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A THEORY OF MONEY AND MARKETPLACES*

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This article considers an infinitely repeated economy with divisible fiat money. The economy has many marketplaces that agents choose to visit. In each marketplace, agents are randomly matched to trade goods. There exist a variety of stationary equilibria. In some equilibrium, each good is traded at a single price, whereas in another, every good is traded at two different prices. There is a continuum of such equilibria, which differ from each other in price and welfare levels. However, it is shown that only the efficient single-price equilibrium is evolutionarily stable.

1. INTRODUCTION

In a transaction, one needs to have what his trade partner wants. However, it is hard for, say, an economist who wants to have her hair cut to find a hairdresser who wishes to learn economics. In order to mitigate this problem of a lack of double coincidence of wants, money is often used as a medium of exchange. If there is a generally acceptable good called money, then the economist can divide the trading process into two: first, she teaches economics students to obtain money, and then finds a hairdresser to exchange money for a haircut. The hairdresser accepts the money since he can use it to obtain what he wants. Money is accepted by many people as it is believed to be accepted by many.

Focusing on this function of money, Kiyotaki and Wright (1989) formalized the process of monetary exchange. In their model, agents are randomly matched to form a pair and trade their goods when they both agree to do so. This and the subsequent models, called the search theoretic models of money, have laid a foundation of monetary economics. The purpose of the present article is to further develop the foundation by introducing the concept of *marketplaces* into the model with divisible fiat money presented by Green and Zhou (1998).

Marketplaces are the places that agents choose to visit to meet trade partners. They capture the following two aspects of actual trading processes. First, matching rarely occurs in a completely random fashion in the real economy.

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People go to a fish market as buyers to buy fish; many potential workers use particular channels for job openings instead of simply walking on streets to meet potential employers. There are marketplaces where agents look for their trade partners.

The second aspect is related to price competition. In order for price competition to take place, there must be a possibility that a price cut leads to an increase in sales. The standard search model incorporates this possibility only in a limited way. Although a seller may increase sales per visit by lowering its price, it cannot increase the number of visits itself. In other words, even if the seller cuts the price, it cannot differentiate itself from its competitors and attract more customers. If there are many marketplaces, sellers can differentiate themselves from their competitors by visiting a new marketplace. It turns out that this function of marketplaces activates the price-adjustment mechanism in the present analysis.

In order to see how marketplaces function, let us now briefly explain the model and the results of the article. Roughly speaking, the present model is described as follows. It has an infinite number of periods and a continuum of infinitely lived agents. There are infinitely many marketplaces, each of which consists of two physically identical sides, A and B. In each period, all the agents simultaneously decide whether to produce a unit of goods or not and choose a marketplace and one of its sides. In each marketplace, the agents on side A are matched with those on side B in a random fashion to form pairs, with the long side being rationed. When two agents are matched and find that one of them holds the good that the other wishes to consume, they negotiate the price, which is modeled as a simultaneous offer game.

We adopt two approaches for the main analysis, an equilibrium approach and an evolutionary approach. We first examine stationary equilibria of this model. There exist various classes of stationary equilibria. Equilibria can be classified based upon two characteristics, the degree of specialization of marketplaces, i.e., the number of goods traded in one marketplace, and the degree of price dispersion, especially, whether each good is traded at a single price or not. In some equilibria, all the agents visit the same marketplace, whereas in others, marketplaces are completely specialized, i.e., only one type of good is traded in each active marketplace. In some equilibria, each good is traded at a single price, whereas in others, each good is traded at two different prices. Each class itself consists of a continuum of equilibria that correspond to different price and welfare levels.

As we mentioned above, the equilibrium approach admits a multitude of outcomes. The reason is the following. In the present model, the only way that a seller can increase the matching probability is to switch to a new marketplace to differentiate itself from other sellers, but no buyer visits such a place if no seller is expected to visit there. Therefore, no unilateral deviation to an empty marketplace is profitable. In other words, the equilibrium approach cannot have agents utilize empty places to start a new transaction pattern, including the one with a price cut.

The evolutionary approach overcomes this coordination problem. It allows a small group of agents, or mutants, to jointly visit a new marketplace and start a new transaction pattern. An equilibrium is said to be evolutionarily stable if no group of mutants fares better than the original population. It is shown that only efficient single-price equilibria with complete specialization are evolutionarily stable. An inefficient equilibrium is upset by the mutants who visit a new marketplace to establish a more efficient trading pattern than before.

The present article serves as a microfoundation for the trading post approach. This approach was initiated by Shapley and Shubik (1977), and applied to a situation with fiat money by Hayashi and Matsui (1996). Trading posts are the places in each of which prespecified pairs of goods are traded. People submit their goods to the designated trading posts. The goods they submit to one side of a post are traded with the other type of goods submitted to the opposite side. Agents obtain the goods on the opposite side in proportion to the amount they submitted. The trading mechanism at trading posts is put in a black box. On the other hand, the present article explicitly models the trading processes. At the same time, it prepares sufficiently many marketplaces that can be used for transactions, but does not specify which place is used for which goods to be traded. Specialization of marketplaces may emerge endogenously. It is verified that the evolutionarily stable outcome of the present model corresponds to the stationary equilibrium examined in Hayashi and Matsui.[2]

Some mention has to be made of the existing search theoretic models of money. In the beginning, these models (e.g., Kiyotaki and Wright, 1989) assume indivisible commodities and fiat money, if any, mainly due to the analytical difficulty of tracking inventory as different agents have different experiences. Avoiding this difficulty, Trejos and Wright (1995) and Shi (1997) addressed the issues related to price levels. In order to do so, Trejos and Wright introduced divisible commodities, whereas Shi presented a model in which each household can simultaneously engage in infinitely many transaction activities. However, each transaction involves an indivisible unit of money. Green and Zhou (1998) presented a model with divisible fiat money. They partially succeeded in solving it, restricting their analysis to a certain class of equilibria.

A new problem arose in Green and Zhou: There exists a continuum of equilibria with different price and welfare levels. Green and Zhou expanded the frontier of the search theoretic models of money, but at the same time, they revealed the fundamental problem of indeterminacy associated with these models.[3]

A crucial reason for this indeterminacy is that, as we mentioned above, the probability of matching is exogenous, and therefore, say, a seller cannot attract more customers by lowering its price even if there is excess supply in the market. The existence of marketplaces allows the possibility of changes in the probability of matching so that the price-adjustment mechanism works.

[2] Iwai's (1996) trading zone model is also related to the present article. Given the number of commodities n, each agent chooses one of $n(n-1)/2$ trading zones in which random matching takes place. The matching probability in a certain zone is proportional to the number of agents visiting the zone. Each agent can hold one unit of indivisible commodities storable with some costs. He examined which commodity becomes a medium of exchange.

[3] Different from the literature on labor search, Kamiya and Shimizu (2002) showed the real indeterminacy in a wide range of money search models.

Models with endogenous matching are not new in other fields. Directed search models in labor economics and local interaction models in evolutionary game theory both have dealt with endogenous matching.[4] Among them, the closest to the present article in terms of formulation of matching technology are Moen (1997) in labor economics, and Mailath et al. (1997) in evolutionary game theory. Moen constructed a model in which firms with different wage offers are assigned to different submarkets, and workers choose a submarket to be matched with a firm in the same submarket. Mailath et al. considered a situation in which players decide to go to certain locations, in which they are randomly matched to play a prespecified game. The present article adopts a specific matching technology in order to cope with additional complexity due to changes in agents' money holdings. This complexity is common to search theoretic models of money.[5]

The rest of the article is organized as follows. Section 2 presents our framework. Section 3 defines and characterizes stationary equilibria, which is followed by the welfare analysis. Section 4 identifies the essentially unique evolutionarily stable equilibrium. Section 5 concludes the article.

