

# Intensive vs Extensive Margin Tradeoffs in a Simple Monetary Search Model\*

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## Abstract

We introduce ex-post heterogeneity into monetary search models with lotteries. Heterogeneity allows lotteries over goods to exist in equilibrium. These lotteries over goods create an intensive margin (expected production in a match) that is non-existent in all indivisible goods monetary search models. We then show there can be a tradeoff between the intensive margin and extensive margin (number of matches) when choosing the optimal monetary stock.

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# 1 Introduction

In the Kiyotaki and Wright (1991, 1993) search model of money, both goods and money are indivisible and trade at the price of one-for-one. Subsequently, Berentsen, Molico and Wright (2002), examined the use of lotteries (randomized tradings) over money and goods to allow the price,  $p$ , in a match to be determined endogenously. Berentsen *et al.* (2002) showed that the only equilibrium in which  $p$  differs from one is when money is exchanged with a probability less than one, and goods exchange with probability equal to one. This means the quantity of goods traded within a match is constant and independent of the money stock. Consequently, changes in the money stock only affect the extensive margin (the number of trades that occur).

In this note, we show that by introducing ex post heterogeneity in production costs, monetary equilibria exist in which goods are exchanged with probability less than one. Ex-post heterogeneity means sellers experience match specific i.i.d. cost shocks.<sup>1</sup> We show that in these equilibria there is a tradeoff between the extensive and intensive margins – increasing the money stock can expand the extensive margin but reduces the probability of goods trades with high cost sellers, thereby lowering the intensive margin.

# 2 Environment

The environment is similar to Kiyotaki and Wright (1993). There are a continuum of infinitely lived agents on the unit interval. Agents discount at rate  $r$ . They are also specialized in production and consumption such that there is the usual double coincidence of wants problem making barter impossible. The probability of meeting an agent who can produce one's desired consumption good is  $\alpha$ . Goods are indivisible. Agents get utility  $U$  from consuming their desired consumption good and incur costs  $C$  from producing their production good. For trade to occur we need  $U > C$ , which we assume from here on. Due to the absence of a double coincidence of wants, agents need money to trade. Let  $M$  be the stock of money in the economy, and agents are constrained in their inventory of money holdings such that they can hold one indivisible unit of money. Consequently,  $M$  is the fraction of agents holding money in the economy. We call these agents buyers while sellers are those without money. We only consider stationary equilibria such that the value of holding money and goods is constant over time. Let  $V_1$  denote the value function for an agent holding one

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<sup>1</sup>Thus, at the beginning of every period agents are identical ex ante, as in Shevchenko and Wright (2004) or Curtis and Wright (2004)

unit of money and  $V_0$  denote the value function for an agent without money.

### 3 Homogeneous Agents

In this section, we briefly review the Berentsen *et al.* (2002) model with buyer-take-all bargaining as a benchmark. This is without loss of generality since all of their results hold, and are strengthened, with this form of bargaining.

We assume that when a buyer meets an appropriate seller he makes a take-it-or-leave-it offer to the seller. The offer consists of a pair of probabilities  $(\lambda, \tau)$  where  $\lambda$  is the probability goods are traded while  $\tau$  is the probability money changes hands.<sup>2</sup> The buyer's problem is:

$$\max_{\lambda, \tau} [\lambda U - \tau(V_1 - V_0)] \quad s.t. \quad -\lambda C + \tau(V_1 - V_0) \geq 0, \quad \tau \leq 1; \quad \lambda \leq 1.$$

It means that the buyer wants to maximise the difference between the utility he gets for consuming the good (which happens with probability  $\lambda$ ) and his change of state from buyer to seller (which happens with probability  $\tau$ ). The constraint means that the seller must be willing to produce for an exchange to take place. Since the buyer will extract the entire surplus of the seller, the constraint holds with equality. Using the constraint to substitute out for  $\tau$  in the objective yields:

$$\max_{\lambda} \lambda(U - C) \quad s.t. \quad \tau = \frac{\lambda C}{V_1 - V_0} \leq 1$$

It is clear that  $\lambda = 1$  is the solution as long as money is highly valued or  $C \leq V_1 - V_0$ . This simply says that acquiring money has greater value than the cost of producing. However, for  $C > V_1 - V_0$ , money has a low value and the constraint is violated. Nevertheless, the buyer can still offer an acceptable lottery: he offers  $\tau = 1$  and  $\lambda = (V_1 - V_0)/C \leq 1$ . The buyer will choose to offer this value of  $\lambda$  since his surplus is still positive and given by  $\lambda(U - C) > 0$ , while the seller is indifferent and thus is assumed to accept the offer.

