Economic Theory **24**, 907–932 (2004)
DOI: 10.1007/s00199-003-0465-1

**Economic Theory**



# Money, price dispersion and welfare*

**Brian Peterson**[1] **and Shouyong Shi**[2]

[1] Department of Economics, Indiana University, Bloomington, IN 47405, USA
(e-mail: bripeter@indiana.edu)
[2] Department of Economics, University of Toronto, 150 St. George Street,
Toronto, Ontario, CANADA, M5S 3G7 (e-mail: shouyong@chass.utoronto.ca)

Received: July 25, 2003; revised version: December 12, 2003

**Summary.** We introduce heterogeneous preferences into a tractable model of monetary search to generate price dispersion, and then examine the effects of money growth on price dispersion and welfare. With buyers' search intensity fixed, we find that money growth increases the range of (real) prices and lowers welfare as agents shift more of their consumption to less desirable goods. When buyers' search intensity is endogenous, multiple equilibria are possible. In the equilibrium with the highest welfare level, money growth reduces welfare and increases the range of prices, while having ambiguous effects on search intensity. However, there can be a welfare-inferior equilibrium in which an increase in money growth increases search intensity, increases welfare, and *reduces* the range of prices.

**Keywords and Phrases:** Inflation, Price dispersion, Search, Efficiency.

**JEL Classification Numbers:** E31, D60.

## 1 Introduction

We study the theoretical relationship between inflation, welfare and price dispersion. Empirically it has been found that higher rates of inflation increase price

* We thank Gabriele Camera, Aleksander Berentsen and an anonymous referee for useful suggestions. We have also received valuable comments from the participants of the workshop at Michigan State, the Purdue Conference on Monetary Theory (2003) and the Midwest Macro Meeting (Chicago, 2003). Shi gratefully acknowledges financial support from the Bank of Canada Fellowship and the Social Sciences and Humanities Research Council of Canada. The opinion expressed here is the authors' own and does not reflect the view of the Bank of Canada.

*Correspondence to*: S. Shi

dispersion.[1] This regularity is important for efficiency concerns because price dispersion may create an inefficiency by driving a wedge between marginal costs and marginal utilities. If inflation widens price dispersion, it could then exacerbate the inefficiency. Although many theoretical models have tried to capture this intuitive welfare consequence of inflation, they often lack a microfoundation for money that is necessary for a coherent welfare analysis.[2] In the current paper, we use the Kiyotaki-Wright [12] matching/search framework to provide a strong microfoundation for money. We discard their rigid assumptions of indivisible money and goods, and endogenize search intensity, so that we can analyze the consequences of money growth on price dispersion and search decisions.

The structure of the model is a blend between [12] and [18, 19]. From [12] we borrow the fundamental trading frictions that make money essential. That is, agents are anonymous, barter is difficult, and the frequency of meetings between agents is finite. These frictions are cast in a decentralized market where buyers and sellers are matched bilaterally to determine the terms of trade through bargaining. Also borrowed from [12] are heterogeneous preferences, whose role will be described later. We embed these elements into the structure in [18, 19], where the basic decision unit is a large household whose members share the matching risks. This integrated structure allows us to tractably analyze how money growth and inflation affect money's ability to efficiently allocate goods across heterogeneous consumers.[3]

Price dispersion in this paper is generated by heterogeneous preferences and trading frictions. There is a continuum of goods. A household derives utility from all goods but has a smooth preference ordering over the goods. The further away a good's type is from the household's preferred good, the lower the marginal utility of consuming the good. Different households have different preference orderings over the goods. This heterogeneity by itself does not generate price dispersion but, in the presence of trading frictions, it does. When a seller meets a buyer who values the good very much, the seller can sell the good for a high price. However, when the buyer in the match does not value the good much, the seller may choose to sell the good to the buyer at a low price rather than withholding the good for a future

[1] For instance, see [13, 17, 8]. Reinsdorf [17] finds that unexpected inflation has negative effects on price dispersion, while the positive relationship is preserved between expected inflation and price dispersion. Reinsdorf also presents a concise survey of the theoretical literature.

[2] Most of previous models in this field combine consumer search with money in the utility function, e.g., [1, 2]. Costly search by consumers gives firms some monopoly power to set price above marginal cost, and the cost to adjust prices induces firms to adjust prices in the $S$-$s$ fashion. When all firms do not adjust their prices at the same time, money growth increases price dispersion across firms and increases search. Intensive search reduces firms' monopoly power and, as Benabou [2] shows, this efficiency gain can outweigh the increased search cost.

[3] In the Kiyotaki-Wright [12] framework, [16, 7] are among the first to analyze the relationship between inflation and price dispersion, but they assume that there is a smallest (indivisible) unit of money. The papers most similar to ours are [4, 11]. Berentsen and Rocheteau [4] assume heterogeneous preferences as we do, but they do not focus on price dispersion. Head and Kumar [11] specifically look at price dispersion, by exploring the mechanism of price dispersion from [6]. This mechanism, complementary to ours, assumes that some buyers have more information about prices than other buyers.

match, because the matching rate is finite. In our model, the distribution of prices is the same for every good.