2. MODEL

We consider an infinite repetition of an economy that is inhabited by a continuum of agents with measure one. Time is discrete and indexed by $t = 1, 2, \ldots$ There are K types of agents, $1, \ldots, K$. The generic element is denoted by k. Assume $K \geq 3$. The mass of each type is $1/K$. There are K types of commodity goods, $1, \ldots, K$, and good 0, or fiat money. An agent of type k obtains utility $u > 0$ if he consumes one unit of good k. Every commodity good is perishable and indivisible. He can produce at most one unit of good $k + 1 \pmod K$ in each period. Its production cost is zero. We assume that agents do not produce goods unless they expect to sell the goods with a positive probability.[6] On the other hand, fiat money is nonperishable and divisible. Each agent can hold any amount of fiat money with no cost. We assume that agents immediately discard fiat money that they never expect to use.[7] M is the total nominal stock of fiat money.

There are countably many *marketplaces*, indexed by $z = 1, 2, 3, \ldots$ Each marketplace has two physically identical sides, A and B.

Each period consists of the following four stages.

Stage 1: Agents simultaneously decide whether to produce goods or not and choose a marketplace and one of its sides.

[4] The literature on directed search is extensive, including Peters (1991).

[5] Recently Corbae et al. (2003) wrote an article on monetary economics with endogenous matching. They considered a situation in which agents are matched to form pairs in the way formalized by Gale and Shapley (1962), i.e., the concept of core is used to find optimal matching. In this sense, matching is not only endogenous but also nonrandom. They mixed cooperative and noncooperative concepts in their analysis in the sense that core is used within each period, whereas across periods, a noncooperative equilibrium concept is adopted.

[6] This assumption corresponds to the existence of an infinitesimal production cost. It eliminates equilibria in which gift giving occurs between anonymous agents.

[7] This assumption corresponds to the existence of an infinitesimal holding cost of fiat money.

Stage 2: In each marketplace, a random matching takes place. The matching technology is frictionless, though the long side is rationed. Also, the matching is uniform. Formally speaking, if the measure of the agents visiting side A is n_A, that of the agents visiting side B is n_B, and among those visiting side B are the agents who belong to set S with its measure being n_S, then the probability that an agent visiting side A meets someone in S is $\min\{n_S/n_A, n_S/n_B\}$.[8]

Stage 3: If a type-k agent and a type-$(k+1)$ (mod K) agent are matched, the type-k agent offers a price p_S, and the type-$(k+1)$ (mod K) agent bids a price p_B. The type of each agent is observable to his partner, but not his money holdings. We assume that $p_S = \infty$ if the type-k agent did not produce his good at Stage 1. No trade takes place in any other type of matching, and in such a case, agents do not make any further move, i.e., they skip Stage 4.

Stage 4: If $p_S \leq p_B$, then the type-k agent exchanges his good for p_S units of fiat money, and the type-$(k+1)$ (mod K) agent exchanges p_S units of fiat money for the good, and consumes it.[9]

From now on, we say "a seller meets a buyer" when a type-k agent meets a type-$(k+1)$ (mod K) agent.

The subsequent analysis uses Markov strategies, according to which actions depend only on the current money holdings of the agent in question.[10] Formally, a *Markov strategy* is defined to be a triple $\sigma = (\lambda, o, \beta)$, where

(i) $\lambda : \mathbb{R}_+ \to \mathbb{N} \times \{A, B\}$: a *location strategy*;
(ii) $o : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{\infty\}$: an *offer strategy*; and
(iii) $\beta : \mathbb{R}_+ \to \mathbb{R}_+$: a *bidding strategy*.

In this expression, $\lambda(\eta) = (z, s)$ implies that the agent who takes λ and holds η units of money chooses side s of marketplace z. Production decisions are reflected in offer strategies, i.e., $o(\eta) = \infty$ implies that the agent does not produce, whereas $o(\eta) = p \leq \infty$ implies that the agent produces a good and offers p if he meets a buyer. We assume $\beta(\eta) \leq \eta$, i.e., the buyer cannot bid beyond his current money holdings. The set of all Markov strategies is denoted by Σ. In the sequel, we allow deviating agents to take full-fledged strategies. A full-fledged strategy is a function from the set of the entire personal histories, into the set of appropriate actions.[11] The set of all strategies is denoted by $\bar{\Sigma}$.

[8] Although we can construct a one-to-one and a onto mapping between two sets of agents with different measures, we assume that rationing still occurs if the measure of one set is different from that of the other.

[9] The subsequent analysis will not be affected at all even if we change the rule on which price to use as long as the price is between p_S and p_B.

[10] We use the word "Markov" more restrictively than used in some other contexts in the sense that Markov strategy in our definition is independent of the current location and the current distribution of other agents' money holdings. However, even if such an alternative definition is adopted, the subsequent results remain unchanged.

[11] A personal history contains his past transaction records, especially the current money holdings and the type of the current trade partner, and some observable aggregate data. We do not specify which aggregate data are observable since it does not affect the subsequent analysis.

Moreover, we impose symmetry across types on Markov strategies in the subsequent arguments.[12] However, we allow different agents of the same type to take different strategies. Henceforth, we represent a symmetric strategy profile by the strategy of type-k agents. For example, for a location strategy λ, $\lambda(\eta) = (1, A)$ means that the agents of all types with money holdings η visit $(1, A)$, and $\lambda(\eta) = (K + k + 1, B)$ means that type-k agents with η visit side B of marketplace $K + (k + 1 \pmod{K})$, type-$(k + 1)$ agents with η visit side B of marketplace $K + (k + 2 \pmod{K})$, and so on.

We denote by μ a distribution on money holdings and strategies: $\mu(\{\eta\}; \{\sigma\})$ is the fraction of the agents who take σ and hold η units of money, which we write $\mu(\eta; \sigma)$ whenever it causes no confusion. Notice that we have extended the notion of symmetry, imposing it on distributions. Given μ, μ_Σ is its marginal distribution on strategies, i.e., $\mu_\Sigma(\Sigma') \stackrel{\text{def}}{=} \mu(\mathbb{R}_+; \Sigma')$ is the fraction of the agents taking strategies in $\Sigma' \subset \Sigma$. Similarly, μ_H is its marginal distribution on money holdings.

The transition of an agent's money holdings η is straightforward. Suppose that the agent takes σ. If he meets a seller with $(\sigma', \eta') = ((\lambda', o', \beta'), \eta')$, and if $\beta(\eta) \geq o'(\eta')$, then his money holdings become $\eta - o'(\eta')$. If, on the other hand, the agent meets a buyer with $(\sigma', \eta') = ((\lambda', o', \beta'), \eta')$, and if $\beta'(\eta') \geq o(\eta)$, then his money holdings become $\eta + o(\eta)$. Otherwise, η remains unchanged.

Given t, each agent tries to maximize the discounted average of future stage payoffs, i.e.,

$$E\left[(1-\delta)\sum_{\tau=t}^{\infty}\delta^{\tau-t}u_\tau \;\middle|\; \Omega_t\right],$$

where $\delta \in (0, 1)$ is a common discount factor, u_τ is u if he obtains his consumption good at period τ and zero otherwise, and Ω_t is the information available at period t. In particular, we denote the above expression by $V^t(\sigma, \eta^t; \mu^t)$ if he takes σ with η^t, money holdings at period t, and the distribution at period t is μ^t. Hereafter, we suppress superscripts ts to write $V(\sigma, \eta; \mu)$ whenever it causes no confusion.

3. STATIONARY EQUILIBRIA

3.1. *Equilibrium Concept.* To begin with, we define stationary distribution. A distribution μ is said to be *stationary* if μ is transformed into μ when all the agents do not revise their strategies, and their money holdings follow the above transition rule.[13] We are now in the position to define our equilibrium concept.

[12] We call a Markov strategy profile $\sigma = (\sigma^1, \ldots, \sigma^K)$ *symmetric* if, when all type-k agents take σ^k and money holdings distributions of all types are identical, the probability that a type-k agent is matched with a type-$(k + i) \pmod{K}$ agent is the same as the probability that a type-$(k + 1) \pmod{K}$ agent is matched with a type-$(k + i + 1) \pmod{K}$ agent for any $i = 1, \ldots, K - 1$ and offer prices and bid prices are common.

