since mixed monetary equilibria are formally just like equilibria in a nonmonetary economy, one might expect that a second universally accepted money could be introduced into a mixed monetary economy (say, a second kind of shell in addition to the existing shell or a different color paper money). The result would be dual fiat

[10] For generic values of M, the set of solutions to $M = M(m)$ and the set of solutions to $M = \hat{M}(m)$ have odd, finite numbers of elements. For each solution m to $M = M(m)$ there is a unique pure monetary equilibrium, while for each solution to $M = \hat{M}(m)$ there is a unique mixed monetary equilibrium. Hence, for generic values of M there exist a finite number of monetary equilibria, an odd number that are pure and an odd number that are mixed (as well as the nonmonetary equilibrium).

currencies circulating with different degrees of acceptability. See Kiyotaki and Wright [12] for an analysis of this possibility in a (much simplified) version of the model.

4. Robustness

In this section we demonstrate that not only can fiat money take on value in this economy, but also this feature is *robust*, at least to small changes in the underlying environment. A particular example that should be of substantial interest to monetary economists concerns *rate of return dominance*—a phenomenon Hicks [6, p. 5] called the "central issue in the pure theory of money." The issue is, why do individuals voluntarily accept or hold "barren" money when there are other assets that yield certain positive nominal interest? In other words, in the context of our framework, the issue is whether it is possible to have equilibria where rational agents use money as a medium of exchange even when real commodities have a greater interest rate or yield.[11]

To investigate this, we now include risk free yields on commodities and money, ω_1 and ω_m, which provide utility (or disutility, if ω_j is negative) per unit of time while in storage. Let $\omega \equiv \omega_1 - \omega_m$. Then $\omega > 0$ implies money is dominated in rate of return. At the same time, we want to allow money to have a positive transaction cost, η, as suggested in the previous section. Although it may actually be reasonable to assume $\eta < \varepsilon$, we want to emphasize that the value of fiat money here follows from its medium of exchange function and not because we have somehow endowed it with desirable intrinsic properties. Thus, we will construct equilibria where money circulates even though $\eta > \varepsilon$ and $\omega > 0$.

Mathematically the argument is straightforward, based on Lemma 1 and Proposition 2, where a nondegenerate pure monetary equilibrium is characterized in terms of a solution to $T(y) = 0$ in the domain (y_1, y_2). As long as both the domain and the solution are continuous functions of some perturbation, we will still have a qualitatively similar equilibrium after a small perturbation. The generalized versions of (3.5) and (3.6) are

$$\begin{aligned} rV_1 &= \omega_1 + \beta(1-m)\, xs_1(x) + \beta my \max_{\pi} \left[\pi(V_m - V_1 - \eta)\right] \\ rV_m &= \omega_m + \beta(1-m)\, s_1(y). \end{aligned} \tag{4.1}$$

[11] As Hicks [6, p. 6] noted back in 1935, economists "would have put it down to 'frictions,' and since there was no adequate place for frictions in the rest of their economic theory, a theory of money based on frictions did not seem for them a promising field for economic analysis." The difficulty of modeling frictions explicitly has led some, like Bryant and Wallace [2], who agree that dominance is a crucial anomaly, to suggest instead that it is "to be explained by deviations from laissez-faire, for example, legal restrictions."

Note the flow utilities ω_j and the monetary transaction cost η, which implies $\pi = 1$ only if $V_m \geqslant V_1 + \eta$. The equilibrium function $T(\cdot)$ constructed in Lemma 1 becomes

$$\begin{aligned} T(y; \omega, \eta) = {} & (r + \beta m y)\{u(y) - u[\varphi(y)]\} \\ & - \beta(1-m)\{s_1(y) - \varphi(y)\, s_1[\varphi(y)]\} - \omega - \beta \eta m y. \quad (4.2) \end{aligned}$$