We first study the version of the model where buyers' search intensity is exogenous. In this model, a household divides the matches into desirable ones and mediocre ones. In desirable matches where the buyer likes the seller's good very much, the buyer is constrained by his money holdings; namely, his holdings are not enough to compel the producer to produce as much as the buyer wishes. Meanwhile, in mediocre matches the buyer does not like the seller's good enough to spend all his money. An increase in money growth (or inflation) reduces the real value of money, reduces the quantity of goods produced in desirable matches, and hence increases (real) prices of such goods.[4] By contrast, the lowest price of goods remains at zero, because it occurs in the least desirable match where the buyer does not want to buy the good. Thus, an increase in money growth widens the range of real prices. The variance of prices increases with money growth in a wide range of parameter values, but the general response is ambiguous.

In the basic model, money growth reduces efficiency and increases velocity of money. As the value of money falls with money growth, households can purchase less of the desirable goods. This causes households to substitute consumption from desirable goods into mediocre goods. The result is a reduction in quality-weighted consumption and hence in welfare. Common to most models of money, the Friedman rule is optimal in this setting. Also, the shift in consumption to mediocre goods requires that the buyers in mediocre matches spend a higher percentage of their money holdings. Thus, velocity of money rises, as in [18, 19, 3, 20].

Next, we endogenize buyers' search intensity. In this environment, multiple equilibria can arise from the interaction between search intensity and the inefficiency in the allocation of goods. If a household believes that other households will search intensively so that the efficiency in the allocation of goods will be high, then a household will choose to search intensively. The reverse happens if a household believes that other households will not search so quite intensively. The possible equilibria can be ranked by the inefficiency in the allocation of goods. Across equilibria the larger is the inefficiency the lower is welfare and the lower is search intensity.

In the equilibrium with the highest welfare level, money growth increases the range of prices and reduces welfare. These effects happen for the same reasons as in the model with constant search intensity. However, search intensity responds to money growth ambiguously. For low growth rates of money, an increase can raise search intensity, while for high growth rates, an increase lowers search intensity. The result will depend upon whether increasing the inefficiency in the allocation of goods raises the surplus to a buyer in a match or lowers it. When the surplus increases, search intensity increases. When the surplus decreases, search intensity decreases.

Search intensity necessarily rises with money growth only in the equilibria ranked the second, fourth, etc., in welfare. In these equilibria, money growth also increases efficiency and welfare. Thus, only an inferior equilibrium necessarily

[4] Throughout this paper, the term "price" refers to the price of a good normalized by the money stock.

generates a positive association of search intensity with both inflation and welfare. In such an equilibrium, however, money growth reduces the range of prices. In no equilibrium does money growth (or inflation) simultaneously increase search intensity, the range of prices, and welfare.

The remainder of this paper is organized as follows. Section 2 describes the basic model with fixed search intensity. The equilibrium in this economy is described in Section 3 and the effects of money growth in Section 4. Section 5 endogenizes search intensity. Section 6 discusses the stability of steady states and uneven allocations of money among buyers. Section 7 concludes the paper and the appendices provide necessary proofs.

## 2 The model

### *2.1 Environment*

The model incorporates the setup of heterogeneous goods and preferences from [12] into the search monetary framework in [18, 19] with divisible money and goods. Goods are perfectly divisible, non-storable, and their types are identified by points on a circle of circumference 2. A continuum of households with unit mass are uniformly distributed and indexed by points on the same circle. A household located at point $h$ on the circle can produce good $h$ and only good $h$. The cost of producing $q$ units is $c\,(q)$, where $c' > 0$, $c'' > 0$, $c(0) = 0$, $c'(0) = 0$, and $c'(\infty) = \infty$. Each household is composed of an infinite number of members, who are exogenously divided into a fraction $N$ of buyers and a fraction $1 - N$ producers/sellers. Buyers carry money and sellers productive capacity to the market to exchange, as described in detail below.[5]

Each household has a preferred type of good from which they derive the most utility. For any other good, the household's preference decreases in the distance (i.e., the shortest arc length) between the good and the preferred good. Denote this distance by $z$. The quality of the good to the household is $a(z)$, with $a' < 0$, $a(0) < \infty$ and $a(1) = 0$. Also, a household's output is a distance 1 from its preferred good, so that there is no utility from consuming one's own output. Let $q\,(z)$ be the quantity consumed of a good of quality $a(z)$. The household's quality-weighted consumption in each period is:

$$y = \alpha N \int_0^1 a(z)q(z)dz,$$

where $\alpha$ is the probability with which a buyer meets a seller. Normalize $\alpha = 1/N$. Utility per period from consumption is $u(y)$. Assume $u' > 0$, $u'' \leq 0$, and $u'(0) \geq u_0 \geq u'(\infty)$, where $u_0$ is a sufficiently large, positive number.