[13] This definition implies that agents never discard money on the path of a stationary distribution.

DEFINITION 1. A stationary distribution μ is a *stationary* (*symmetric Markov perfect*) *equilibrium* iff

(i) only Markov strategies are taken, i.e., $\mu_\Sigma(\Sigma) = 1$,
(ii) on any equilibrium path, at least K marketplaces are empty,
(iii) no agent has an incentive to deviate with any money holdings, i.e., for all σ in the support of μ_Σ, for all $\tilde{\sigma} \in \bar{\Sigma}$, and all $\eta \in \mathbb{R}_+$,

$$V(\sigma, \eta; \mu) \geq V(\tilde{\sigma}, \eta; \mu)$$

This definition implies that an equilibrium should satisfy a requirement similar to subgame perfection with regard to money holdings. Note that in any Markov perfect equilibrium, the agent produces his production good if and only if he expects to meet a buyer whose bid is no less than his intended offer with a positive probability.

3.2. *Equilibria with No Specialization.* First of all, there exist equilibria in which only one marketplace is active. These equilibria correspond to those found in the complete random matching model of Green and Zhou (1998). Suppose that all the agents go to the same marketplace, and moreover, they are evenly distributed between sides A and B, and offer and bid a common price p, if possible. Then no agent has an incentive to change his location strategy since a visit to another marketplace gives him no utility, and the situation is exactly the same if he visits the other side. It is also verified that they have no incentive to change their offer and bidding strategies if the price p is sufficiently high.

In this class of equilibria, each agent may end up in selling a good even if he has a sufficient amount of money to buy his consumption good. Consequently, the support of the distribution of money holdings is a countable set.[14] There is a continuum of equilibria with different price and welfare levels.

3.3. *Single-Price Equilibria with Complete Specialization.* In this subsection and the next, we consider equilibria with *complete specialization of marketplaces*, i.e., in each active marketplace, only a single type of good is traded, and the sellers of that good go to one side and the buyers go to the other side.

Given a price level $p > 0$, a *single-price equilibrium with complete specialization and with* p (henceforth, a p-SPE) is a stationary equilibrium in which every good is traded at p. The canonical p-SPE, μ_p, is defined as follows:

(i) $\mu_p(0; \sigma_p) = 1 - m$,
(ii) $\mu_p(p; \sigma_p) = m$,

where $m = M/p$ is the total real stock of fiat money, and $\sigma_p = (\lambda_p, o_p, \beta_p)$ is a Markov strategy such that

(i) $\lambda_p(\eta) = \begin{cases} (k, B) & \text{if } \eta \geq p \\ (k+1, A) & \text{if } \eta < p \end{cases}$

[14] The reader should refer to Green and Zhou for the formal description of this class of equilibria.

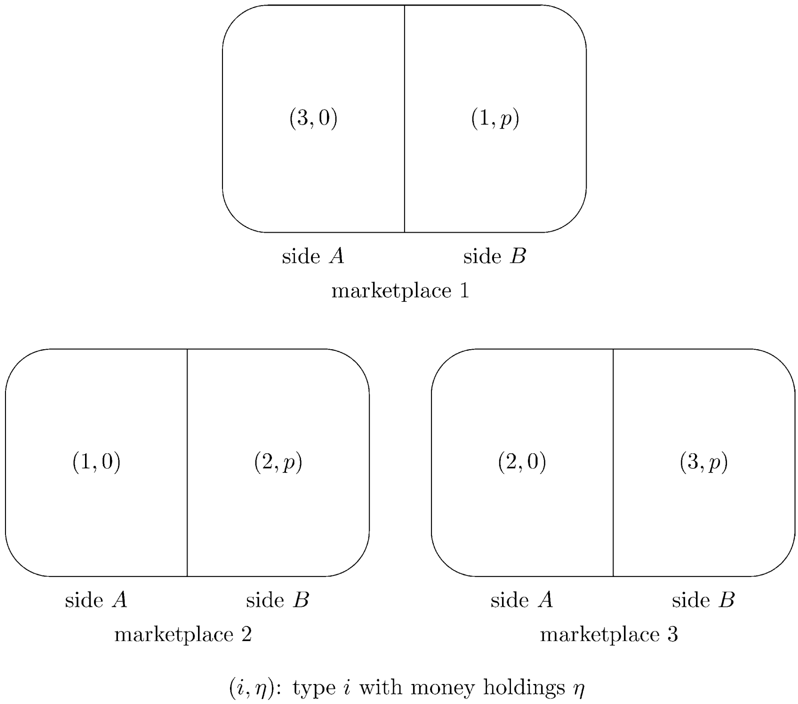

FIGURE 1

WHO GOES WHERE IN A p-SPE; $K = 3$

(ii) $o_p(\eta) = \begin{cases} p & \text{if } \eta < p \\ \infty & \text{if } \eta \geq p \end{cases}$

(iii) $\beta_p(\eta) = \begin{cases} p & \text{if } \eta \geq p \\ \eta & \text{if } \eta < p \end{cases}$

In short, non-money holders go to side B of an appropriate marketplace to meet the buyers of their production goods, whereas money holders, who have p units of money, go to side A of an appropriate marketplace to meet the sellers of their consumption goods. Figure 1 illustrates who goes where on the equilibrium path of the canonical p-SPE, and Figure 2 gives the transition of each agent's money holdings. Note that μ_p is stationary. We now state and prove that these canonical distributions are indeed stationary equilibria.

THEOREM 1. *For all $p > M$, and all δ, the canonical p-SPE μ_p is a stationary equilibrium.*

PROOF. We denote $V(\sigma_p, \ell p; \mu_p)$ by V_ℓ for $\ell \in \mathbb{N}_+$.[15] We divide the proof into two cases:

[15] For general η, we have $(\sigma_p, \eta; \mu_p) = V(\sigma_p, \lceil \frac{\eta}{p} \rceil p; \mu_p)$ where $\lceil x \rceil$ is the integer part of x.

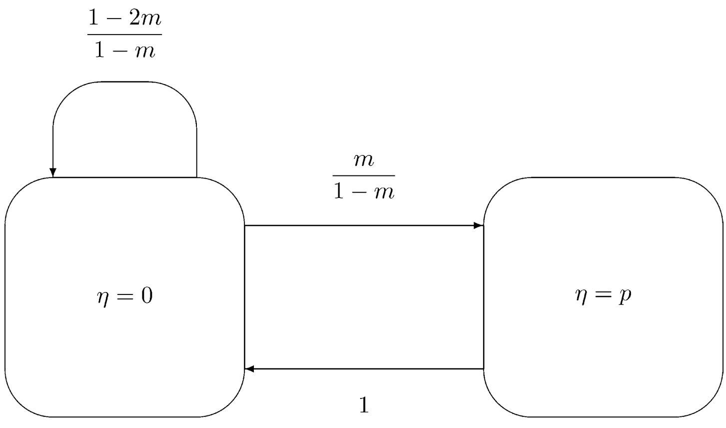

FIGURE 2

TRANSITION OF AN AGENT IN A p-SPE: CASE OF $m = \frac{M}{P} < \frac{1}{2}$

Case 1: $m \geq \frac{1}{2}$.

In this situation, the sellers are lying on the short side. Therefore, the buyers are rationed (except in the case of $m = 1/2$), whereas the sellers are not. Let $r = \frac{1-m}{m}$. Then we have the following value functions:

$$V_0 = \delta V_1,$$
$$V_\ell = r((1-\delta)u + \delta V_{\ell-1}) + (1-r)\delta V_\ell, \quad \ell \geq 1$$

Solving this system of equations, we obtain

$$V_\ell = r\left(1 - \left[\frac{\delta r}{1-\delta+\delta r}\right]^{\ell-1} \frac{\delta r}{1+\delta r}\right) u, \quad \ell \geq 0$$

The only incentive compatibility condition that we need to verify is the one under which no money holder becomes a seller. It is given by

$$V_\ell \geq \delta V_{\ell+1}, \quad \ell \geq 1$$

which is equivalent to

$$(1-\delta)r\left(1 - \left[\frac{\delta r}{1-\delta+\delta r}\right]^{\ell}\right) u \geq 0, \quad \ell \geq 1$$

Since we have $\delta < 1$ and $r > 0$, this inequality holds.

