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Dynamics, cycles, and sunspot equilibria in 'genuinely dynamic, fundamentally disaggregative' models of money

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Abstract

This paper pursues a line of Cass and Shell, who advocate monetary models that are "genuinely dynamic and fundamentally disaggregative" and incorporate "diversity among households and variety among commodities". Recent search-theoretic models fit this description. We show that, like overlapping generations models, search models generate interesting dynamic equilibria, including cycles, chaos, and sunspot equilibria. This helps us understand how alternative models are related, and lends support to the notion that endogenous dynamics and uncertainty matter, perhaps especially in monetary economies. We also suggest such equilibria in search models may be more empirically relevant than in some other models.

JEL classification: E31; D51

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1. Introduction

Cass and Shell [3] advocate macro and monetary models with an explicit microeconomic structure. The framework they had in mind was the overlapping generations model of fiat money (see [22,23,27]). They argue "this basic structure has

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two general features which we believe are indispensable to the development of macroeconomics as an intellectually convincing discipline... *First it is genuinely dynamic... Second it is fundamentally disaggregative*". They also say that they "firmly believe that a satisfactory general theory must, at a minimum, encompass some diversity among households as well as some variety among commodities", although for technical reasons these features "will typically impose a significant constraint on our ability to derive substantial propositions concerning qualitative effects".

Since they wrote these words a new monetary model has been developed that is also genuinely dynamic and fundamentally disaggregative and takes seriously diversity among households and variety among commodities—the search model. While ostensibly quite different, on some dimensions search models deliver very similar predictions to overlapping generations models. We show here that, like overlapping generations models, search models generate interesting dynamic equilibria—including cycles, chaos, and sunspot equilibria. Knowing this helps us understand how alternative monetary theories are connected, and lends further support to the notion that extrinsic dynamics and uncertainty matter, perhaps especially in monetary economies (see [3,24]).

There are several search-based monetary models in the literature. Many assume severe restrictions on how much money agents can carry, typically $m \in \{0,1\}$; e.g., see [13,25] or [26]. This is convenient for the same reason two-period-lived agents are convenient in the overlapping generations model: at any point in time we can partition the set of agents into a subset with money who want to spend it and a subset without money who want to get it. Yet $m \in \{0,1\}$ is not an appealing assumption. Recent search models allow agents to carry any $m \in \mathcal{R}_+$. Some versions of this model, such as the one in Molico [19], are notoriously difficult to analyze. Here we focus on the much more tractable version in Lagos and Wright [15], referred to hereafter as LW.

Section 2 reviews the basic LW model. Section 3 goes beyond previous analyses by considering dynamic equilibria, including cycles and chaos. Section 4 considers sunspots. Section 5 concludes.[1]

2. The basic model

Time is discrete. The set of agents $\mathcal{A}$ has measure 1. Agents live forever and have discount factor $\beta \in (0,1)$. There is a set of nonstorable goods $\mathcal{G}$. Our version of Cass and Shell's "diversity among households [and] variety among commodities" is as follows. Each $i \in \mathcal{A}$ consumes a subset of goods $\mathcal{G}^i \subset \mathcal{G}$, and produces one good g^i where $g^i \notin \mathcal{G}^i$. Also, given any two agents i and j drawn at random from $\mathcal{A}$, the probability of a single coincidence is $prob(g^i \in \mathcal{G}^j) = \sigma \in (0, \frac{1}{2})$ and the probability of a

[1] A textbook treatment of cycles and sunspots in overlapping generations (and some other) models is [1]. Nonmonetary search models can also display interesting dynamics, but only if the meeting technology has increasing returns, as in [6], which is *not* the case here. Dynamics have been studied in monetary search models in [29,25,5,16,8,9,18]; these papers all assume $m \in \{0,1\}$.

double coincidence is $prob(g^i \in \mathcal{G}^j \wedge g^j \in \mathcal{G}^i) = 0$. In a single-coincidence meeting, if $g^i \in \mathcal{G}^j$ we call j the *buyer* and i the *seller*.[2]

Let $u(q)$ be i's utility from consuming any good in $\mathcal{G}^i$ and $c(q) = q$ the disutility of production. The assumption that c is linear is what makes things tractable. Assume u is $\mathcal{C}^n$ (n times continuously differentiable) with $u' > 0$ and $u'' < 0$. Also, $u(0) = 0$ and $u(\bar{q}) = \bar{q}$ for some $\bar{q} > 0$. Let q^* denote the *efficient* quantity, which solves $u'(q^*) = 1$. We sometimes need $u''' \leqslant (u'')^2/u'$, which holds if u' is log-concave. In addition to goods, there is another object called *money* that is storable, divisible, and can be held in any quantity $m \geqslant 0$. For now, it has no intrinsic value—it is fiat money.

Each period has two sub-periods, day and night. During the day there is a decentralized market with anonymous bilateral matching and bargaining; at night there is a centralized frictionless market. Agents do not discount between sub-periods here but this is easily generalized (see [21]). All trade in the decentralized market must be quid pro quo, as there can be no credit between anonymous agents (see [14] or [28]). We could allow intertemporal trade in the centralized market, but in equilibrium it will not happen since we cannot find one agent who wants to save and another to borrow at the same interest rate.

Let F_t be the distribution of money holdings in the decentralized market at t, where $M_t = \int m \, dF_t(m) = M$ for all t (we keep M_t constant so we can focus on dynamics due exclusively to beliefs). Let $V_t(m)$ be the value function of an agent with m dollars in the decentralized market and $W_t(m)$ the value function in the centralized market. Let $q_t(m, \tilde{m})$ and $d_t(m, \tilde{m})$ be the quantity of goods and dollars traded in a single-coincidence meeting between a buyer with m and a seller with $\tilde{m}$ dollars. Let α be the probability of meeting someone in the decentralized market in any given period, and to reduce notation let $\lambda = \alpha\sigma$. Then Bellman's equation is[3]

$$\begin{aligned} V_t(m) = \lambda \int & \{u[q_t(m, \tilde{m})] + W_t[m - d_t(m, \tilde{m})]\} \, dF_t(\tilde{m}) \\ & + \lambda \int \{-q_t(\tilde{m}, m) + W_t[m + d_t(\tilde{m}, m)]\} \, dF_t(\tilde{m}) \\ & + (1 - 2\lambda) W_t(m). \end{aligned} \tag{1}$$

We normalize the price of goods in the centralized market to 1 and let ϕ_t be the price of money. Then the problem of an agent in this market is

$$\begin{aligned} W_t(m_t) = & \max_{x_t, y_t, m_{t+1}} \{u(x_t) - y_t + \beta V_{t+1}(m_{t+1})\} \\ & \text{s.t. } x_t = y_t + \phi_t(m - m_{t+1}), \end{aligned} \tag{2}$$

[2] In LW, double coincidence meetings can occur with positive probability and the results are basically unchanged. Also, in LW there are some general goods that everyone consumes, but this is not important for anything here.