As in the proof of Lemma 1, we seek a solution to $T(y; \omega, \eta) = 0$ in $(y_1, y_2(\eta))$, where y_1 is the same as before, but now $y_2(\eta)$ solves $u(y_2) = u[\varphi(y_2)] + \eta$. Note that $y_2(\eta) > y_1$ as long as η is not too large. If $\omega = \eta = 0$, then $T(y_1; 0, 0)$ and $T[y_2(0); 0, 0]$ are strictly positive and strictly negative, respectively, and therefore there exists a solution to $T(y; 0, 0) = 0$ in $(y_1, y_2(0))$. By continuity, for all (ω, η) in an open neighborhood of zero there also exists a solution to $T(y; \omega, \eta) = 0$ in $(y_1, y_2(\eta))$. Hence, Lemma 1 and therefore Theorem 2 go through, and there exists a nondegenerate pure monetary equilibrium for strictly positive (ω, η). Now, note that we can always shift $u(\cdot)$ and ε together, so that $u_\varepsilon(\cdot) = u(\cdot) - \varepsilon$ is the same, without affecting the system. If we shift $u(\cdot)$ and ε down until $\eta > \varepsilon > 0$, we have a pure monetary equilibrium in which money has a higher transaction cost than commodities, as well as a lower rate of return.

We have demonstrated that money can be a generally acceptable medium of exchange even if $\omega_1 > \omega_m$. Of course, the subjective expected discounted utility of holding currency, measured by V_m, cannot fall short of that on real commodities, V_1, by the definition of a monetary equilibrium. But the objective rates of return on money and commodities are the flow yields ω_1 and ω_m (note that there are no capital gains associated with a change in exchange rates in equilibrium here). This does seem to genuinely capture the phenomena of agents using one thing for a medium of exchange, even when it objectively appears to be intrinsically inefficient, simply because it is universally acceptable. However, and this is important, we point out that if ω or η gets *too* large, then money will definitely not have value. For example, suppose $\eta > u_\varepsilon(y_1)$; then $y_1 > y_2(\eta)$, and therefore monetary equilibrium simply cannot exist.

5. Welfare

In this section we examine welfare. Although producers, commodity traders, and money traders all have different expected discounted utilities (based on their different current states), we focus on what seems to be a natural criterion for welfare analysis, $W = N_0 V_0 + N_1 V_1 + N_m V_m$ (note that $rV_j \to rW$ as $r \to 0$, for $j = 0, 1, m$). Figure 3 displays W and the two

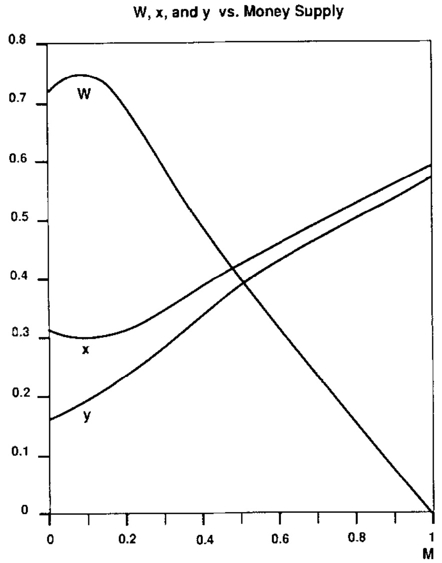

FIGURE 3

reservation goods, x and y, for an example using $u(z) = .6 - z$, $c = 0$ with probability 1, $r = .001$, $\alpha = .5$, $\beta = 10$, $\varepsilon = \eta = .01$, and $\omega = 0$. The most interesting feature of this example is that W is maximized at a strictly positive value of M, which we call the *optimal quantity of real balances*. Using arguments analogous to those in the previous section, we can show by continuity that the optimal quantity of money could be positive even if money were dominated in rate of return or had a greater transaction cost than barter. In other examples, however, the optimal quantity of money can be zero.