Case 2: $m < \frac{1}{2}$.

In this situation, the buyers are lying on the short side. Therefore, the sellers are rationed, whereas the buyers are not. Let $r = \frac{m}{1-m}$. Then, we have the following value functions:

$$V_0 = r\delta V_1 + (1-r)\delta V_0$$
$$V_\ell = (1-\delta)u + \delta V_{\ell-1}, \quad \ell \geq 1$$

Solving this system of equations, we obtain

$$V_\ell = \left(1 - \delta^\ell \frac{1}{1+\delta r}\right) u, \quad \ell \geq 0$$

The only incentive compatibility condition that we need to verify is the one under which no money holder becomes a seller. It is given by

$$V_\ell \geq r\delta V_{\ell+1} + (1-r)\delta V_\ell, \quad \ell \geq 1$$

which is equivalent to

$$(1-\delta)(1-\delta^\ell)u \geq 0, \quad \ell \geq 1$$

Thus, this condition holds. ■

3.4. *Dual-Price Equilibria with Complete Specialization.* In the present model, since there can be more than one marketplace for a specific transaction, there is no a priori reason that a single price prevails. In fact, there are equilibria in which the same goods are traded at different prices. The simplest class of such equilibria is given below.

A *dual-price equilibrium* is a stationary equilibrium in which each good is traded at two different prices. In particular, given $p > 0$ and an integer $n \geq 2$, we consider a *dual-price equilibrium with complete specialization and with two price levels p and np* (henceforth, we call it (p, np)-DPE) which has two classes of active marketplaces, *low-price* marketplaces, in which goods are traded at price p, and *high-price* marketplaces, in which goods are traded at price np.

In such an equilibrium, the low-price marketplaces are in excess demand; for if not, all the buyers go there. On the other hand, the high-price marketplaces are in excess supply; for if not, all the sellers go there.[16]

The canonical (p, np)-DPE, $\mu_{(p,np)}$, is given in Table 1 together with the following description. In Table 1, the entry for (σ, η) $(\sigma = \sigma_1, \sigma_n;\ \eta = 0, p, np)$ is $\mu_{(p,np)}(\sigma, \eta)$, e.g., $\mu_{(p,np)}(\sigma_n, 0) = h_{0n}$. Their values are determined in the sequel.

[16] In labor economics, Montgomery (1991) shows the existence of equilibria with wage dispersion by using a similar idea.

TABLE 1
THE CANONICAL (p, np)-DPE

$\sigma\backslash\eta$	0	p	np
σ_1	h_{01}	h_1	0
σ_n	h_{0n}	0	h_n

In this distribution, $\sigma_i = (\lambda_i, o_i, \beta)(i = 1, n)$ is given by

$$\text{(i)}\ \ \lambda_1(\eta) = \begin{cases} (K+k, B) & \text{if } \eta \geq np \\ (k+1, A) & \text{if } np > \eta > \ell^* p \\ (k, B) & \text{if } \ell^* p \geq \eta \geq p \\ (k+1, A) & \text{if } \eta < p \end{cases}$$

$$\text{(ii)}\ \ \lambda_n(\eta) = \begin{cases} (K+k, B) & \text{if } \eta \geq np \\ (k+1, A) & \text{if } np > \eta > \ell^* p \\ (k, B) & \text{if } \ell^* p \geq \eta \geq p \\ (K+k+1, A) & \text{if } \eta < p \end{cases}$$

$$\text{(iii)}\ \ o_1(\eta) = \begin{cases} p & \text{if } \eta < p \text{ or } \ell^* p < \eta < np \\ \infty & \text{otherwise} \end{cases}$$

$$\text{(iv)}\ \ o_n(\eta) = \begin{cases} np & \text{if } \eta < p \\ p & \text{if } \ell^* p < \eta < np \\ \infty & \text{otherwise} \end{cases}$$

$$\text{(v)}\ \ \beta(\eta) = \begin{cases} np & \text{if } \eta \geq np \\ p & \text{if } np > \eta \geq p \\ \eta & \text{if } \eta < p \end{cases}$$

for some integer ℓ^* such that $n > \ell^* \geq 1$. Given p, n, and δ, the value of ℓ^* is determined in Appendix A so as to satisfy agents' incentive constraints off the equilibrium path. In this description, marketplaces $1, \ldots, K$ correspond to the low-price marketplaces, whereas $K+1, \ldots, 2K$ correspond to the high-price marketplaces.

The above strategies look complicated partly because they specify agents' behavior not only on the equilibrium path, but also off the equilibrium path. On the equilibrium path, we have

$$\sigma_1(0) = ((k+1, A), p, \cdot)$$
$$\sigma_1(p) = ((k, B), \cdot, p)$$

and

$$\sigma_n(0) = ((K+k+1, A), np, \cdot)$$
$$\sigma_n(np) = ((K+k, B), \cdot, np)$$

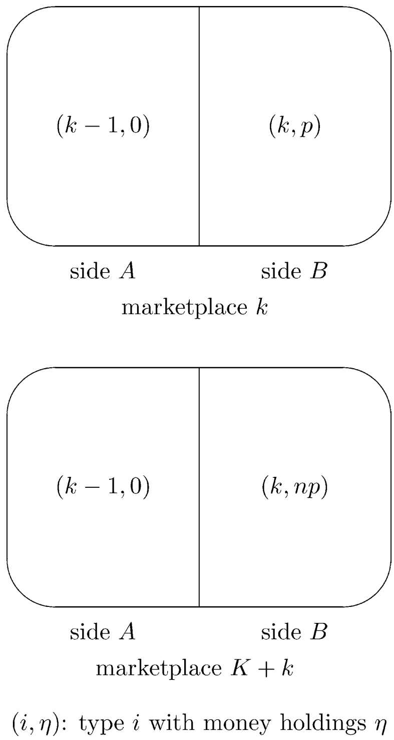

FIGURE 3

THE CANONICAL (p, np)-DPE

There are two groups of agents in the marketplace. The first group consists of the agents who take σ_1, using the low-price marketplaces, whereas the second group consists of those who take σ_n, using the high-price marketplaces. Each agent in both groups probabilistically alternates between a buyer and a seller. Figure 3 shows where agents go in this equilibrium, and Figure 4 gives the transition of each agent's money holdings. Some agents are rationed and stay in the same state, which is omitted in the figure.

We show that the canonical DPE is a stationary equilibrium for all δ if $r = 1/n$. We give the proof in Appendix A.

When $r = 1/n$, we have

$$\frac{M}{p} = h_1 + nh_n = \frac{n}{n+1}$$

so the low price is determined uniquely as $\frac{1}{n}(n+1)M$.