[3] The first term is the expected payoff from buying $q_t(m, \tilde{m})$ and going to the centralized market that night with $m - d_t(m, \tilde{m})$ dollars. The second term is the expected payoff from selling $q_t(\tilde{m}, m)$ and going to the centralized market with $m + d_t(\tilde{m}, m)$ dollars. Notice the roles of m and $\tilde{m}$ are reversed in these two terms. The final term is the payoff from going to the centralized market without trading in the decentralized market.

where x_t is total consumption of goods in $\mathcal{G}^i$, y_t is production, and m_{t+1} is money left over after trading. We impose nonnegativity on m_{t+1} and x_t but *not* on y_t. Our approach is to allow $y_t < 0$ when solving (2); then, after finding equilibrium, one can introduce conditions to insure $y_t \geqslant 0$ (see LW).

If we insert y_t from the budget equation,

$$W_t(m) = \phi_t m_t + \max_{x_t, m_{t+1}} \{u(x_t) - x_t - \phi_t m_{t+1} + \beta V_{t+1}(m_{t+1})\}. \tag{3}$$

From (3), $x_t = q^*$ where $u'(q^*) = 1$. Also, $W_t(m)$ is linear:

Lemma 1. $W_t(m) = W_t(0) + \phi_t m.$

Given this, we can simplify (1) to

$$\begin{aligned} V_t(m) = \lambda \int & \{u[q_t(m, \tilde{m})] - \phi_t d_t(m, \tilde{m})\} \, dF_t(\tilde{m}) \\ & + \lambda \int \{-q_t(\tilde{m}, m) + \phi_t d_t(\tilde{m}, m)\} \, dF_t(\tilde{m}) + W_t(m). \end{aligned} \tag{4}$$

We now turn to the terms of trade in the decentralized market. Consider a single-coincidence meeting when the buyer has m and the seller $\tilde{m}$ dollars. We use the generalized Nash solution,

$$\max_{q,d} \; [u(q) + W_t(m-d) - W_t(m)]^{\theta} [-q + W_t(\tilde{m} + d) - W_t(\tilde{m})]^{1-\theta}$$

s.t. $d \leqslant m$. By Lemma 1, this reduces to

$$\max_{q,d} \; [u(q) - \phi_t d]^{\theta} [-q + \phi_t d]^{1-\theta} \tag{5}$$

s.t. $d \leqslant m$. The solution (q, d) to (5) does not depend on $\tilde{m}$, and depends on m iff the constraint $d \leqslant m$ binds. Thus, we write $q_t(m, \tilde{m}) = q_t(m)$ and $d_t(m, \tilde{m}) = d_t(m)$ in what follows.

Lemma 2. *The bargaining solution is*

$$q_t(m) = \begin{cases} \hat{q}_t(m) & \text{if } m < m_t^*, \\ q^* & \text{if } m \geqslant m_t^*, \end{cases}$$

$$d_t(m) = \begin{cases} m & \text{if } m < m_t^*, \\ m^* & \text{if } m \geqslant m_t^*, \end{cases} \tag{6}$$

where $m_t^* = [\theta q^* + (1-\theta) u(q^*)]/\phi_t$ *and* $\hat{q}_t(m)$ *solves*

$$\phi_t m = \frac{\theta q u'(q) + (1-\theta) u(q)}{\theta u'(q) + 1 - \theta}. \tag{7}$$

Proof. One can easily check that the proposed solution satisfies the first-order conditions from (5), which are necessary and sufficient in this problem. □

Lemma 3. *For all* t, *for all* $m < m_t^*$, *we have* $q_t'(m) > 0$ *and* $q_t(m) < q^*$.

Proof. The implicit function theorem implies $\hat{q}_t(m)$ is C^{n-1} and

$$\hat{q}_t' = \frac{\phi_t(\theta u' + 1 - \theta)}{u' - \theta(\phi_t m - q)u''}$$
$$= \frac{\phi_t(\theta u' + 1 - \theta)^2}{u'(\theta u' + 1 - \theta) - \theta(1-\theta)(u - q)u''} > 0$$

for all $m < m_t^*$, where the second equality follows by virtue of (7). Since $\lim_{m \to m_t^*} \hat{q}_t(m) = q^*$, we have $q_t(m) = \hat{q}_t(m) < q^*$ for all $m < m_t^*$. □

Inserting (6) and (3) into (4), we have

$$V_t(m) = v_t(m) + u(q^*) - q^* + \phi_t m + \max_{m_{t+1}} \{-\phi_t m_{t+1} + \beta V_{t+1}(m_{t+1})\}, \tag{8}$$

where

$$v_t(m) = \lambda\{u[q_t(m)] - \phi_t d_t(m)\} + \lambda \int [\phi_t d_t(\tilde{m}) - q_t(\tilde{m})]\, dF_t(\tilde{m}) \tag{9}$$

can be interpreted as the expected return from 1 day of decentralized trade, and the other terms in (8) can be interpreted as the return from going to the centralized market that night. In LW we show there exists a unique $V_t(m)$ in a certain class of functions satisfying (8).[4] To avoid these details, in this paper, we work with the sequence problem,

$$V_0(m_0) = \max_{\substack{\{m_{t+1}\} \\ m_0 \text{ given}}} \sum_{t=0}^{\infty} \beta^t [v_t(m_t) + u(q^*) - q^* + \phi_t(m_t - m_{t+1})], \tag{10}$$

where the agent takes as given the entire path for (ϕ_t, F_t).