The intuition for this is quite clear; if money is valued sufficiently high, then the buyer demands the good with probability one and offers the money with probability less than one. This allows buyers to give up a fraction of a unit of money on average for the good. However, when the value of money is low, the buyer must give up the money with probability one and ask for goods with a probability less than one. Expected monetary prices are given by  $p = \tau/\lambda$ . Thus, when money is highly valued  $p < 1$  and when money has low value  $p > 1$ .

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<sup>2</sup>Lotz, Shevchenko and Waller (2006) look at a more general bargaining problem where dependence between the two lotteries is possible. We show that the optimal lottery structure is what we have here.

Now, we have to prove existence of the two previous possible equilibria. Since the buyer makes an offer that extracts the entire expected surplus of the seller,  $V_0 = 0$  and the buyer's value function is given by:

$$V_1 = \rho(1 - M) [\lambda U - \tau (V_1 - V_0)] \quad (1)$$

where  $\rho = \alpha/r$ .

We first consider equilibria where  $\lambda = 1$  and  $\tau \leq 1$ . It follows that

$$V_1 = \rho(1 - M) (U - C), \quad \tau = C/\rho(1 - M) (U - C)$$

where  $V_1$  is monotonically decreasing in  $M$ . We assume that at  $M = 0$ ,  $\tau|_{M=0} < 1$ . Since  $\tau$  is increasing in  $M$  and approaches infinity as  $M \rightarrow 1$ , there is a unique value of  $\bar{M} < 1$ , such that  $\tau = 1$  and  $V_1 = C$ . For values of  $M \in (0, \bar{M})$ ,  $V_1 > C$  and  $\tau < 1$ . From an economic point of view, this seems intuitive; when there is a sufficiently low quantity of money in the economy, money will be highly valued. Thus, trading one unit of money for one unit of goods - while beneficial to the buyer - involves a price that is too high, i.e  $p = 1$ . So the buyer exploits his bargaining power to offer the seller a lottery over the money in order to trade at a lower expected price,  $p = \tau < 1$ , which the seller is willing to accept. On average, buyers spend less than a unit of money and it is in this sense that money is 'divisible'.

Now consider the case where  $M \in [\bar{M}, 1)$ . From the bargaining problem, we know that  $\tau = 1$  and  $\lambda = V_1/C \leq 1$ . Substituting this into (1) yields  $CV_1 = \rho(1 - M) (U - C) V_1$  implying  $V_1 = 0$  unless  $C = \rho(1 - M) (U - C)$ . The value of  $M$  that solves this expression is  $\bar{M}$ . For  $M > \bar{M}$ , the only solution is  $V_1 = 0$  and so a monetary equilibrium does not exist. This is the result in Berentsen *et al.* (2002) - monetary equilibria do not have lotteries over goods.

## 4 Cost heterogeneity

In this section, we introduce *ex post* heterogeneity across agents by assuming there is a continuum of cost types  $C^j$ . By *ex post* heterogeneity, we mean that agents experience match specific i.i.d. cost shocks. Thus, at the beginning of every period agents are identical *ex ante*.<sup>3</sup> By assumption, information is symmetric, i.e.  $C^j$  is observable by the buyers. Let  $C^j \sim F(C)$  with density  $dF(C)$ ,

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<sup>3</sup>The advantage of assuming *ex post* instead of *ex ante* heterogeneity is that it has no distributional consequences in terms of money balances across agent types, and so it simplifies the model without affecting our basic conclusions. To understand how *ex ante* heterogeneity would affect the model, see Lotz, Shevchenko and Waller (2006).