[5] The infinite number of members in each household ensures that even though members may have different outcomes in their individual matches, the randomness from the matching process is smoothed out at the household level, which makes the model tractable. Furthermore, household members are assumed to act in the best interests of the household. For more analysis of the household assumption, such as the endogenous determination of the division $N$, see [18, 19].

In addition to goods there is an intrinsically worthless, divisible object called money. Let $M$ be the stock of money per household and $m$ the money holdings of a particular household.

Time is discrete and the discount factor is $\beta \in (0, 1)$. At the start of each period a household allocates an equal amount of money, $m/N$, to each of its buyers. Then the producers enter the market, setting up production at fixed locations to sell goods to buyers. Buyers enter the market to buy goods. A buyer meets a seller with probability $\alpha\,(= 1/N)$ and a seller meets a buyer with probability $\alpha N/(1-N) = 1/(1-N)$. We assume that two producers never meet with each other. This implies that barter does not arise, and so every trade is an exchange of money for goods. This is a simplifying assumption, not a necessary one for our analysis.[6]

A match is characterized by the distance of the producer's good from the buyer's preferred good, $z$, and the buyer's money holdings, $m/N$. The buyer makes a take-it-or-leave-it offer to the producer, which specifies the amounts of goods, $q$, and money, $x$, to be exchanged. If the seller accepts the offer, he immediately produces the quantity of goods specified in the offer for the specified amount of money. After trade the producers and buyers return to the household. The household collects money and goods from the members. All members consume the same amount. Before the next period the household receives a lump-sum transfer of money, $\tau$.

In the remainder of this section, we will analyze a particular household's decisions. We use lower-case variables to denote this household's decisions, and capital-case letters the aggregate variables. The state variable for a particular household is $m$, the amount of money it possesses at the start of a period. Let $v(m)$ denote the value function, where the dependence on aggregate variables is suppressed. Let $\omega$ be the value of next period's money, discounted to the current period. Then,

$$\omega = \beta v'(m_{+1})$$

where the subscript $+1$ indicates that the variable is one period ahead.

### *2.2 A particular household's decisions*

We analyze the household's trade decisions first and then its decisions on $(c, m_{+1})$. The trade decisions consist of the acceptance strategies for producers and proposal strategies for buyers. Since the buyer in a match makes take it or leave it offers, the producer's household instructs the producer to accept an offer if and only if the offer generates a non-negative surplus. So, we omit the notation for the seller's strategy and focus on the decision by the buyer's household on the quantities of trade, $(q, x)$. When choosing $(q, x)$, the household takes as given other households' value of money, $\Omega$, proposals, $(X, Q)$, and acceptance strategies. In addition, since an agent is atomistic in his household, his trade has no effect on the household's marginal utility of consumption.

[6] As [9, 4] have shown in similar environments, money can still be valuable even when every match has a double coincidence of wants. Two randomly matched agents may have very asymmetric tastes for each other's goods, in which case they will choose to exchange with money as the medium of exchange. They barter only in matches where tastes are not very asymmetric.

Consider a match in which the producer's good is of quality $a(z)$ to the buyer's household. An offer of $x$ units of money for $q$ units of goods yields a surplus $[u'(y)a(z)q - x\omega]$ to the buyer and a surplus $[x\Omega - c(q)]$ to the producer. The offer maximizes the buyer's surplus, subject to the producer's acceptance and the constraint on money, $x \leq m/N$. Because the producer accepts an offer as long as the surplus is non-negative, the offer will set the producer's surplus to zero, resulting in $x = c(q)/\Omega$. Thus, the offer $(q, x)$ maximizes $[u'(y)a(z)q - x\omega]$ subject to

$$x = c(q)/\Omega \leq m/N. \tag{2.1}$$

To describe the solution, define $q^*(z)$, $x^*(z)$ and $\bar{z}$ by the following equations (for given $(\omega, \Omega, y, m)$):

$$u'(y)a(z) = c'(q^*(z))\frac{\omega}{\Omega}, \tag{2.2}$$

$$x^*(z) = c(q^*(z))/\Omega, \tag{2.3}$$

$$c(q^*(\bar{z})) = \Omega m/N. \tag{2.4}$$

Denote $\bar{x} = m/N$ and $\bar{q} = q^*(\bar{z})$. Then, the optimal offer satisfies a cut-off rule detailed in the following lemma (the proof is omitted):

**Lemma 2.1.** *For a match of type $(z, m/N)$, the optimal offer will solve*

$$(q(z), x(z)) = \begin{cases} (\bar{q}, \bar{x}), & \textit{if } z \leq \bar{z} \\ (q^*(z), x^*(z)), & \textit{if } \bar{z} < z \leq 1. \end{cases} \tag{2.5}$$

The cut-off level $\bar{z}$ divides the continuum of goods into two subsets, $(\bar{z}, 1]$ and $[0, \bar{z}]$. If the good in a match has $z \in (\bar{z}, 1]$ the buyer's household does not like the good very much, and so the buyer will trade only a fraction of his money for the good. In this case, the quantity of goods traded maximizes total surplus in the trade, $[u'(y)a(z)q - c(q)]$. If the good in a match is very valuable to the buyer's household, i.e., if $z \in [0, \bar{z}]$, the buyer likes to purchase a large quantity of the good, but his money holdings constrain how much he can purchase. In this case, the buyer spends all his money, and the quantity of goods traded is less than the amount which maximizes total surplus in the trade.