Since $q_t(m)$ and $d_t(m)$ are C^{n-1} for all $m \neq m_t^*$, so is $v_t(m)$, and we have the first-order conditions

$$-\phi_t + \beta v_{t+1}'(m_{t+1}) + \beta\phi_{t+1} \leqslant 0, = 0 \quad \text{if } m_{t+1} > 0 \quad \forall t. \tag{11}$$

In a monetary equilibrium at least one agent must choose $m_{t+1} > 0$, and for him (11) holds with equality. Indeed, we claim that all agents choose the same $m_{t+1} = M > 0$ in any equilibrium. This will be verified in several steps. First we use a simple arbitrage argument to show that ϕ_t cannot grow too fast, and then we use this to establish that $m_t < m_t^*$.

Lemma 4. *In any equilibrium*, $\beta\phi_{t+1} \leqslant \phi_t$ *for all* t.

[4] This is a nonstandard dynamic programming problem because it is nonstationary and because $V_t(m)$ is unbounded due to the term $\phi_t m$. The problem can be rendered stationary by including ϕ_t as an aggregate state variable as in [7]. The fact that V_t is unbounded can be finessed by working in the space of functions $\bar{V}(m) = \bar{v}(m) + \phi m$, where $\bar{v}$ is continuous and bounded. See LW.

Proof. From (9) we have

$$v'_{t+1}(m) = \begin{cases} \lambda u'[\hat{q}_{t+1}(m)]\hat{q}'_{t+1}(m) - \lambda\phi_{t+1} & \text{if } m < m^*_{t+1}, \\ 0 & \text{if } m > m^*_{t+1} \end{cases} \tag{12}$$

where $\hat{q}'_{t+1}$ is given above. If $\beta\phi_{t+1} > \phi_t$ then the left-hand side of (11) is strictly positive for $m_{t+1} > m^*_{t+1}$ and so the problem has no solution. □

Lemma 5. *In any equilibrium, $d_t(m_t) = m_t < m^*_t$ and $q_t(m_t) = \hat{q}_t(m_t) < q^*$ for all $t \geqslant 1$.*

Proof. One can easily check $\lim_{m_{t+1} \to m^*_{t+1}} v'_{t+1}(m_{t+1}) < 0$. This combined with Lemma 4 implies that the left-hand side of (11) is strictly negative for all $m_{t+1} \in (m^*_{t+1} - \varepsilon, m^*_{t+1})$ for some $\varepsilon > 0$, and weakly negative for all $m_{t+1} > m^*_{t+1}$. Hence, we have $m_{t+1} < m^*_{t+1}$. The rest follows from Lemma 2. □

We now know that agents choose $m_{t+1} < m^*_{t+1}$. If we could show $v'' < 0$ for $m_{t+1} < m^*_{t+1}$ then we could conclude they all choose the same m_{t+1}. While v'' cannot be signed in general, because it depends on u''', we do have the following result.

Lemma 6. *Assume that $\theta \approx 1$ or that u' is log-concave. Then in any equilibrium, for all $t \geqslant 1$, we have $v''(m_{t+1}) < 0$ for all $m_{t+1} < m^*_{t+1}$, and F_t is degenerate at M.*

Proof. Differentiation implies v'' has the same sign as $\Gamma + (1-\theta)[u'u''' - (u'')^2]$, where Γ is a function of parameters that is strictly negative but otherwise need not concern us. Hence, $v'' < 0$ for all m in the relevant range if either we assume $u'u''' \leqslant (u'')^2$ for all q, which follows from log-concavity, or we assume $\theta \approx 1$. Given $v'' < 0$, there is a unique solution to (11) at equality, and so all agents choose the same m_{t+1}. That is, F is degenerate. □

The next step is to reduce the model to one equation in one unknown. To begin, insert v'_{t+1} from (12) into (11) to get

$$\phi_t = \beta\phi_{t+1}[1 - \lambda + \lambda e(q_{t+1})], \tag{13}$$

where $e(q_{t+1}) = u'[q_{t+1}(m_{t+1})]\hat{q}'_{t+1}(m_{t+1})/\phi_{t+1}$. Inserting $\hat{q}'_{t+1}$, we have

$$e(q) = \frac{u'(q)[\theta u'(q) + 1 - \theta]^2}{u'(q)[\theta u'(q) + 1 - \theta] - \theta(1-\theta)[u(q) - q]u''(q)}. \tag{14}$$

Notice $e(q)$ does not depend on t or ϕ_{t+1} directly.

Now (7) with $m = M$ implies $\phi_{t+1} = f(q_{t+1})/M$, where

$$f(q) = \frac{\theta q u'(q) + (1-\theta)u(q)}{\theta u'(q) + 1 - \theta}. \tag{15}$$

Inserting (15) into (13), we get a simple difference equation:

$$f(q_t) = \beta f(q_{t+1})[1 - \lambda + \lambda e(q_{t+1})]. \tag{16}$$

A *monetary equilibrium* is a path $\{q_t\}$ satisfying (16) that remains in $(0, q^*)$ for all t. A steady state q^s is a solution to

$$1 = \beta(1-\lambda) + \beta\lambda e(q^s). \tag{17}$$

Proposition 1. *Assume either that $\theta \approx 1$ or that u' is log-concave. Then there can be at most one monetary steady state q^s. A steady state q^s exists iff $e(0) > (1-\beta+\lambda\beta)/\lambda\beta$. When it exists, q^s is increasing in θ, λ and β. Also, $q^s \to q^*$ as $\beta \to 1$ iff $\theta = 1$.*

Proof. Let $T(q)$ denote the right-hand side of (17). Calculation shows the stated conditions imply $e' < 0$ and hence $T' < 0$, so there cannot be more than one solution to $T(q) = 1$. Notice

$$T(q^*) = \frac{\beta\lambda}{1-\theta(1-\theta)[u(q^*)-q^*]u''(q^*)} + \beta(1-\lambda).$$