THEOREM 2. *For all integers $n \geq 2$, and all δ, the canonical $(\frac{1}{N}(n+1)M$, $(n+1)M)$-DPE $\mu_{(\frac{1}{n}(n+1)M,(n+1)M)}$ is a stationary equilibrium.*

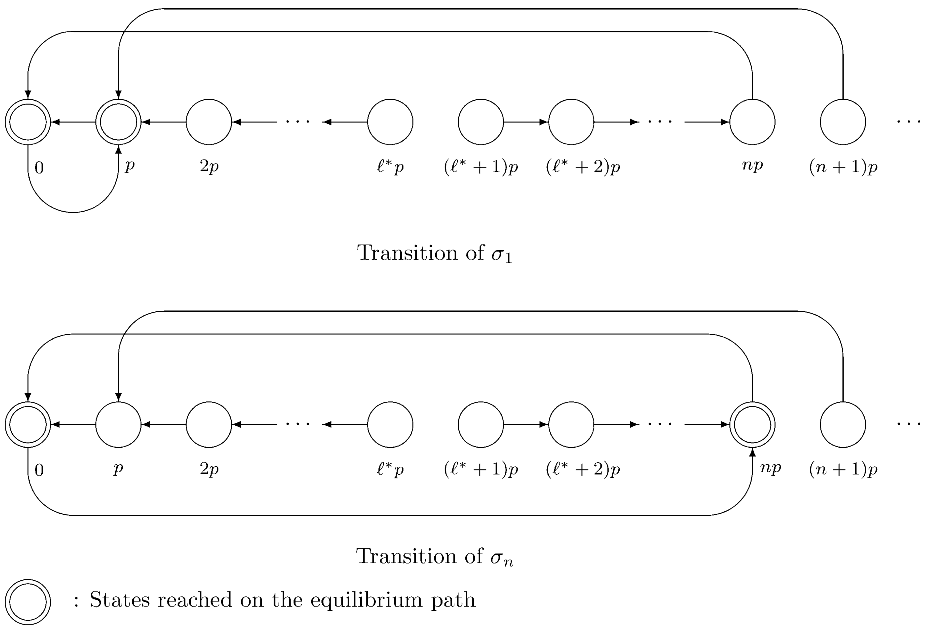

FIGURE 4

TRANSITIONS IN A (p, np)-DPE

3.5. *Welfare Analysis.* This subsection examines the welfare of various stationary equilibria. We define

$$V(\mu, \hat{\mu}) \overset{\text{def}}{=} \int_{\bar{\Sigma}\times\mathbb{R}_+} V(\sigma, \eta; \hat{\mu})\, d\mu(\sigma, \eta)$$

and

$$W(\mu) \overset{\text{def}}{=} V(\mu, \mu)$$

We regard W as the welfare of the economy. In other words, welfare is assumed to be measured by the average value of all the agents. Moreover, W is used as the criterion of efficiency. Formally, we call a stationary distribution μ *efficient* if μ maximizes $W(\mu)$. The maximum value of $W(\mu)$ is $\frac{1}{2}u$ due to the assumptions on production and matching technologies, according to which one cannot produce and consume in the same period, and therefore, in each period, at most a half of the entire population obtains u.

First, if only one marketplace is used as in the case of no specialization, then the probability of a type-k agent being matched with an agent of either type $k-1$ or $k+1$ is at most $2/K$. Furthermore, at most only a half of the matched agents can obtain u. Therefore, the welfare for this case is at most $\frac{1}{K}u$.

Next, we calculate the welfare of canonical single- and dual-price equilibria with complete specialization.

(i) p-SPE:

(a) *Case 1*: $m \geq \frac{1}{2}$, i.e., $p \leq 2M$.

$$W(\mu_p) = (1-m)V_0 + mV_1$$
$$= (1-m)u$$

(b) *Case 2*: $m \leq \frac{1}{2}$, i.e., $p \geq 2M$.

$$W(\mu_p) = mu$$

In particular, $W(\mu_p)$ attains the maximum value when $p = 2M$, i.e.,

$$W(\mu_{2M}) = \frac{1}{2}u$$

(ii) $(\frac{1}{n}(n+1)M, (n+1)M)$-DPE:

$$W\big(\mu_{(\frac{1}{n}(n+1)M,(n+1)M)}\big) = (h_{01} + h_{0n})V_0 + h_1V_1 + h_nV_n$$
$$= \frac{1}{n+1}u$$

To sum up, the canonical $2M$-SPE is efficient, whereas the other canonical single-price equilibria and all the canonical dual-price equilibria are inefficient.

THEOREM 3. *The canonical* $2M$*-SPE is efficient.*

Without any qualification, the reverse of the above statement is not true.[17] However, it is verified that any efficient stationary equilibrium that is robust to

[17] Let μ be defined as follows:
(i) $\mu(0;\sigma) = \frac{1}{2}r$
(ii) $\mu(p;\sigma) = \frac{1}{2}$
(iii) $\mu(3p;\sigma) = \frac{1}{2}(1-r)$
where $\sigma = (\lambda, o, \beta)$ is a Markov strategy such that

(i) $\lambda(\eta) = \begin{cases} (k-1, A) & \text{if } \eta \geq 2p \\ (k, B) & \text{if } 2p > \eta \geq p \\ (k+1, A) & \text{if } \eta < p \end{cases}$

(ii) $o(\eta) = \begin{cases} 2p & \text{if } 2p > \eta \geq p \\ p & \text{if } \eta < p \\ \infty & \text{otherwise} \end{cases}$

(iii) $\beta(\eta) = \begin{cases} 2p & \text{if } \eta \geq 2p \\ p & \text{if } 2p > \eta \geq p \\ \eta & \text{if } \eta < p \end{cases}$

Then it is verified that, given δ, μ is a stationary equilibrium if r is sufficiently small and efficient.

discount factor is $2M$-SPE. The following is the formal result. Its proof is tedious and therefore omitted. Interested readers may refer to the discussion paper version of the present article (Matsui and Shimizu, 2001).

THEOREM 4. *If*

(*i*) *there exists* $\hat{\delta}$ *such that* μ *is a stationary equilibrium for any* $\delta \in (\hat{\delta}, 1)$, *and*
(*ii*) μ *is efficient*,

then μ *is a* $2M$-SPE.

4. EVOLUTIONARY STABILITY

4.1. *Stability Concept.* This section examines the evolutionary stability of stationary distributions. In order for a distribution to be evolutionarily stable, we require that the original population fare at least as well as any small group of mutants in the long run provided that the agents are sufficiently patient. The formal definition is given below.

DEFINITION 2. A distribution μ is said to be *evolutionarily stable* if

(i) there exists $\hat{\delta} \in (0, 1)$ such that μ is a stationary (symmetric Markov perfect) equilibrium for all $\delta \in (\hat{\delta}, 1)$, and
(ii) for all $\gamma > 0$, there exists $\bar{\delta} \in (0, 1)$ such that for all $\delta \in (\bar{\delta}, 1)$, there exists $\bar{\epsilon} > 0$ such that for all $\epsilon \in (0, \bar{\epsilon})$, and all $\tilde{\mu}$ such that $\tilde{\mu}_H = \mu_H$, the following equation holds:

$$V(\mu, (1-\epsilon)\mu + \epsilon\tilde{\mu}) + \gamma > V(\tilde{\mu}, (1-\epsilon)\mu + \epsilon\tilde{\mu})$$

This definition is similar to the definition of evolutionarily stable strategy (Maynard Smith and Price, 1973). However, the present definition has three differences from it. The first, and the most significant, difference is that agents are patient in the present model. Therefore, comparison between the original population and the mutants is made in terms of discounted average payoffs instead of instantaneous payoffs. In calculating these values, it is assumed that the fraction of the mutants remains "small." Incorporating a possibility of a growing population of mutants complicates the analysis, which we do not deal with in the present article.

Second, the original population survives even if it is "a little" worse than the mutants. In fact, in the present definition, mutants cannot invade the population unless they fare better than the original population in the long run. This reflects the idea that a "small" one-shot gain is considered negligible, and only constant gains over time would be counted as a threat to the original population.

Finally, the mutant population may include "dummies," i.e., those who do not actually mutate. This way we save some cumbersome notation.

4.2. *Result.* This subsection shows that the class of the efficient single-price equilibria is the only evolutionarily stable distributions. To begin with, the following theorem shows that any inefficient equilibrium cannot be evolutionarily stable.

THEOREM 5. *Every evolutionarily stable equilibrium is efficient, i.e., if μ is an evolutionarily stable equilibrium, then $W(\mu) = \frac{1}{2}u$ holds.*

For the detail of the proof, see Appendix B.