We will refer to goods in $[0, \bar{z}]$ as the household's *desirable goods* and those in $(\bar{z}, 1]$ *mediocre goods*. These divisions are the household's choices. Similarly, we refer to a match with $z \in [0, \bar{z}]$ as a desirable match and a match with $z \in (\bar{z}, 1]$ a mediocre match. Figure 1 illustrates $q(z)$. In desirable matches, the constant quantity $\bar{q}$ is traded. In mediocre matches, the quantity traded declines until no goods are traded in the limit at $z = 1$. That is, $q^{*\prime}(z) < 0$ and $q^*(1) = 0$. Similarly, $x^{*\prime}(z) < 0$ and $x^*(1) = 0$.

Following the above trade decisions, the household will receive the following quality-weighted quantity of goods:

$$y = \bar{q}J(\bar{z}) + \int_{\bar{z}}^{1} a(z)q^*(z)dz, \tag{2.6}$$

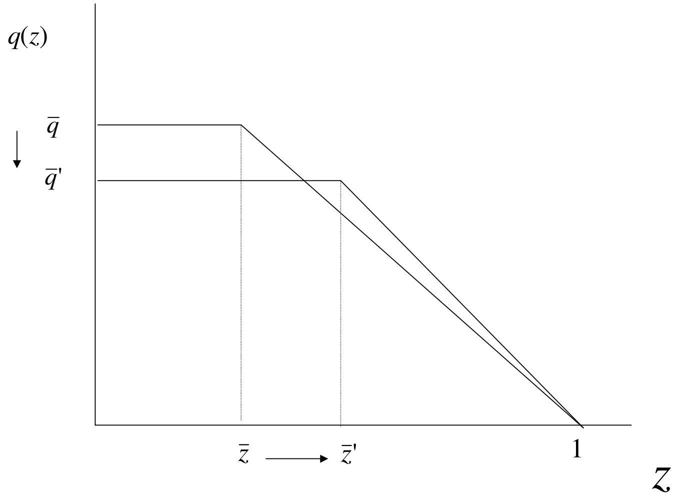


**Figure 1.** Quantities $q(z)$ in a match of type $z$ and the effect of money growth

where $J(z)$ is defined by:

$$J(z) \equiv \int_0^z a(t)dt. \tag{2.7}$$

Because goods are non-storable, the household will consume all of the goods, and so $y$ is also the household's quality-weighted consumption. The household's value function satisfies

$$v(m) = u(y) - \int_0^1 c(Q(z))\, dz + \beta v(m_{+1}),$$

where $m_{+1}$ denotes the household's money holdings at the beginning of the next period given as:

$$m_{+1} = m + \tau + \left(\bar{Z}\frac{M}{N} + \int_{\bar{Z}}^1 X^*(z)dz\right) - \left(\bar{z}\frac{m}{N} + \int_{\bar{z}}^1 \frac{c(q^*(z))}{\Omega}dz\right) \tag{2.8}$$

The two terms following the transfer $\tau$ are the total amount of money obtained in the current period by the household's sellers and the amount spent by the buyers. Note the distinction between the household's own choices $(\bar{z}, q, m)$ and other households' choices $(\bar{Z}, Q, M)$.

Using (2.2) and the notation $\omega = \beta v'(m_{+1})$, we can express the envelope condition for $m$ as

$$\omega_{-1} = \beta\left[\omega + \frac{\omega}{N}\left(\frac{J(\bar{z})}{a(\bar{z})} - \bar{z}\right)\right]. \tag{2.9}$$

This equation requires that the current value of money, $\omega_{-1}/\beta$, be equal to the sum of the future value of money and the non-pecuniary service or return that money yields in the current trades. The service, given by the term following $\omega$ in the above equation, comes from money's role in relaxing the money constraint (2.1). For given $\omega$, this amount of service is an increasing function of $\bar{z}$ because, the wider the range of trades in which the money constraint binds, the more frequently a marginal unit of money serves the role of relieving the constraint.