Hence, $T(q^*) \leqslant \beta < 1$, and there is a solution $q^s \in (0, q^*)$ to $T(q) = 1$ iff $T(0) > 1$ iff $e(0) > (1-\beta+\lambda\beta)/\lambda\beta$. Also, $T(q^*) = \beta$ if $\theta = 1$ and $T(q^*) < \beta$ if $\theta < 1$. Hence, $q^s \to q^*$ as $\beta \to 1$ iff $\theta = 1$. The rest is routine. □

3. Dynamics

We now consider equilibria where q changes over time. To begin, denote the right-hand side of (16) by $R(q_{t+1})$. Notice $R(q_{t+1}) \geqslant 0, f(0) = 0$, and $\lim_{q\to\infty} f(q) = \infty$ (at least if we assume u' is bounded away from 0). Also notice

$$f'(q) = \frac{u'(q)[\theta u'(q) + 1 - \theta] - \theta(1-\theta)[u(q)-q]u''(q)}{[\theta u'(q) + 1 - \theta]^2} > 0.$$

Hence, $\forall q_{t+1} \geqslant 0$ there exists a unique $q_t = g(q_{t+1})$ such that $f(q_t) = R(q_{t+1})$. Thus g is single-valued, although $h = g^{-1}$ is generally a correspondence.

Clearly, $q = 0$ is a (nonmonetary) steady state.[5] Under the conditions in Proposition 1 there is a unique monetary steady state $q^s = g(q^s) > 0$. Hence g intersects the 45° line exactly twice, at 0 and at q^s.

Lemma 7. *Assume a unique monetary steady state q^s exists; then $g'(q^s) < 1$ and $g'(0) > 1$.*

Proof. Differentiation implies

$$g'(q_{t+1}) = \frac{\beta}{f'(q_t)}\{f'(q_{t+1})[1-\lambda+\lambda e(q_{t+1})] + \lambda e'(q_{t+1})f(q_{t+1})\}.$$

[5] Notice $e(q) \leqslant \theta u'(q) + 1 - \theta$. Hence, $f(q)e(q) \leqslant \theta q u'(q) + (1-\theta)u(q)$. Therefore the right-hand side of (16) is 0 at $q = 0$.

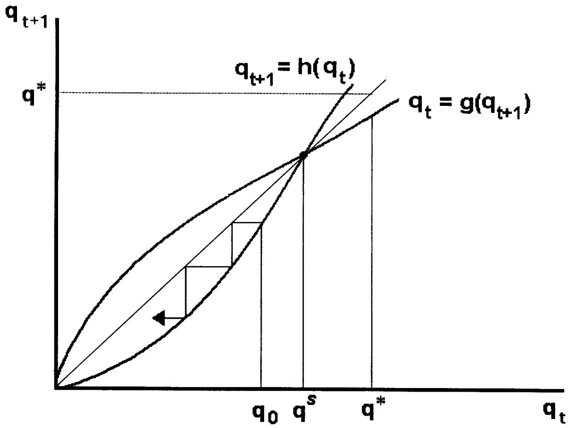

Fig. 1. The case where h is a function.

At $q_t = q_{t+1} = q^s$ we have

$$g'(q^s) = \beta(1-\lambda) + \beta\lambda e(q^s) + \frac{\beta\lambda e'(q^s)f(q^s)}{f'(q^s)}.$$

Inserting (17), we have

$$g'(q^s) = 1 + \frac{\beta\lambda e'(q^s)f(q^s)}{f'(q^s)}. \tag{18}$$

As in Proposition 1, $T(q^*) < 1$ where $T(q) = \beta\lambda e(q) + \beta(1-\lambda)$. Hence, if there exists a unique solution to $T(q^s) = 1$ we must have $T'(q^s) < 0$. This means $e'(q^s) < 0$, so (18) implies $g'(q^s) < 1$ and $g'(0) > 1$. □

Fig. 1 shows $q_t = g(q_{t+1})$ and $q_{t+1} = h(q_t)$ in the (q_t, q_{t+1}) plane when h is single valued. Since $g'(0) > 1$ we have $h'(0) < 1$. This means that there exists a continuum of dynamic equilibria: for all $q_0 \in (0, \bar{q}_0)$, where $\bar{q}_0 \geqslant q^s$, there is a path from q_0 staying in $(0, q^*)$.[6] We summarize this observation in the next proposition. Note that we state things assuming uniqueness, but if there are multiple monetary steady states, the same result holds if we interpret q^s as the lowest one.

Proposition 2. *If a unique steady state $q^s > 0$ exists, there is a $\bar{q}_0 \geqslant q^s$ such that $\forall q_0 \in (0, \bar{q}_0)$ there is an equilibrium starting at q_0 where $q_t \to 0$.*

In Fig. 2, h is a correspondence: given q_t there can be multiple values of q_{t+1} consistent with equilibrium. The figure is drawn with $g'(q^s) < -1$, which implies there is a 2-period cycle. We state this formally, but as the result is standard we only sketch the proof (see [1] for a textbook treatment).

[6] In the case shown in the figure we have $\bar{q}_0 = q^s$ because h is single valued; in general, when h is not single valued we can have $\bar{q}_0 > q^s$ (see Fig. 2).

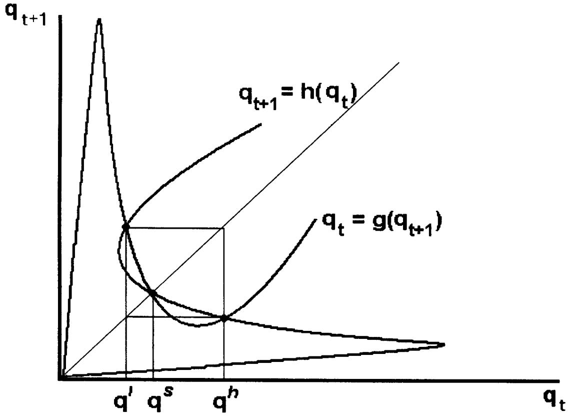

Fig. 2. The case where h is a correspondence.

Proposition 3. *Assume a monetary steady state* q^s *exists. If* $g'(q^s) < -1$ *then* $\exists(q^l, q^h)$, *with* $q^l < q^s < q^h$, *such that* $q^i = g^2(q^i)$, $i = l, h$, *and if* $g'(q^s)$ *is close to* -1 *then* $q^i \in (0, q^*)$.