To see the intuition behind the result, consider the following example. Suppose that the economy is trapped in the canonical $4M$-SPE, μ_{4M}. Then there is an excess supply due to a high price. Suppose now that a small fraction ϵ of money holders mutate at the beginning of period 1 to take strategy $\tilde{\sigma} = (\tilde{\lambda}, \tilde{o}, \tilde{\beta})$ given by

$$\text{(i)}\ \tilde{\lambda}(\eta) = \begin{cases} (K+k, B) & \text{if } \eta \geq 2M \\ (K+k+1, A) & \text{otherwise} \end{cases}$$

$$\text{(ii)}\ \tilde{o}(\eta) = \begin{cases} 2M & \text{if } \eta < 2M \\ \infty & \text{otherwise} \end{cases}$$

$$\text{(iii)}\ \tilde{\beta}(\eta) = \begin{cases} 2M & \text{if } \eta \geq 2M \\ \eta & \text{otherwise} \end{cases}$$

The mutant distribution is denoted by $\tilde{\mu}$. It is given by

$$\tilde{\mu}(4M; \tilde{\sigma}) = \frac{1}{4}$$

$$\tilde{\mu}(0; \tilde{\sigma}) = \frac{3}{4}$$

Let $\hat{\mu}^1 = (1-\epsilon)\mu_{4M} + \epsilon\tilde{\mu}$ be the entire distribution after the mutation at period 1. Since the mutants do not interact with the agents in the original population, the distribution of the money holdings of the mutants evolves independently of the original population. Note that the distribution of the original population remains unchanged. On the other hand, the mutant distribution at period 2 changes from $\tilde{\mu}$ to $\tilde{\mu}^2$, which is given by

$$\tilde{\mu}^2(2M; \tilde{\sigma}) = \frac{1}{2}$$

$$\tilde{\mu}^2(0; \tilde{\sigma}) = \frac{1}{2}$$

It remains $\tilde{\mu}^2$ thereafter. The entire distribution after period 2 is given by

$$\hat{\mu}^t = (1-\epsilon)\mu_{4M} + \epsilon\tilde{\mu}^2 \quad t \geq 2$$

Thus, we obtain

$$V(\mu_{4M}, \hat{\mu}) = \frac{1}{4}u$$

and

$$V(\tilde{\mu}, \hat{\mu}) = \frac{1+\delta}{4}u$$

which implies $V(\tilde{\mu}, (1-\epsilon)\mu_{4M} + \epsilon\tilde{\mu}) > V(\mu_{4M}, (1-\epsilon)\mu_{4M} + \epsilon\tilde{\mu}) + \gamma$ for a sufficiently small γ and a sufficiently large δ. Hence, the canonical $4M$-SPE is not immune to the invasion of these mutants.

Theorem 5 together with Theorem 4 implies the following corollary. It states that any evolutionarily stable outcome is not only efficient but also a $2M$-SPE.

COROLLARY 1. *If μ is evolutionarily stable, then μ is a $2M$-SPE.*

Our next result states that the canonical $2M$-SPE is evolutionarily stable. Also this result implies the existence of evolutionarily stable equilibrium.

THEOREM 6. *The canonical $2M$-SPE is evolutionarily stable.*

For the detail of the proof, see Appendix C. Here, we draw a sketch of the proof. First of all, it is verified, using Theorem 1, that the canonical $2M$-SPE satisfies condition (i) of Definition 2.

Note that, without mutation, every agent in the original population deterministically alternates producing with consuming, and obtains $\frac{1}{2}u$ on average. It is verified that, as long as the size ϵ of mutants is sufficiently small, every agent in the original population continues to keep the same trading pattern for a long period of time with a probability close to one, and therefore, obtains a value close to $\frac{1}{2}u$ (Lemma 1 in Appendix C).

On the other hand, if the mutant population does not interact with the original population, they obtain at most $\frac{1}{2}u$. In order to increase the average value, the mutants need to increase the frequency of consumption, which requires that they buy goods from agents in the original population more often than sell goods to them. However, this cannot be done infinitely often on average since the transaction price is fixed at $2M$ when they trade with the original population, and each mutant's initial money holdings are finite. Thus, if the discount factor δ is sufficiently large, any group of mutants obtain the average value of no more than $\frac{1}{2}u$ (Equation (C.6)). Lemma 1 and Equation (C.6) imply that the canonical $2M$-SPE satisfies condition (ii) of Definition 2.

5. CONCLUDING REMARKS

We have analyzed a search theoretic model of money with marketplaces. We have adopted two solution concepts in the main analysis, stationary equilibrium and evolutionarily stable distribution.

We have viewed the equilibrium approach as a proxy for the short-run situation. In some equilibria, one marketplace is used for all transactions, whereas in others, marketplaces are specialized; only one commodity good is traded in each active marketplace. There is a continuum of equilibria with different price and welfare levels. There also exist dual-price equilibria, in which the same good is traded at two different prices.

We have adopted the evolutionary approach as a proxy for the long-run situation. In the long run, those who fare better than others survive, whereas those who do not do well shift from one strategy to a better one or disappear from the market. As a result, an efficient single-price equilibrium prevails. This evolutionary view of the market is found in, among others, Alchian (1950) and Friedman (1953).[18]

When destroying some equilibrium by mutation, mutants utilize some unused places to create a new market. This idea of using unused places for deviation is similar to the ideas of the secret hand shake in Robson (1990) and the cheap-talk in Matsui (1991). In Matsui, an unused message is sent to signal others that one is a new type. They take a more efficient strategy profile than before only if they both send this new message. In a similar manner, successful mutants of the present model choose a new marketplace to establish a more efficient trading pattern than before.[19]

To conclude the article, four remarks are in order. First, the restriction to Markov perfect equilibria plays a crucial role in the proof of the essential uniqueness result of the evolutionarily stable outcome. For example, Markov perfection excludes the possibility of punishment against mutants, e.g., sellers' price cut. Without the restriction, agents may take a strategy according to which they trail the mutants who go to a new place and behave as sellers so that the mutants cannot increase the probability of matching with buyers even if they cut the price. We do not think that this change in the results undermines our analysis. Rather, the lack of retaliation and punishment against price cut is an essential feature that makes the price-adjustment mechanism work, and the concept of Markov perfection expresses it in a simple form.

Second, the matching technology of the present model exhibits constant returns to scale, i.e., the matching probability depends only on the relative size of agents visiting side A and those visiting side B. Although it serves a benchmark, one may wonder how the results would change if the matching technology is that of increasing returns to scale, i.e., the larger the absolute size of a market place, the greater is the matching probability. In such a case, some results, especially, the one in Section 4 on evolution may be modified since creating a small new market may not pay off. Indeed, such a new market never appears if the degree of increasing returns to scale is too large as in Iwai (1996), in which the probability of matching goes to zero as the size of the market tends to zero. If, on the other hand, the scale economy exists but is not too large, then the further the price is away from $2M$, the more likely is the corresponding equilibrium to be destroyed by the mutants

[18] It should be noted that the assumption of simultaneous trials is made for the sake of analytical simplicity instead of a realistic description of changes in behavior. In reality, even if a seller cuts its price, it has to wait for a while to attract new customers. People gradually realize that there is the new seller who sells goods cheaper than other stores. Effectively speaking, when a seller tries a new marketplace with a new price, buyers need to visit the seller not necessarily right after the seller's trial, but only before it disappears from the market. The effect of a price cut will appear only gradually.

[19] See also Mailath et al. (1997).

creating a new market since a gain from a better seller–buyer ratio exceeds a loss caused by the effect of scale economy.

Third, we have analyzed the effects of evolutionary pressure without any specification of explicit dynamics. Although we have obtained the efficiency result, analyses with explicit dynamics would deepen our understanding of the process of price adjustment.

Fourth, in the discussion paper version of the present article (Matsui and Shimizu, 2001), the model is extended to an economy with multiple currencies, in which the effect of change of money supply when the current money stock is overissued differs from when it is underissued. To further study such an economy, we may introduce issuers explicitly, examining their incentives to issue fiat money, and the way they interact with each other. We leave these studies for future research.