where  $F$  is continuous in  $C$ , with  $C \in [C^L, C^H]$  and expectation  $C^e = \int_{C^L}^{C^H} C dF(C)$ . It follows that the probability a buyer meets an appropriate seller with production costs less than some value  $\hat{C}$  is  $\alpha(1-M) \int_{C^L}^{\hat{C}} dF(C)$ . We assume that all agents get the same utility,  $U$ , from consuming their desired consumption good where  $U > C^H$ . This ensures that there are gains from trade with all sellers who produce the buyer's desired consumption good. It also eliminates market participation issues present in Johri (1999), and Camera and Vesely (2006). Since agents are identical before trading begins, let  $V_1$  denote the value function for an agent holding one unit of money, and  $V_0$  denote the value function for an agent without money.

With cost heterogeneity, the offer consists of a pair of probabilities  $(\lambda^j, \tau^j)$  where  $\lambda^j$  is the probability goods are traded with a type  $j$  seller while  $\tau^j$  is the probability money changes hands with this  $j$  seller. Conjecture that, for a given value of  $M$ , there is a cutoff value  $\bar{C} \in [C^L, C^H]$  such that the bargaining solutions yield:

$$\begin{aligned} \forall C^j > \bar{C}, \quad \tau^j = 1 \text{ and } \lambda^j = \frac{V_1}{C^j} < 1 \\ \forall C^j < \bar{C}, \quad \tau^j = \frac{C^j}{V_1} < 1 \text{ and } \lambda^j = 1 \\ \forall C^j = \bar{C}, \quad \tau^j = 1, \lambda^j = 1 \text{ and } V_1 = \bar{C}. \end{aligned}$$

Once again, conjecture an equilibrium in which buyers can choose appropriate values of  $\lambda^j$  and  $\tau^j$  such that both buyers and sellers are willing to trade at the price  $\tau^j/\lambda^j$ . The value function of a buyer is:

$$V_1 = \rho(1-M) \int_{\bar{C}}^{C^H} (\lambda^j U - V_1) dF(C) + \rho(1-M) \int_{C^L}^{\bar{C}} (U - \tau^j V_1) dF(C)$$

Substituting the lotteries into this expression and rewriting gives:

$$V_1 = \rho(1-M) V_1 \int_{\bar{C}}^{C^H} (U/C^j - 1) dF(C) + \rho(1-M) \int_{C^L}^{\bar{C}} (U - C^j) dF(C) \quad (2)$$

The first term is the expected surplus received when goods are acquired from high cost sellers, which occurs with probability  $\lambda^j = V_1/C^j$ , while the second term is the expected surplus from trading with low cost sellers.

For the cutoff seller it must be the case that  $V_1 = \bar{C}$ , so  $\bar{C}$  solves:

$$T(\bar{C}) \equiv \bar{C} \left[ 1 - \rho(1-M) \int_{\bar{C}}^{C^H} (U/C^j - 1) dF(C) \right] - \rho(1-M) \int_{C^L}^{\bar{C}} (U - C^j) dF(C) = 0 \quad (3)$$

We can now state the following proposition:

**Proposition 1** For  $M \in [0, \hat{M}_1]$ ,  $\bar{C} = C^H$  and  $\tau \leq 1, \lambda = 1$  for all sellers. For  $M \in (\hat{M}_1, \hat{M}_2)$ , there exists a unique value  $\bar{C} \in (C^L, C^H)$  where for  $C^j > \bar{C}$ ,  $\tau^j = 1$  and  $\lambda^j < 1$ , while for  $C^j < \bar{C}$ ,  $\tau^j < 1$  and  $\lambda^j = 1$ . For  $M = \hat{M}_2$ ,  $\bar{C} = C^L$  and  $\tau = 1, \lambda \leq 1$  for all sellers. For  $M > \hat{M}_2$ , no monetary equilibrium exists.

The proof is in the appendix. Figure 1 illustrates Proposition 1.