Proof. If $g'(q^s) < -1$ then in the (q_t, q_{t+1}) plane g is steeper than h where they intersect on the 45° line. Given $g(0) = h(0) = 0$, the curves must intersect off the 45° line at some point (q^l, q^h). This means $g^2(q^l) = q^l$ and $g^2(q^h) = q^h$. Hence the system has a 2-cycle. Technically, as g' passes through -1 the system undergoes a flip bifurcation that gives rise to the cycle, which guarantees q^l and q^h will be close to q^s if g' is close to -1. □

In (18) we found $g'(q^s) = 1 + \beta\lambda f(q^s)e'(q^s)/f'(q^s)$. For instance, if $\theta = 1$ then $g'(q^s) = 1 + \beta\lambda q^s u''(q^s)$, which suggests that sufficient curvature in u is required for $g'(q^s) < -1$. To consider an explicit example, suppose

$$u(q) = \frac{(b+q)^{1-\eta} - b^{1-\eta}}{1-\eta},$$

where $b \in (0, 1)$ and $\eta > 0$. With $\theta = 1$, we can solve explicitly for

$$q^s = \left(\frac{\lambda\beta}{1-\beta+\lambda\beta}\right)^{1/\eta} - b.$$

For $b \approx 0$ we have $g'(q^s) \approx 1 - \eta(1-\beta+\beta\lambda)$, and so $g'(q) < -1$ when $\eta > \eta_0 = 2/(1-\beta+\beta\lambda)$. For η close to η_0 the cycle stays in $(0, q^*)$. More generally, given $b > 0$ there is a critical η_b such that as we increase η beyond η_b the system bifurcates and a 2-cycle emerges. As η increases further cycles of other periodicity emerge; e.g. with $\alpha = 1$, $\sigma = 0.5$, $b = 0.01$ and $\beta = 1/1.1$, cycles of period 3 emerge at $\eta_b \approx 3.9$, as can be verified numerically by looking for fixed points of g^3. Once we have cycles of period 3 we have cycles of all periods, and we have chaos (see [17]).

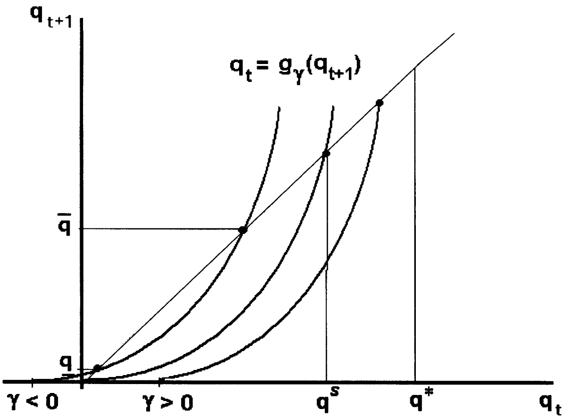

Fig. 3. Equilibria for different γ.

Similar results hold in other models, including overlapping generations models, although there seems to be a view that "unrealistic" parameter values are required in that context. In our model, we need $q_{t+1} = h(q_t)$ to be backward bending, just like in an overlapping generations model, but in that model the condition means the *saving function* is nonmonotone in the rate of return, while here it means the *demand for liquidity* is nonmonotone in inflation, which seems to be a different requirement.[7]

We now introduce a flow return γ per nominal unit of money. This can have interesting effects on the equilibrium set in our model, as it does in overlapping generations models (see [10]), and it will also be useful in the next section. One can interpret γ as interest on currency if $\gamma > 0$ and as a storage cost or tax if $\gamma < 0$. Interpreting it as interest on currency will allow us below to relate to Friedman's [11] policy discussion. Note that γ is a *real* return per unit of currency: the 'government' pays γm goods to an agent holding m dollars (it does not matter which good since every $g \in \mathcal{G}$ has the same price). We balance the budget with lump sum taxes.

To derive the equilibrium conditions with $\gamma \neq 0$, simply add the term γm_t minus the lump sum tax to the right-hand side of the budget constraint in (2), and notice that now $W_t(m) = W_t(0) + (\phi_t + \gamma)m$. The bargaining solution is as in (6) except we replace ϕ_t with $\phi_t + \gamma$ in the definitions of m_t^* and $\hat{q}_t(m)$. The steps leading to (16) now yield

$$f(q_t) = \beta f(q_{t+1})[1 - \lambda + \lambda e(q_{t+1})] + \gamma M, \tag{19}$$

where e and f are as before, although now $\phi_t = f(q_t)/M - \gamma$. Hence, $q_t = g_\gamma(q_{t+1})$ where g_γ is defined by (19). Fig. 3 shows the dynamical system for three cases: $\gamma < 0$, $\gamma = 0$, and $\gamma > 0$. The $\gamma = 0$ case was analyzed above, and we now consider the cases $\gamma < 0$ and $\gamma > 0$ in turn.

[7] The essential point is that here money is a genuine medium of exchange, not merely a store of value, and so the empirical implications have different interpretations. Moreover, our model can be easily and naturally calibrated to an arbitrarily period length—a year, a day, or whatever—which also changes the empirical implications.

If $\gamma < 0$ then g_γ is shifted to the left of g_0. There is some $\underline{\gamma} < 0$ such that if $\gamma < \underline{\gamma}$ there is no monetary steady state, or indeed any monetary equilibria. If $\gamma \in (\underline{\gamma}, 0)$, there are generically an even number of monetary steady states, all below the q^s that prevailed with $\gamma = 0$. The lowest monetary steady state $\underline{q}$ inherits the stability properties of the nonmonetary steady state from the case $\gamma = 0$, so it is locally stable. Now $q = 0$ is still a steady state (where agents dispose of money, given $\gamma < 0$) but it is unstable; indeed, for any $q_0 \in (0, \underline{q})$ we have $q_t \to \underline{q}$. Although $\underline{q}$ is locally stable, q_t may not converge to it if $q_0 > \underline{q}$; it is possible that q_t will converge to another steady state $\bar{q}$ or cycle around $\bar{q}$ (this cannot happen in Fig. 3, because h is single valued, but it can happen in general).