APPENDIX

A. *Proof of Theorem 2.* It suffices to show that there exists an integer ℓ^* $(1 \leq \ell^* < n)$ such that σ_1 and σ_n are the best responses to $\mu_{(p,np)}$ for any δ whenever $r = 1/n$. For this purpose, we introduce the following auxiliary strategies, value functions, and conditions:

Let $\tilde{r} = h_{01}/h_1$. Then let

$$\tilde{V}_0 = \delta \tilde{V}_1, \quad \text{and}$$

$$\tilde{V}_\ell = \tilde{r}((1-\delta)u + \delta \tilde{V}_{\ell-1}) + (1-\tilde{r})\delta \tilde{V}_\ell, \quad \ell \geq 1$$

This is an auxiliary value function. It corresponds to the auxiliary strategy according to which an agent uses the low-price marketplaces only, irrespective of the agent's money holdings. Also, let $\hat{r} = h_n/h_{0n}$. Then let

$$\hat{V}_0 = \hat{r}\delta \hat{V}_n + (1-\hat{r})\delta \hat{V}_0$$

$$\hat{V}_\ell = \delta^{n-\ell} \hat{V}_n, \quad 1 \leq \ell < n$$

$$\hat{V}_n = (1-\delta)u + \delta \hat{V}_0$$

This is another auxiliary value function. It corresponds to another auxiliary strategy. If one has money holdings less than p, he goes to the high-price marketplace as a seller. If he has money holdings greater than or equal to p but less than np, he goes to the low-price marketplace as a seller. If he has money holdings greater than or equal to np, he goes to the high-price marketplace as a buyer.

We choose ℓ^*, $\tilde{r}$, and $\hat{r}$ so that the following conditions are satisfied:

(C0) $\tilde{V}_0 = \hat{V}_0$.
(C1) $\tilde{V}_\ell \geq \hat{V}_\ell$ if $1 \leq \ell \leq \ell^*$.
(C2) $\tilde{V}_\ell \leq \hat{V}_\ell$ if $\ell^* < \ell \leq n$.

From the description of $\tilde{V}$s and (C1), we obtain

$$\tilde{V}_0 = \frac{\delta\tilde{r}}{1+\delta\tilde{r}}u = \frac{\delta\hat{r}}{1+\delta\hat{r}}u = \hat{V}_0 \tag{A.1}$$

Then we obtain $\tilde{r} = \hat{r}$. From now on, this common ratio is denoted by r. Then sequentially applying (A.1) to $\tilde{V}$s, we obtain

$$\tilde{V}_\ell = ru - \left[\frac{\delta r}{1-\delta+\delta r}\right]^{\ell-1} \frac{\delta r}{1+\delta r} ru, \quad \ell \geq 1 \tag{A.2}$$

Similarly, using the description of $\hat{V}$s and (A.1), we obtain

$$\hat{V}_\ell = \delta^{n-\ell}\frac{1-\delta+\delta r}{1+\delta r}u, \quad 1 \leq \ell \leq n \tag{A.3}$$

It is verified that $\tilde{V}_\ell$ is concave in ℓ, and that $\hat{V}_\ell$ is convex in ℓ. Therefore, if we prove that $\tilde{V}_1 \geq \hat{V}_1$ and $\tilde{V}_n \leq \hat{V}_n$ hold for some r, then there exists ℓ^* between 1 and n such that (C1) and (C2) hold. Indeed, it is verified that $\tilde{V}_1 \geq \hat{V}_1$ holds if $r \geq 1/n$. After tedious calculation, it is also verified that $\tilde{V}_n \leq \hat{V}_n$ holds if $r = 1/n$. Thus, if $r = 1/n$, then there exists ℓ^* between 1 and n such that (C1) and (C2) hold. Moreover, if δ is close to 1, then $\tilde{V}_1 \geq \hat{V}_1$ is approximately equivalent to $r \geq 1/n$, whereas $\tilde{V}_n \leq \hat{V}_n$ is approximately equivalent to $r \leq 1/n$ (still, both inequalities hold if $r = 1/n$). Thus, in the limit of δ going to 1, $r = 1/n$ necessarily holds.

Using $\tilde{V}$ and $\hat{V}$ defined above, we can write the value function in the canonical DPE as follows:

$$V_\ell^* = \begin{cases} \tilde{V}_0 = \hat{V}_0 & \text{if } \ell = 0 \\ \tilde{V}_\ell & \text{if } 1 \leq \ell \leq \ell^* \\ \hat{V}_\ell & \text{if } \ell^* < \ell \leq n \\ (1-\delta)u + \delta V_{\ell-n}^* & \text{if } \ell > n \end{cases}$$

Then we can show that all incentive compatibility conditions are satisfied whenever (C0)–(C2) are all satisfied. ■

B. *Proof of Theorem 5.* Note first that, in upsetting the original population, we do not necessarily use the most "plausible" mutants. Which mutants are plausible is often situation dependent, and the proof becomes too complicated to handle if we start addressing the plausibility of mutation. One example that we think is plausible is described in the main text.

Suppose that μ is an evolutionarily stable equilibrium and $W(\mu) < \frac{1}{2}u$. By the definition of the stationary equilibrium, we assume marketplaces $1, \ldots, K$ are empty without loss of generality.

We construct a mutant distribution $\tilde{\mu}$ such that any strategy in the support of $\tilde{\mu}_\Sigma$ visits only marketplaces $1, \ldots, K$. Then the payoff of the original population is not affected by the mutants and vice versa, i.e., we have

$$V(\mu, (1-\epsilon)\mu + \epsilon\tilde{\mu}) = W(\mu) < \frac{1}{2}u$$
$$V(\tilde{\mu}, (1-\epsilon)\mu + \epsilon\tilde{\mu}) = W(\tilde{\mu})$$

Thus, it suffices to show that for any small γ, if δ is large enough and ϵ is small enough, then $W(\tilde{\mu}) \geq \frac{1}{2}u - \gamma$.

We define $\bar{\eta}$ as follows:

$$\bar{\eta} \stackrel{\text{def}}{=} \inf\left\{\eta' \mid \mu_H(\{\eta \leq \eta'\}) \geq \frac{1}{2}\right\}$$

and divide the proof into two cases, $\bar{\eta} > 0$ and $\bar{\eta} = 0$. In both cases, the mutants choose new marketplaces and start a new transaction pattern with a different price p^*.

Case 1: $\bar{\eta} > 0$.

In this case, we let $p^* = \bar{\eta}$. We then partition the mutants into two sets S_1 and S_2 of equal sizes so that a mutant with money holdings η at the time of mutation belongs to S_1 if $\eta < p^*$, and S_2 if $\eta > p^*$, respectively (if there is a mass at $\eta = p^*$, then we divide them so that the sizes of the two sets are equal). Such two sets can be found by way of the definition of $p^* = \bar{\eta}$.

Suppose now that an agent holds η_0 units of money at the time of mutation and belongs to S_1. Then he ignores η_0, starts with producing his production good, and alternates production and consumption, trading goods at the price of p^*. On the other hand, if he belongs to S_2, which implies $\eta_0 > p^*$, then he ignores $\eta_0 - p^*$, and starts with consuming his consumption good with the rest of the behavior being the same as those in set S_1.