Unfortunately, a general welfare characterization here would be difficult. Therefore, from now on we concentrate on the special case where production is instantaneous ($\alpha = \infty$), with a degenerate distribution, $c = \bar{c}$ with probability 1, where $0 < \bar{c} < u_\varepsilon(0)$. We also assume that $\eta = \omega = 0$. Effectively, now there are only two states, $j = 1$ or m, so a nondegenerate pure monetary equilibrium can be characterized by

$$rV_1 = \beta(1-m)\, xs_1(x) + \beta my(V_m - V_1) \tag{5.1}$$

$$rV_m = \beta(1-m)\, s_1(y) \tag{5.2}$$

plus $N_m = m = M$, $N_1 = 1 - m$, $u_\varepsilon(x) = \bar{c}$, and $u_\varepsilon(y) = V_m - V_1 + \bar{c}$. Now we can compute welfare explicitly as

$$W = \frac{\beta}{r}(1-m)\{(1-m)\, xs_1(x) + ms_1(y) + my[u(y) - u(x)]\}. \tag{5.3}$$

The first observation is that when there is too much real money (i.e., $M = m \to 1$), welfare is low, because there are too many money traders and not enough commodity traders with whom to trade. Clearly, this results from our simplifying assumption that agents can carry only one unit of either indivisible commodities or money and not both. A more interesting question is whether the introduction of a little fiat money can improve welfare, compared to the pure barter economy. Since the pure barter equilibrium is equivalent to the limit of pure monetary equilibria as $M \to 0$, we evaluate the derivative of welfare near $M = 0$,

$$\left.\frac{dW}{dM}\right|_{M=0} = \frac{\beta}{r}\{(y + r/\beta)[u(y) - u(x)] - xs_1(x)\}. \tag{5.4}$$

Using (5.1)–(5.3), one can show that as $x = u_\varepsilon^{-1}(\bar{c})$ gets close to zero this expression becomes positive. Also, when the time preference rate r increases relative to the arrival rate of traders (i.e., r/β gets large), the likelihood of a welfare improvement from the introduction of money will increase. Intuitively, the more difficult is barter exchange or the more costly it is to wait, the more likely it is that using money improves welfare. On the other hand, we can also show that as $x = u_\varepsilon^{-1}(c)$ gets closer to one, (5.4) becomes negative. If barter becomes easy enough, the advantage of generally acceptable fiat money is dominated by its disadvantage, the crowding out of the real commodities in circulation. The point is that money can, but need not, improve welfare. Further, this potential welfare improving role does *not* depend on money having superior intrinsic properties, such as a small transaction cost, a high rate of return, or a low storage cost, as we assumed in Kiyotaki and Wright [11]. It simply results from the fact that money speeds up trade, by helping to overcome the double coincidence problem.[12]

6. Conclusion

A model with search frictions and many commodities highlights the double coincidence problem with pure barter and therefore provides a natural framework within which to think about money as a medium of exchange. We have shown that both pure barter and monetary equilibria exist in our model and characterized the role of money in terms of liquidity. We have further demonstrated that these monetary equilibria are robust.

[12] We point out that mixed monetary equilibria are always inefficient, since in such an equilibrium, money does nothing to speed up exchange but only crowds out commodities. This is easy to understand from the observation that a mixed monetary equilibrium is equivalent to a pure barter equilibrium with the arrival rate of traders slowed down from β to $(1 - m)\beta$, and reducing this arrival rate can only reduce welfare.

Even if fiat money is endowed with properties making it a less than ideal asset (e.g., a relatively low rate of return) or a less than ideal medium of exchange (e.g., a relatively high transaction cost), it can continue to circulate and play a role in facilitating trade and improving welfare. Of course, if the intrinsic properties of fiat money become too unfavorable, then naturally it will not circulate in equilibrium.

The model presented here is obviously special, and there is room for much future research. We think the key assumptions underlying the results are the following. First, the transaction cost ε was critical in reducing the nonmonetary economy to a pure barter economy; with $\varepsilon = 0$ all trades would be acceptable, so all goods might serve as media of exchange. Second, symmetry in the set of goods and the set of agents led us to focus on equilibria with no commodity money. Third, the indivisibility of real commodities combined with the restriction on storage (implying inventories always consist of one unit of one thing) kept the model tractable, but precluded a potentially interesting analysis of the distribution of prices. Our hope is that the results we have come up with so far may convince researchers that models like this one, and its potential generalizations, are worth pursuing.[13]