If $\gamma > 0$ then g is shifted to the right. Then there is a unique monetary steady state, and it is above the q^s that prevailed with $\gamma = 0$. One can generalize Lemma 3 to show that q_t can never exceed q^*: as we will argue shortly, equilibria only exist for γ below some threshold $\hat{\gamma}$, and given $\theta = 1$ and $\gamma = \hat{\gamma}$, we have $q = q^*$ is a steady state, while otherwise $q_t < q^*$ $\forall t$. First, notice that $\gamma > 0$ implies $q = 0$ is *not* an equilibrium. Thus, $\gamma > 0$ eliminates the nonmonetary steady state, and any path converging to $q = 0$. Also, suppose a monetary steady state does not exist with $\gamma = 0$; i.e., suppose $q_t = g_0(q_{t+1})$ in Fig. 3 does not intersect the 45° line when $\gamma = 0$. When $\gamma > 0$, it is obvious from (19) that g_γ must intersect the 45° line at some $q > 0$, so there necessarily is a monetary steady state.[8]

To complete the analysis of $\gamma > 0$ we characterize the maximum feasible interest payment $\hat{\gamma}$. First, with $\gamma \neq 0$, the generalization of equilibrium condition (17) becomes

$$e(q) = 1 + \frac{1-\beta}{\lambda\beta} - \frac{M}{\lambda\beta f(q)}\gamma. \tag{20}$$

Second, we can generalize Lemma 5 to show $\beta(\phi_{t+1} + \gamma) \leqslant \phi_t$. In steady state this reduces to $\gamma \leqslant \frac{1-\beta}{\beta}\phi$ or, since $f(q) = (\phi + \gamma)M$,

$$\gamma \leqslant \frac{1-\beta}{M} f(q) = \gamma(q). \tag{21}$$

Any equilibrium has to satisfy this condition.

Suppose we set γ to the maximum feasible γ, satisfying (21) with equality. Then (20) becomes

$$e(q) = 1 + \frac{1-\beta}{\lambda\beta} - \frac{M}{\lambda\beta f(q)}\gamma(q) = 1.$$

When $\theta = 1$, $e(q) = 1$ holds at $q^s = q^*$ and $\gamma = \gamma(q^*) = (1-\beta)q^*/M = \gamma^*$; for any $\theta < 1$, however, $e(q) = 1$ holds at $q^s < q^*$ and $\gamma = \gamma(q^s) < \gamma^*$. That is, given $\theta = 1$ the maximum interest we can pay is γ^*, and it implies $q^s = q^*$ $\forall t$ is an equilibrium (indeed, the unique equilibrium); given $\theta < 1$, however, the maximum interest we can pay is $\gamma < \gamma^*$, and it implies the steady state is $q^s < q^*$.

[8] Intuitively, even if you believe money has no exchange value, you would still be willing to give some positive q for it so as to get the interest γ. Similar results hold in models where $m \in \{0, 1\}$ (see [16]).

We interpret $\gamma = \gamma^*$ as the Friedman Rule for interest on currency. One can derive similar results if, rather than paying interest, we contract the money supply so ϕ_t increases with t: the maximum deflation rate consistent with monetary equilibrium ($\phi_{t+1}/\phi_t = 1/\beta$) is the optimal policy, and it achieves the efficient outcome iff $\theta = 1$. Yet there is one significant difference: with deflation and fiat money ($\gamma = 0$) the nonmonetary equilibrium always exists; with interest payment γ^*, given $\theta = 1$ the efficient outcome is the *unique* equilibrium.[9] We summarize the key results as follows:

Proposition 4. *Suppose $\gamma > 0$. Then there is some maximum feasible $\hat{\gamma}$. If $\theta = 1$, $\hat{\gamma} = \gamma^*$ and it implies the unique equilibrium is $q_t = q^*\ \forall t$. If $\theta < 1$, $\hat{\gamma} < \gamma^*$ and $q_t < q^*\ \forall t$. In any case, $\gamma > 0$ implies there is no equilibrium with $q_t = 0$ or $q_t \to 0$. Now suppose $\gamma < 0$. Then there is some $\underline{\gamma} < 0$ such that $\gamma \in (\underline{\gamma}, 0)$ implies multiple equilibria, all of which have $q_t < q^*\ \forall t$, while $\gamma < \underline{\gamma}$ implies there are no monetary equilibria.*

4. Sunspots

Let s_t be a stochastic process with no effect on fundamentals but potentially with an effect on behavior due to expectations. Assume s_t is known at t before the decentralized market opens, but s_{t+1} is random with distribution function $G(s_{t+1}|s_t)$. In general, all variables now need to be indexed by s as well as t. In a stationary equilibrium we need to index things by s but not by t. Much of this section will focus on stationary equilibria, but we will also consider briefly nonstationary sunspot equilibria.

We begin with an example where $\gamma = 0$. The example constructs a nonstationary sunspot equilibrium along the lines of Cass and Shell [4] by randomizing across deterministic equilibria. Assume the graph of $q_{t+1} = h(q_t)$ is a correspondence, so that given some q_0 there are two distinct values, q_1^H and q_1^L, both in $(0, q^*)$ and both satisfying $q_1 = h(q_0)$. This means that

$$f(q_0) = \beta f(q_1)[1 - \lambda + \lambda e(q_1)] \tag{22}$$

holds at both $q_1 = q^H$ and $q_1 = q^L$.

Now suppose we set $q_1 = q^H$ with probability π and $q_1 = q^L$ with probability $1 - \pi$. To be more precise, we set $prob(s = s_1) = \pi$ and $prob(s = s_2) = 1 - \pi$, and the equilibrium is as follows: $s = s_1$ implies $q_1 = q^H$ while $s = s_2$ implies $q_1 = q^L$, and for $t > 1$ we follow a deterministic equilibrium path from q_1. Proceeding as above, the generalized equilibrium condition is

$$f(q_0) = \beta\pi f(q^H)[1 - \lambda + \lambda e(q^H)] + \beta(1 - \pi)f(q^L)[1 - \lambda + \lambda e(q^L)]. \tag{23}$$

[9] We already argued that $\gamma = \gamma^*$ implies the steady state is q^* and that for any $\gamma > 0$ there are no equilibrium paths leading to $q = 0$. If $h' > 0$ at q^* then there clearly can be no nonstationary equilibria. If $h' < 0$ at q^* then paths starting near q^* must oscillate around q^*, which violates the result that $q_t \leqslant q^*$. Hence $q_t = q^*$ for all t is the unique equilibrium.