## 3 Symmetric equilibrium

### *3.1 Definition and existence*

We focus on symmetric monetary equilibria. A symmetric monetary equilibrium consists of an individual household's decisions $(q(z), x(z), m_{+1})$, the implied value $\omega$, and other households' decisions and values, $(Q(z), X(z), \Omega)$, that meet the following requirements: (i) The quantities of trade in a symmetric equilibrium, $(q(z), x(z))$, are optimal given $(Q(z), X(z), \Omega)$, i.e., they satisfy (2.5); (ii) $\omega$ satisfies (2.9); (iii) Individual decisions equal aggregate decisions, i.e., $q(z) = Q(z)$, $x(z) = X(z)$, and $\omega = \Omega$; (iv) The value of the money stock is positive and bounded, i.e., $0 < \omega_{-1}m/\beta < \infty$ for all $t$.

Of interest is the steady state of the equilibrium under a constant money growth rate. Monetary transfer in each period is $\tau = m_{+1} - m = (\gamma - 1)m$, where $\gamma > 0$ is the (gross) money growth rate. In such a steady state, the total value of money ($\omega M$) is constant. So that $\omega_{-1}/\omega = m_{+1}/m = \gamma$. Then, (2.9) becomes:

$$\frac{\gamma}{\beta} - 1 = \frac{1}{N}\left(\frac{J(\bar{z})}{a(\bar{z})} - \bar{z}\right). \tag{3.1}$$

A steady state can be determined recursively, by determining $\bar{z}$ first. In fact, (3.1) determines $\bar{z}$ independently of all other variables. Under the maintained assumptions on $a(.)$, it is easy to show that (3.1) has a unique solution for $\bar{z}$ iff $\beta \leq \gamma < \infty$. Denote this solution as $\bar{Z}(\gamma)$. Then, $\bar{Z}(\gamma) > 0$ for all $\gamma > \beta$, $\bar{Z}(\beta) = 0$ and $\bar{Z}(\infty) = 1$.

Next, we determine quality-weighted consumption, $y$. To do so, express all other variables as functions of $(y, \bar{z})$. Emphasizing the dependence of the quantity $q^*$ on $(y, z)$, we write:

$$q^*(z) = Q(y, z) \equiv c'^{-1}\left(u'(y)a(z)\right). \tag{3.2}$$

Then, $\bar{q} = Q(y, \bar{z})$. Similarly, we can rewrite (2.6) as follows:

$$y = J(\bar{z})Q(y, \bar{z}) + \int_{\bar{z}}^{1} a(z)Q(y, z)dz. \tag{3.3}$$

The following lemma (proven in Appendix A) states that (3.3) has a unique solution for $y$. This lemma and the unique solution for $\bar{z}$ imply the ensuing proposition.

**Lemma 3.1.** *For any given $\bar{z} \geq 0$, (3.3) has a unique solution for $y$. Denote this solution as $Y(\bar{z})$. Then $Y(\bar{z}) > 0$ and $Y'(\bar{z}) \leq 0$ (where the equality holds only when $\bar{z} = 0$).*

**Proposition 3.2.** *There is a unique monetary steady state iff $\gamma \in [\beta, \infty)$. In the steady state, $\bar{z} > 0$ if and only if $\gamma > \beta$.*

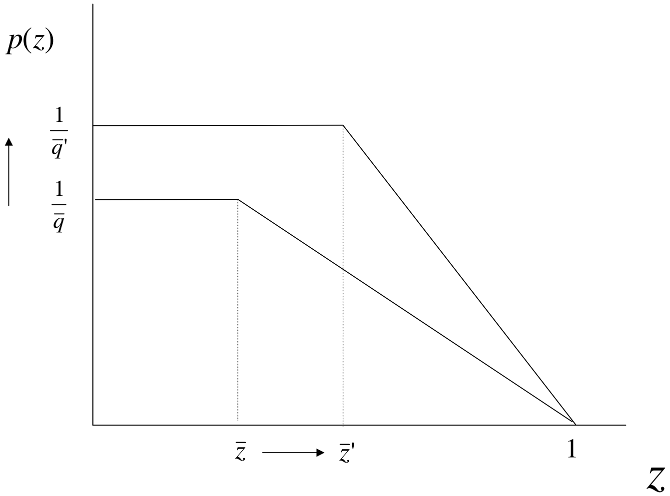


**Figure 2.** Prices in a match of type $z$, $p(z)$, and the effect of money growth

### 3.2 Price dispersion

The term "price" refers to the price of a good normalized by the money stock per buyer. Fix a particular type of good. If the seller of the good encounters a buyer whose preferred good is a distance $z$ from the seller's good, the normalized price in the trade is

$$p(z) = \frac{x(z)}{q(z)} \bigg/ \frac{M}{N} .$$

Since $z$ is uniformly distributed over the circle, there is a distribution of prices over the same type of good. This distribution is identical for all types of goods because goods are symmetric.