APPENDIX 1: PROOF OF PROPOSITION 3

First, observe that $u_\varepsilon(x) = V_1 - V_0$ and $u_\varepsilon(y) = V_m - V_0$ imply $x > y$ iff $V_m > V_1$, which is always true in pure monetary equilibrium. In order to show $y > x^2$, we begin by evaluating the function $T(y)$ constructed in the proof of Lemma 1 at $y = x^2$, which gives

$$T(x^2) = (r + \beta m x^2)[u(x^2) - u(x)] + \beta(1-m)[x s_1(x) - s_1(x^2)],$$

where $x = \varphi(y) = \varphi(x^2)$. The first term on the right side of this equation is positive since $x < 1$. To check the second term, we use a Taylor's expansion of $s_1(x)$ around x^2: for some $x_0 \in (x^2, x)$,

$$\begin{aligned} x s_1(x) - s_1(x^2) &= x[s_1(x^2) + (x - x^2)\, s_1'(x^2) + \tfrac{1}{2}(x - x^2)^2 s_1''(x_0)] - s_1(x^2) \\ &\geqslant x[s_1(x^2) + (x - x^2)\, s_1'(x^2)] - s_1(x^2) \\ &= (1 - x)[x^2 s_1'(x^2) - s_1(x^2)] \geqslant 0. \end{aligned}$$

[13] Some further applications are explored, in a much simplified version of the model, in Kiyotaki and Wright [12]. There, we discuss endogenizing the interaction between specialization and monetary exchange, we construct equilibria with dual fiat currencies circulating with different degrees of acceptability, we pursue some of the welfare implications in more detail, and we characterize "sunspot equilibria" where extrinsic uncertainty matters for the use of money.

Both inequalities follow from the convexity of $s_1(x)$, which follows from the assumption $u'(z)+zu''(z)\leqslant 0$. Hence, we conclude $T(x^2)>0$, and so the solution to $T(y)=0$ must lie to the right of x^2, as seen in Fig. 2.

APPENDIX 2: DERIVATION OF λ_1

Let p_{ij} denote the instantaneous transition rate from state i to state j. Then, in particular,

$$p_{10}=\beta(1-m)\,x^2, \qquad p_{1m}=\beta my, \qquad p_{m0}=\beta(1-m)\,y. \tag{A.1}$$

Now beginning in state 1, let T be the time of the first transition, a random variable with the CDF $F(t)=\mathrm{pr}(T\leqslant t)=1-\exp[-(p_{10}+p_{1m})t]$. The transition at t can be either to state 0 or to state m. If we let T_0 and T_m be independent random variables with cumulative distribution functions $F_j(t)=\mathrm{pr}(T_j\leqslant t)=1-\exp[-p_{1j}t]$, $j=0$ and m, then the actual transition is to state m iff $T_m<T_0$. This occurs with probability

$$\begin{aligned}\mathrm{pr}(T_m<T_0)&=\int_0^\infty \mathrm{pr}(T_0>t\mid T_m=t)\,dF_m(t)\\&=\int_0^\infty [1-F_0(t)\mid T_m=t]\,dF_m(t)=\frac{p_{1m}}{p_{1m}+p_{10}}.\end{aligned}$$

Let D_1 be the random time it takes to get from state 1 to 0. Given the first transition is at t, then conditional on knowing whether the transition was to m or 0, we have either $E[D_1\mid T_0>T_m=t]=t+1/p_{m0}$ or $E[D_1\mid T_m>T_0=t]=t$. Thus,

$$\begin{aligned}E[D_1\mid T=t]&=\mathrm{pr}(T_m<T_0)\left[t+\frac{1}{p_{m0}}\right]+\mathrm{pr}(T_0<T_m)\,t\\&=t+\frac{p_{1m}}{(p_{10}+p_{1m})\,p_{m0}}.\end{aligned}$$

Hence, the unconditional expectation is

$$ED_1=\int_0^\infty E[D_1\mid T=t]\,dF(t)=\frac{p_{m0}+p_{1m}}{(p_{10}+p_{1m})\,p_{m0}}. \tag{A.2}$$

Substituting the values for p_{ij} given by (A.1) into (A.2) and using $\lambda_1=1/ED_1$ yields the desired formula.

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