Clearly, if the equilibrium condition for the deterministic economy holds at both q^H and q^L then the equilibrium condition for the economy with sunspots also holds.[10]

We now consider stationary sunspot equilibria, where q is a time-invariant function of s. We omit time subscripts and write, e.g., $q_t = q$ and $q_{t+1} = q_{+1}$ if we need to differentiate one period from the next. The generalization of (3) is

$$W(m,s) = u(q^*) - q^* + \phi(s)m + \gamma m + \max_{m_{+1}} \{-\phi(s)m_{+1} + \beta E V_{+1}(m_{+1}, s_{+1})\},$$

where E is the expectation with respect to $G(s_{+1}|s)$. As in Lemma 1, $W(m,s)$ is linear. The bargaining solution is as in (6), except $m^*(s)$ and $\widehat{q}(m,s)$ are indexed by s, and (8) becomes

$$V(m,s) = v(m,s) + u(q^*) - q^* + \phi(s)m + \gamma m + \max_{m_{+1}} \{-\phi(s)m_{+1} + \beta E V_{+1}(m_{+1}, s_{+1})\},$$

where $v(m,s)$ is the obvious generalization of (9).

The relevant first-order conditions are

$$-\phi(s) + \beta E[v'_{+1}(m_{+1}, s_{+1}) + \phi(s_{+1}) + \gamma] = 0 \quad \forall s, \tag{24}$$

where

$$v'(m,s) = \begin{cases} \lambda u'[\hat{q}(m,s)]\hat{q}'(m,s) - \lambda[\phi(s) + \gamma] & \text{if } m < m^*(s), \\ 0 & \text{if } m \geqslant m^*(s) \end{cases}$$

and $\hat{q}'$ is as above. Notice that $m_{+1} \geqslant m^*(s_{+1})$ with probability 1 implies $v'_{+1}(m_{+1}, s_{+1}) = 0$ with probability 1, and thus (24) implies $E[\phi(s_{+1}) + \gamma] = \beta^{-1}\phi(s) > \phi(s)$ with probability 1. At least with $\gamma \leqslant 0$, this is cannot be—we cannot expect $\phi(s)$ to rise in every state in a stationary equilibrium—and so we must have $m_{+1} < m^*(s_{+1})$ with positive probability. However, it is not necessarily true that $m_{+1} < m^*(s_{+1})$ with probability 1.

The conditions in Lemma 6 now imply $v'' \leqslant 0$ $\forall s$ with strict inequality if $m_{+1} < m^*(s_{+1})$. Hence, there is a unique solution to (24), and F is again degenerate at $m_{+1} = M$. Let $B = \{s\colon M < m^*(s)\}$ be the set of realizations for s such that $q < q^*$.

[10] Following Peck [20] one can also construct a sunspot equilibrium which is not a randomization over deterministic equilibria, even if h is a function. Consider (22), and suppose that $R' > 0$ where $R(q_1)$ denotes the right-hand side (it is not hard to make assumptions to guarantee this). Then let $q^H = q_1 + \varepsilon$ and $q^L = q_1 - \varepsilon$. Then clearly $R(q^L) < f(q_0) < R(q^H)$. Therefore, we can choose the probability π so that the equilibrium condition for the sunspot economy (23) will hold.

Inserting v' into (24), we have

$$\phi(s) = \beta \int_B [\phi(s_{+1}) + \gamma]\{1 - \lambda + \lambda e[Q(s_{+1})]\}\, dG(s_{+1}|s) + \beta \int_{B^c} [\phi(s_{+1}) + \gamma]\, dG(s_{+1}|s),$$

where $Q(s)$ is the equilibrium value of q in state s and $e(q)$ is as defined above.

Inserting $\phi(s) = f[Q(s)]/M - \gamma$, where f is as defined above, we have a functional equation in $Q(s)$:

$$f[Q(s)] = \beta \int_B f[Q(s_{+1})]\{1 - \lambda + \lambda e[Q(s_{+1})]\}\, dG(s_{+1}|s) + \beta \int_{B^c} f[Q(s_{+1})]\, dG(s_{+1}|s) + \gamma M.$$

It is always possible for agents to ignore sunspots; e.g. if $\gamma = 0$ one solution is $Q(s) = q^s\ \forall s$ where q^s is the steady state in the model without sunspots, and another is $Q(s) = 0\ \forall s$. Or, if $\gamma < 0$ but $|\gamma|$ is not too big, then there are two monetary steady states without sunspots, called $\underline{q}$ and $\bar{q}$ in the previous section. Here we are interested in a *proper* sunspot equilibrium, or a non-constant solution $Q(s)$ to the functional equation.

At this point, mainly to reduce notation, we set $\theta = 1$, which implies $e(q) = u'(q)$ and $f(q) = q$. Now consider the case $s \in \{s_1, s_2\}$, with $prob(s_{+1} = s_2|s = s_1) = H_1$ and $prob(s_{+1} = s_1|s = s_2) = H_2$ (H_j is the probability of leaving state s_j), and let q_j denote $Q(s_j)$. One possibility is that B contains both states—say, $q_1 < q_2 < q^*$—in which case we have

$$q_1 = \beta(1 - H_1)q_1[1 - \lambda + \lambda u'(q_1)] + \beta H_1 q_2[1 - \lambda + \lambda u'(q_2)] + \gamma M,$$

$$q_2 = \beta H_2 q_1[1 - \lambda + \lambda u'(q_1)] + \beta(1 - H_2)q_2[1 - \lambda + \lambda u'(q_2)] + \gamma M.$$

We seek a solution to this system with $q_1 < q_2 < q^*$. For this, we mimic what one does in overlapping generations models.