From Lemma 2.1, $p(z)$ satisfies

$$p(z) = \begin{cases} \bar{p} \equiv 1/\bar{q}, & \text{if } z \leq \bar{z}, \\ p^*(z) \equiv \frac{c(q^*(z))}{q^*(z)c(\bar{q})}, & \text{if } \bar{z} < z \leq 1. \end{cases} \tag{3.4}$$

Prices are constant for $z \in [0, \bar{z}]$. For $z \in (\bar{z}, 1)$, $p^{*\prime}(z) < 0$.[7] Also, the lowest price occurs at $z = 1$, and it is $\underline{p} \equiv c'(0)/c(\bar{q}) = 0$. Thus, $p^* \searrow \underline{p}$ as $z \nearrow 1$. Figure 2 illustrates $p(z)$.

## 4 Effects of money growth

We now examine the effects of a permanent increase in the money growth rate on the steady state. All proofs for the results in this section appear in Appendix B.

[7] The negative sign of $p^{*\prime}(z)$ follows from the assumptions that $c(0) = 0$ and $c$ is convex, along with the fact that $q^*(z)$ is decreasing.

### *4.1 On trade decisions and price dispersion*

Money growth has the following effects on the trade decisions:

**Proposition 4.1.** *$d\bar{z}/d\gamma > 0$, $dy/d\gamma < 0$, $d\bar{q}/d\gamma < 0$, and $dq^*(z)/d\gamma \geq 0$ for all $z \in (\bar{z}, 1)$. Also, an increase in $\gamma$ increases prices of each type of good. The range and the mean of prices increase, but the standard deviation of prices may either increase or decrease.*

To understand these effects, let us start with $\bar{z}$. A higher money growth rate makes the value of money deteriorate more quickly between periods. To induce a household to hold money in this case, the amount of non-pecuniary service that money generates in trades must increase. Because money generates service by relaxing the money constraint in the range of matches with $z \in [0, \bar{z}]$, for it to generate higher services, $\bar{z}$ must increase to widen this range.

The quantity of goods traded in a desirable match, $\bar{q}$, falls when money growth increases. In a desirable match, the buyer likes to buy a large quantity of the good but is constrained by his money holdings. An increase in money growth exacerbates the money constraint by reducing the value of money (i.e., $\omega m/N$). Thus, the quantity of goods traded in such a match falls.

The reduction in the amount of desirable goods reduces quality-weighted consumption, $y$, because these goods deliver higher utility to the household. To smooth consumption, the household counters the reduction in $y$ by increasing consumption of mediocre goods, i.e., by increasing $q^*(z)$ for each $z \in (\bar{z}, 1)$.[8] The increase in mediocre goods only mitigates, but does not completely offset, the reduction in quality-weighted consumption caused by the fall in $\bar{q}$. Figure 1 illustrates these effects of an increase in money growth on $q(z)$.

Figure 2 illustrates the effect of a higher growth rate of money on prices. Prices of all goods, except for $z = 1$, increase. The price of desirable goods increases, because higher money growth lowers the value to money, as shown by the decrease in $\bar{q}$. However, there is a second effect on goods purchased in mediocre matches. Since agents substitute consumption into mediocre goods, and production costs are increasing in $q$, the higher demand for mediocre goods raises prices of such goods more *proportionally* than those of desirable goods. However, the price of goods at the far end $z = 1$ stays at zero under the assumption $c'(0) = 0$. As a result, the range of prices unambiguously widens with an increase in money growth.

In addition, the shape of the price distribution changes. As money growth increases $\bar{z}$, the mass of the price distribution at the level $\bar{p}$ increases. Thus, higher order statistics of the price distribution, such as the standard deviation, may either increase or decrease with money growth.

### *4.2 On velocity of money and output*

Our model generates endogenous velocity of money, as in related models such as [18, 19, 3, 20]. Denoted $\mathcal{V}$, velocity is defined in the usual way as the ratio

[8] As is clear from the explanation, $q^*(z)$ would remain unchanged for $z \in [\bar{z}, 1]$ if the marginal utility of consumption is constant.

of nominal output to the money stock. Nominal output is $\int_0^1 p(z)\, q(z)\, dz$. (This differs from quality weighted output, $y$, because the quantities here are weighted by prices.) With (2.5) and (3.4), velocity of money is

$$\mathcal{V} = \frac{1}{N}\left[\bar{z} + \frac{1}{c(\bar{q})}\int_{\bar{z}}^{1} c\left(q^*(z)\right) dz\right]. \tag{4.1}$$

Because an increase in the money growth rate reduces $\bar{q}$ and increases $q^*(z)$ for all $z \in [\bar{z}, 1)$, velocity of money rises. Another way to express this result is that an increase in money growth increases the weighted sum of output in matches, where the weights are prices.