Formally, we define Markov strategies $\tilde{\sigma}^k_{1\eta_0} = (\tilde{\lambda}_{1\eta_0}, \tilde{o}, \tilde{\beta})$, $\tilde{\sigma}^k_{2\eta_0} = (\tilde{\lambda}_{2\eta_0}, \tilde{o}, \tilde{\beta})$ as follows:

(i) $\tilde{\lambda}_{1\eta_0}(\eta) = \begin{cases} (k, B) & \text{if } \eta \geq \eta_0 + p^* \\ (k+1, A) & \text{otherwise} \end{cases}$

(ii) $\tilde{\lambda}_{2\eta_0}(\eta) = \begin{cases} (k, B) & \text{if } \eta \geq \eta_0 \\ (k+1, A) & \text{otherwise} \end{cases}$

(iii) $\tilde{o}(\eta) = p^*$

(iv) $\tilde{\beta}(\eta) = \begin{cases} p^* & \text{if } \eta \geq p^* \\ \eta & \text{otherwise} \end{cases}$

A mutant in S_ℓ ($\ell = 1, 2$) takes $\sigma_{\ell\eta_0}$ if his money holdings are η_0 at the time of mutation.

In this mutant distribution, agents in S_1 and S_2 alternate their moves, and a mutant in S_1 is always matched with another in S_2 for transaction, and vice versa. Therefore, we have

$$W(\tilde{\mu}) = \frac{1}{2}u > W(\mu) \qquad \blacksquare$$

Case 2: $\bar{\eta} = 0$.

This case is equivalent to $\mu_H(\eta = 0) > \frac{1}{2}$. In order to construct a distribution in which the buyer–seller ratio is one to one, we need to distribute money from the "rich" to the "poor." Let N and $\tilde{\eta}$ be a pair of a positive integer and a positive number such that $\frac{1}{2N}$-fraction of agents have at least $\tilde{\eta}$ units of money. We can find such a pair since $M > 0$ holds. Among mutants, let these agents constitute set T_1, and let the rest of the mutants be in set T_2. Let $p^* = \tilde{\eta}/N$.

Take an agent in T_1 with the money holdings of η_0 at the time of mutation. Note $\eta_0 \geq Np^*$. His location strategy is

$$\tilde{\lambda}_{1\eta_0}(\eta) = \begin{cases} (k, B) & \text{if } \eta \geq \eta_0 - (N-1)p^* \\ (k+1, A) & \text{otherwise} \end{cases}$$

In other words, he acts as a buyer N times at the beginning as if his initial money holdings were Np^*.

Next, take an agent in T_2; she ignores her initial money holdings η_0, and starts her new life as a seller. To be precise, her location strategy is

$$\tilde{\lambda}_{2\eta_0}(\eta) = \begin{cases} (k, B) & \text{if } \eta \geq \eta_0 + p^* \\ (k+1, A) & \text{otherwise} \end{cases}$$

Every agent in T_1 and T_2 offers p^* and bids p^* if possible.

If these mutants take the above-mentioned strategies, then in N periods, the fraction of buyers becomes a half since the agents in T_1 repeat buying goods for N consecutive times, distributing money to those in T_2. From the Nth period on, they alternate between sellers and buyers. Thus, the average value of the mutants satisfies

$$V(\tilde{\mu}) \geq \delta^N \frac{1}{2}u$$

The right-hand side of the above inequality tends to $\frac{1}{2}u(>W(\mu))$ as δ goes to 1 (note that N does not depend on δ). Hence, the mutant distribution constructed above upsets the original population. $\blacksquare$

C. *Proof of Theorem 6.* Theorem 1 implies that the canonical $2M$-SPE satisfies condition (i) of Definition 2.

We denote the canonical $2M$-SPE by $\mu = \mu_{2M}$, and a candidate mutant distribution by $\tilde{\mu}$. To simplify notation, we denote $\hat{\mu} \stackrel{\text{def}}{=} (1-\epsilon)\mu + \epsilon\tilde{\mu}$. In the following, when we say "the tth period," we mean the tth period after mutation occurred.

LEMMA 1. *For all $\gamma > 0$, and all $\delta \in (0, 1)$, there exists $\bar{\epsilon}(\gamma, \delta) > 0$ such that for all $\epsilon \in (0, \bar{\epsilon}(\gamma, \delta))$, the following equation holds:*

$$V(\mu, \hat{\mu}) > \frac{1}{2}u - \gamma$$

PROOF. We first classify all the agents into two, infected and normal agents. *Infected agents* at the tth period ($t = 1, 2, \ldots$) are either mutants or those who have met infected agents at least once before or in the tth period. Those who are not infected agents are called *normal agents* at a given period. Note that a normal agent at period t is still normal in the next period if he meets another normal agent at that period. Let I_t ($t = 1, 2, \ldots$) be an upper bound of the measure of the infected agents at period t, and let N_t be a lower bound of the measure of normal agents at period t. Then we have the following inequalities:

$$N_1 \geq 1 - \frac{2K\epsilon}{1-\epsilon} \tag{C.1}$$

$$I_1 \leq \frac{2K\epsilon}{1-\epsilon} \tag{C.2}$$

$$N_{t+1} \geq \frac{N_t^2}{1+I_t} \tag{C.3}$$

$$I_{t+1} \leq 1 - N_{t+1} \tag{C.4}$$

N_{t+1} is calculated under the worst scenario, i.e., the one according to which all the infected agents go to one side of existing active marketplaces to maximize the number of infections. Now, let r_t be a lower bound of the probability that one is matched with a normal agent. We have

$$r_t \geq \frac{N_t}{1+I_t} \tag{C.5}$$

Next, we define V_i^t ($i = 0, 1$) as follows:

$$V_0^t = r_t\delta V_1^{t+1}$$
$$V_1^t = r_t\big((1-\delta)u + \delta V_0^{t+1}\big)$$

where V_i^t is a lower bound of the value that a normal agent with money holdings $2Mi$ obtains. If δ is sufficiently close to 1, then

$$V(\mu, \hat{\mu}) \geq \frac{1}{2}\left(V_0^1 + V_1^1\right)$$

From (C.1)–(C.5), we have

$$r_{t+1} \geq \frac{2}{2 - \Pi_{\tau=1}^{t} r_\tau} - 1$$

It is verified that for each $t = 1, 2, \ldots,$

$$r_t \underset{\epsilon \downarrow 0}{\rightarrow} 1$$

For all $T \geq 1$, we have

$$\begin{aligned}\frac{1}{2}\left(V_0^1 + V_1^1\right) &\geq \frac{1-\delta}{2} \sum_{t=1}^{T} \delta^{t-1} \Pi_{\tau=1}^{t} r_\tau u + \frac{1}{2} \delta \Pi_{\tau=1}^{T} r_\tau \left(V_0^{T+1} + V_1^{T+1}\right) \\ &> \frac{1}{2} \frac{r_T(1-\delta)\left(1 - \delta^T r_T^T\right)}{1 - \delta r_T} u\end{aligned}$$

where we make use of $r_{t+1} \leq r_t$. Since the last term converges to $\frac{1}{2}(1 - \delta^T)u$ as r_T approaches 1, we can take a sufficiently large T and find $\bar{\epsilon}(\gamma, \delta) > 0$ such that for all $\epsilon \in (0, \bar{\epsilon}(\gamma, \delta))$,

$$V(\mu, \hat{\mu}) > \frac{1}{2} u - \gamma$$ ■

It is also verified that the following inequality holds:

$$V(\tilde{\mu}, \hat{\mu}) \leq \left(\frac{1}{2} + \frac{1-\delta}{4}\right) u \tag{C.6}$$

Given a sufficiently small γ, let $\bar{\delta}$ be a positive number in $(1 - 2\frac{\gamma}{u}, 1)$. Next, given γ and $\delta \in (\bar{\delta}, 1)$, define $\bar{\epsilon}(\gamma/2, \delta)$ as in the proof of Lemma 1. Then, by Lemma 1 and (C.6), we have

$$V(\mu, \hat{\mu}) - V(\tilde{\mu}, \hat{\mu}) + \gamma > \left(\frac{1}{2}u - \frac{1}{2}\gamma\right) - \left(\frac{1}{2}u + \frac{1}{2}\gamma\right) + \gamma = 0$$

for all $\epsilon \in (0, \bar{\epsilon}(\gamma/2, \delta))$, and all $\tilde{\mu}$ satisfying the equation of condition (ii) of Definition 2, which concludes the proof of the theorem. ■

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