One way to proceed is to follow Guesnerie [12] or Azariadis and Guesnerie [2] by considering the limiting case where $H_1 = H_2 = 1$. In this case the system becomes

$$q_1 = \beta q_2[1 - \lambda + \lambda u'(q_2)] + \gamma M = g_\gamma(q_2),$$

$$q_2 = \beta q_1[1 - \lambda + \lambda u'(q_1)] + \gamma M = g_\gamma(q_1),$$

where $q_t = g_\gamma(q_{t+1})$ is precisely the dynamical system in the previous section. That is, with $H_1 = H_2 = 1$ we have reduced our search for sunspot equilibria to a search for 2-cycles. We know a 2-cycle exists if $g' < -1$ by Proposition 3, and we worked out an explicit example to that effect. Since the system of equations in question is continuous in (H_1, H_2) there also exist stationary sunspot equilibria with $q_1 < q_2 < q^*$ for some $H_j < 1$.

Another way to proceed is to set $\gamma < 0$ and then try to construct sunspot equilibria from the two monetary steady states, similar to Cass and Shell [4]. To begin, set

$H_1 = H_2 = 0$, so that

$$q_1 = \beta q_1[1 - \lambda + \lambda u'(q_1)] + \gamma M,$$

$$q_2 = \beta q_2[1 - \lambda + \lambda u'(q_2)] + \gamma M.$$

These two equations are identical, but can have two different solutions: the two monetary steady states in Fig. 3. Thus, we can set $q_1 = \underline{q} \in (0, q^*)$ and $q_2 = \bar{q} \in (\underline{q}, q^*)$. By continuity, there exists a solution $q_1 < q_2 < q^*$ for some $H_j > 0$.

We summarize the key results of this section as follows:

Proposition 5. *When h is a correspondence, sunspot equilibria exist where, given q_0, we randomize over q_1. When there is a 2-cycle, stationary sunspot equilibria exist when H_j is near 1. When there exist multiple monetary steady states, stationary sunspot equilibria exist when H_j is near 0.*

5. Conclusion

This paper has pursued a line, due to Cass and Shell, that says it is desirable to work with monetary models that are "genuinely dynamic and fundamentally disaggregative" and also allow "diversity among households and variety among commodities". Recent search-theoretic models fit this description well. We showed here that these models, like some other models, generate interesting dynamics. This helps us to understand how alternative monetary theories are related, and gives further support to the notion that extrinsic dynamics and uncertainty matter. One possible advantage of our framework is that the parameter values needed for fluctuating equilibria may not be as empirically implausible, and certainly the implied period length seems more empirically relevant, that in some other models.

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References

[1] C. Azariadis, Intertemporal Macroeconomics, Blackwell, Oxford, 1993.
[2] C. Azariadis, R. Guesnerie, Sunspots and cycles, Rev. Econom. Stud. 53 (1986) 725–737.
[3] D. Cass, K. Shell, In defence of a basic approach, in: J. Kareken, N. Wallace (Eds.), Models of Monetary Economies, Federal Reserve Bank of Minneapolis, Minneapolis, 1980, pp. 251–260.
[4] D. Cass, K. Shell, Do sunspots matter?, J. Political Economy 91 (1983) 193–227.

[5] M. Coles, R. Wright, A dynamic equilibrium model of search, bargaining and money, J. Econom. Theory 78 (1998) 32–54.
[6] P. Diamond, D. Fudenberg, Rational expectations business cycles in search equilibrium, J. Political Economy 89 (1987) 606–619.
[7] D. Duffie, J. Geanakoplos, A. Mas-Colell, A. McLennan, Stationary Markov equilibria, Econometrica 62 (1994) 745–782.
[8] H. Ennis, Search, monetary equilibria and sunspots, Macroeconom. Dynam. 5 (2001) 132–142.
[9] H. Ennis, Macroeconomic fluctuations and bargaining, unpublished manuscript, 2002.
[10] R. Farmer, M. Woodford, Self-fulfilling prophecies and the business cycle, Macroeconom. Dynam. 1 (1997) 74–769.
[11] M. Friedman, The Optimal Quantity of Money and Other Essays, Adeline, Chicago, 1969.
[12] R. Guesnerie, Stationary sunspot equilibria in an N-commodity world, J. Econom. Theory 40 (1986) 103–128.
[13] N. Kiyotaki, R. Wright, A contribution to the pure theory of money, J. Econom. Theory 53 (1991) 215–235.
[14] N. Kocherlakota, Money is memory, J. Econom. Theory 81 (1998) 232–251.
[15] R. Lagos, R. Wright, A unified framework for monetary theory and policy analysis, unpublished manuscript, 2002.
[16] Y. Li, R. Wright, Government transaction policy, media of exchange, and prices, J. Econom. Theory 81 (1998) 290–313.
[17] T. Li, J. Yorke, Period three implies chaos, Amer. Math. Monthly 82 (1975) 985–992.
[18] H. Lomeli, T. Temzelides, Discrete time dynamics in a random matching monetary model, Econom. Theory 20 (2002) 259–269.
[19] M. Molico, The distribution of money and prices in search equilibrium, unpublished manuscript, 1999.
[20] J. Peck, On the existence of sunspot equilibria in an overlapping generations model, J. Econom. Theory 44 (1988) 19–42.
[21] G. Rocheteau, R. Wright, Money in search equilibrium, competitive equilibrium, and competitive search equilibrium, unpublished manuscript, 2002.
[22] P. Samuelson, An exact consumption-loan model of interest with or without the social contrivance of money, J. Political Economy 66 (1998) 467–481.
[23] K. Shell, Notes on the economics of infinity, J. Political Economy 79 (1971) 1002–1011.
[24] K. Shell, Monnaie et allocation intertemporelle, unpublished manuscript, 1977.
[25] S. Shi, Money and prices: a model of search and bargaining, J. Econom. Theory 67 (1995) 467–496.
[26] A. Trejos, R. Wright, Search, bargaining, money and prices, J. Political Economy 103 (1995) 118–141.
[27] N. Wallace, The overlapping generations model of fiat money, in: J. Kareken, N. Wallace (Eds.), Models of Monetary Economies, Federal Reserve Bank of Minneapolis, Minneapolis, 1980, pp. 49–82.
[28] N. Wallace, Whither monetary economics?, Int. Econom. Rev. 42 (2001) 47–870.
[29] R. Wright, A note on sunspot equilibria in search models of fiat money, J. Econom. Theory 64 (1994) 234–241.