To understand the rise in velocity, it is useful to uncover the source of endogenous velocity. For each trade with $z \in [0, \bar{z}]$, the buyer's money constraint binds. In such a trade, nominal output is equal to the buyer's money holdings and so velocity is constant. In contrast, a trade with $z \in (\bar{z}, 1]$ does not suffer from a binding money constraint. Nominal output in such a trade responds to money growth disproportionately relative to the money stock. This is the source of endogenous velocity and the positive response of velocity to money growth.[9]

### *4.3 On social welfare*

To analyze the welfare effect of money growth, we first characterize the efficient allocation chosen by a fictional social planner who is constrained by the matching technology. The social planner chooses a quantity $q^o(z)$ for each $z$ to solve the following problem:

$$\max\left[u(y^o) - \int_0^1 c(q^o(z))dz\right] \text{ s.t. } y^o = \int_0^1 a(z)q^o(z)dz.$$

The allocation $q^o$ satisfies $q^o(z) = c'^{-1}\left(u'(y^o)a(z)\right)$ and the cutoff level $\bar{z}^o$ is zero. The efficient allocation exists iff there is a solution to the following equation:

$$y = \int_0^1 a(z)c'^{-1}\left(u'(y)a(z)\right) dz. \tag{4.2}$$

Similar to Lemma 3.1, there is a unique solution for $y$ to the above equation. Therefore, there is a unique steady state of the efficient allocation.

**Proposition 4.2.** *The equilibrium steady state is efficient iff $\gamma = \beta$. For all $\gamma > \beta$, the equilibrium steady state has the following properties: $y < y^o$, $\bar{z} > \bar{z}^o = 0$, and social welfare is a decreasing function of $\gamma$.*

[9] A change in the money growth rate also changes the range of trades in which the money constraint binds. This can be another source of endogenous velocity when the change in money growth is large. However, when the money growth rate changes only marginally, the effect of $\bar{z}$ itself on velocity is negligible.

Money growth reduces social welfare, despite the fact that it increases nominal output relative to the money stock. This is because an increase in money growth reduces a household's consumption of desirable goods, increases consumption of mediocre goods, and hence reduces quality-weighted consumption. Although output weighted by prices rises, it is output of mediocre goods that rises. The household would prefer to consume more desirable goods and less mediocre goods. This can be achieved by increasing the marginal value of money ($\omega$) through a reduction in money growth. Therefore, the so-called Friedman rule (i.e., $\gamma = \beta$) restores efficiency.

## 5 Endogenous search intensity of buyers

It is sometimes argued that inflation, by widening price dispersion, induces buyers to search and hence increases welfare (e.g., [1]). We examine this issue now by endogenizing buyers' search intensity. All proofs for this section are delegated to Appendix C.

### *5.1 Decisions and optimal conditions*

Consider a particular household again. This household chooses the search intensity for each of its buyers, denoted $i$, in addition to other decisions described earlier. Because all buyers in the household hold the same amount of money, it is optimal for them to have the same search intensity (In Sect. 6 we will show that it is optimal for a household to allocate money and search intensity evenly among buyers.) Let $I$ be the aggregate search intensity per buyer, which individual households take as given. A buyer searching with intensity $i$ gets a match with probability $\alpha i B(I)$ in a period, where $B \geq 0$. With the normalization $\alpha = 1/N$, a buyer's matching probability per search intensity is $B/N$. The total number of matches for the particular household in a period is $\alpha N i B = iB$. Similarly, the total number of matches per household is $IB$. This implies that the matching probability for a seller is $IB/(1 - N)$.

Restrict $IB(I) \leq \min\{N, 1 - N\}$ so that the matching rates $IB/N$ and $IB/(1 - N)$ are indeed probabilities, and assume $\lim_{I \to 0} IB(I) = 0$ so that search intensity must be positive in order to generate a match. In addition, we impose the standard assumptions that $B'(I) < 0$ and $-B'(I)I < B(I)$. These assumptions capture the matching externalities. Namely, an increase in the aggregate search intensity per buyer increases congestion for buyers and reduces congestion for sellers, resulting in a lower matching rate per search intensity for each individual buyer and a higher matching rate for each individual seller. Denote $\eta = -B'I/B$. Then, $\eta \in (0, 1)$.

The disutility of a buyer's search intensity is denoted $L(i)$. Impose the standard assumptions: $L' > 0$, $L'' \geq 0$, and $L'(0) = 0 < L'(\infty)$.

We modify the formulas of a particular household's utility per period (or welfare) $w$, quality-weighted consumption $y$, and the law of motion of money holdings as follows:

$$w = u(y) - NL(i) - IB(I) \int_0^1 c\left(Q(z)\right) dz,$$
$$y = iB(I) \int_0^1 a(z)q(z)dz,$$
$$m_{+1} = m + \tau + IB(I) \int_0^1 X(z)dz - iB(I) \int_0^1 x(z)dz.$$