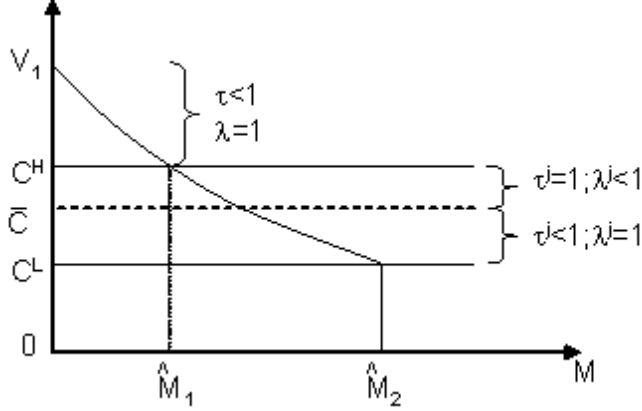


Figure 1: Equilibria with lotteries over goods ( $\lambda$ ) and money ( $\tau$ ).

For low values of the money stock ( $M < \hat{M}_1$ ), money is highly valued so all sellers trade goods with probability one and receive money with a probability less than one. For intermediate values of  $M$ , high cost sellers value money less than the cost of producing so they only want to give up a fraction of the good for money. Consequently, buyers have to offer a lottery over goods for exchange to occur. On the other hand, low cost sellers value money more than the cost of producing so they agree to receive a fraction of the unit of money for their unit of good; buyers offer a lottery over money. It then follows that there is a distribution of expected prices with  $p > 1$  for high cost sellers ( $C^j > \bar{C}$ ) and  $p < 1$  for low cost sellers ( $C^j < \bar{C}$ ). For high values of the money stock, no monetary equilibrium exists.

What is the optimal steady-state money stock in this economy? In Kiyotaki and Wright (1991, 1993), Rocheteau (2000) and Berentsen *et al.* (2002), the optimal value of the money stock is  $M^* = 1/2$ , which maximizes the extensive margin (number of matches). In these models, the intensive margin is the expected production in a match and all monetary equilibria have  $\lambda = 1$  so the intensive margin is constant. Hence, changes in  $M$  do not affect the intensive margin and so there is no tradeoff between the extensive and intensive margins.

In our model, for  $M \in (\hat{M}_1, \hat{M}_2)$ ,  $\lambda < 1$  and, in this range, increasing  $M$  has two negative effects. First, it lowers  $\lambda^j$  for all sellers with costs above  $\bar{C}$  thereby lowering the expected quantity

of goods traded in those matches (it worsens the intensive margin). Second, it lowers the cutoff value  $\bar{C}$  meaning more sellers resort to lotteries over goods rather than lotteries over money (more sellers produce less).

As is standard in the literature, define steady-state welfare to be  $W(M) = MV_1 + (1 - M)V_0$ . Consider the two possible ranges for  $M$ :  $M \in [0, \hat{M}_1]$  with  $C^H = \bar{C}$  and  $M \in (\hat{M}_1, \hat{M}_2)$  with  $\bar{C} \in (C^L, C^H)$ . Using  $V_0 = 0$ ,  $\bar{C} = V_1$  and (2) we have

$$W(M) = \begin{cases} \rho M(1 - M)(U - C^e) & \text{if } M \in [0, \hat{M}_1] \\ \rho M(1 - M)\mathcal{Z} & \text{if } M \in (\hat{M}_1, \hat{M}_2) \end{cases}.$$

where

$$\mathcal{Z} \equiv \int_{\bar{C}}^{C^H} \bar{C}(U/C^j - 1) dF(C) + \int_{C^L}^{\bar{C}} (U - C^j) dF(C). \quad (4)$$

The optimal steady-state money stock,  $M^*$ , satisfies

$$W'(M^*) = \begin{cases} \rho(1 - 2M^*)(U - C^e) = 0 & \text{if } M^* \in [0, \hat{M}_1] \\ \rho(1 - 2M^*)\mathcal{Z} + M^*(1 - M^*)\frac{\partial \mathcal{Z}}{\partial \bar{C}}\frac{\partial \bar{C}}{\partial M} = 0 & \text{if } M^* \in (\hat{M}_1, \hat{M}_2) \end{cases}.$$

From (4),  $\partial \mathcal{Z} / \partial \bar{C} > 0$  and totally differentiating (3) yields  $\partial \bar{C} / \partial M < 0$  for  $M < \hat{M}_2$ . It then follows that if  $1/2 < \hat{M}_1$ ,  $M^* = 1/2$  and the optimal steady-state money stock maximizes the extensive margin since the intensive margin is unaffected for  $\lambda = 1$ . On the other hand, if  $\hat{M}_1 < 1/2$ , then choosing  $M^* \in (\hat{M}_1, 1/2)$  is optimal since  $W'(1/2) < 0$ . Therefore, if there are too few buyers for  $M < \hat{M}_1$ , then it is optimal to increase the number of trades on the extensive margin (by increasing  $M$ ) even though it reduces the intensive margin by lowering  $\lambda$ . But it is not optimal to maximize the number of trades that occur since  $M^* < 1/2$ .

## 5 Conclusion

By extending the Berentsen, Molico and Wright (2002) monetary search model with lotteries to the case where agents are heterogeneous ex post, we first showed that there exist monetary equilibria with lotteries over goods. Second, because goods may be exchanged with probability less than one, a tradeoff between the intensive and extensive margin emerges, modifying the optimal monetary stock.

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## Appendix

### Proof of Proposition 1:

If  $\bar{C} = C^H$ , using (2) gives us:

$$V_1 = \rho(1 - M)(U - C^e); \quad \tau^H = C^H / \rho(1 - M)(U - C^e)$$

Assume  $\tau^H|_{M=0} < 1$ . Let  $\hat{M}_1 \in (0, 1)$  denote the value of the money stock making  $\tau^H = 1$ . Thus, for  $M \in [0, \hat{M}_1]$ ,  $V_1 \geq C^H$  and all sellers give up the good with probability one and receive money with probability less than one. If  $\bar{C} = C^L$ , then  $\lambda^j = V_1/C^j$  for all  $j > L$  and  $\tau_j = 1, \forall j$ . It follows that  $V_1$  solves:

$$V_1 = V_1 \left[ \rho(1 - M) \int_{C^L}^{C^H} (U/C^j - 1) dF(C) \right].$$

So  $V_1 = 0$ , unless  $M = \hat{M}_2$  satisfying:

$$1 = \rho(1 - \hat{M}_2) \int_{C^L}^{C^H} (U/C^j - 1) dF(C) \tag{5}$$

which implies  $V_1$  is indeterminate and any value  $V_1 \in (0, C^L]$  is a monetary equilibrium. For  $M > \hat{M}_2$ , no monetary equilibria exist. It is straightforward to show that  $\hat{M}_2 > \hat{M}_1$  is true  $\forall C$ . As a result for  $M \in (\hat{M}_1, \hat{M}_2)$ , it may be possible to have  $\bar{C} \in (C^L, C^H)$ . We now establish existence and uniqueness of  $\bar{C}$ . Using (5), (3) gives us

$$\begin{aligned} T(C^L) &= C^L (M - \hat{M}_2) / (1 - \hat{M}_2) < 0 \\ T(C^H) &= C^H (M - \hat{M}_1) / (1 - \hat{M}_1) > 0. \end{aligned}$$

Since  $F$  is continuous in  $C$  by assumption, there exists a value  $\bar{C}$  such that  $T(\bar{C}) = 0$ , where  $\bar{C} \in (C^L, C^H)$  for  $M \in (\hat{M}_1, \hat{M}_2)$ . Given that  $T(C^L) < 0$  and  $T(C^H) > 0$ , there may be multiple equilibria but if so there are an odd number of them. We now show that the equilibrium is unique. The proof is by contradiction. If there are multiple equilibria then at least one of the equilibria must have  $T'(\bar{C}) < 0$ . Applying Leibnitz's rule, we obtain:

$$T'(\bar{C}) = 1 - \rho(1 - M) \int_{\bar{C}}^{C^H} (U/C^j - 1) dF(C). \tag{6}$$

Substitute this into (3) and rearrange to get

$$\bar{C}T'(\bar{C}) = \rho(1 - M) \int_{C^L}^{\bar{C}} (U - C^j) dF(C) > 0$$

which requires  $T'(\bar{C}) > 0$  for all  $\bar{C}$ . Hence, the equilibrium is unique. ■