The trade decisions are the same as before (see Lemma 2.1). In the current setup, these decisions yield the following envelope condition for $m$:

$$\frac{\gamma}{\beta} - 1 = \frac{iB(I)}{N} \left( \frac{J(\bar{z})}{a(\bar{z})} - \bar{z} \right), \tag{5.1}$$

where the term $iB/N$ captures the utilization rate of money per buyer. Also, the average quality-weighted consumption per match is:

$$\frac{y}{iB} = \bar{q}J(\bar{z}) + \int_{\bar{z}}^1 a(z)q^*(z)dz. \tag{5.2}$$

The household's optimal decision on search intensity satisfies the following condition:[10]

$$\frac{L'(i)N}{B(I)} = S \equiv u'(y)\frac{y}{iB(I)} - \left[\bar{z}c(\bar{q}) + \int_{\bar{z}}^1 c(q^*(z))dz\right]. \tag{5.3}$$

The left-hand side of (5.3) is the marginal disutility of increasing search intensity, normalized by the buyer's matching rate per search intensity, $B/N$. The right-hand side is the buyer's expected surplus per trade, averaged over the types of matches. Because a buyer makes take-it-or-leave-it offers, his average surplus per match is the utility of the average level of quality-weighted consumption per match minus the average cost of production.[11]

### *5.2 Multiple steady states*

To determine the steady state of a symmetric equilibrium where $i = I$, we express other variables as functions of $(\bar{z}, I)$. Equation (5.1) defines $I$ as an implicit function of $\bar{z}$, $I = F1(\bar{z})$. This is always a decreasing function. Next, for a given $\bar{z}$ and $I$, (5.2) can be solved to determine a function $y = Y(\bar{z}, I)$. This function can then be inserted into (5.3) to obtain $I = F2(\bar{z})$. As shown in Appendix C and explained later, the function $F2(.)$ may be either decreasing or non-monotonic.

[10] To obtain (5.3), note that $\bar{q}$ does not depend on $i$ directly, since $\bar{q} = c^{-1}(\omega m/N)$. Also, the marginal effect of $i$ on utility through $\bar{z}$ is negligible, and so a marginal change in $i$ affects utility entirely through its effects on $q^*(z)$ and $y$.

[11] The second-order partial derivative of net utility with respect to $i$ is negative under the maintained assumptions $L'' > 0$ and $B' < 0$. Thus, the optimal choice of $i$ is interior.

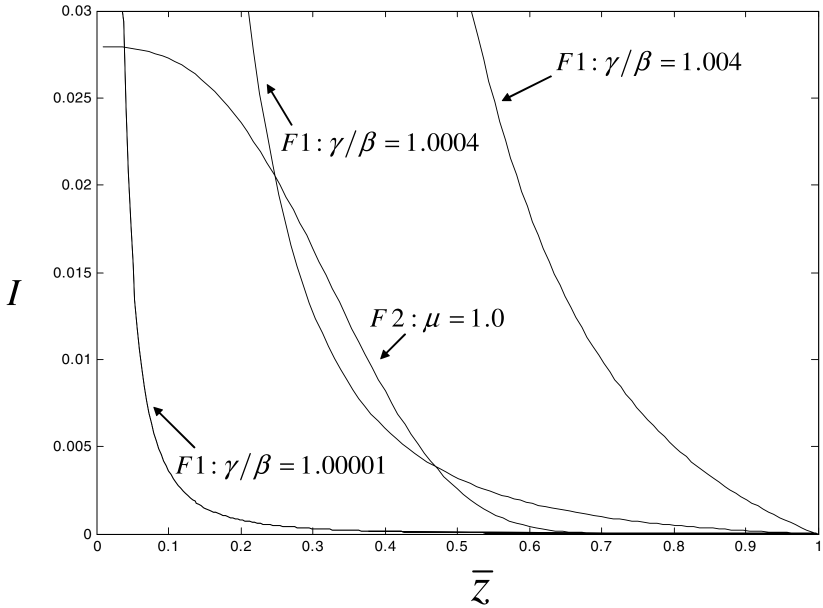


**Figure 3.** Multiple equilibria and the effects of increasing money growth

A steady state is such a $\bar{z}$ that solves $F1(\bar{z}) = F2(\bar{z})$. If $\bar{z} = 1$, then there is no trade and the steady state is non-monetary. If $\bar{z} < 1$, then the steady state is *interior*. In this environment, multiple interior steady states can arise. To illustrate, consider the following example.

*Example 5.1.* Consider the functional forms: $a(z) = 1 - z$, $c(q) = q^{\psi}$, $L(I) = I^{\xi}$, $B(I) = \frac{1}{2I+1}$ and $u(y) = 2y$. Choose $\psi = 1.4$, $\xi = 1.2$ and $N = 0.5$.

With a linear utility function of consumption, we plot the two curves $F1$ and $F2$ in Figure 3 for $\gamma/\beta \in \{1.00001, 1.0004, 1.004\}$. The curve $F1$ changes with the money growth rate, but the curve $F2$ does not. For high values of $\gamma/\beta$, no interior steady state exists, except the non-monetary steady state at $\bar{z} = 1$. For very low values of $\gamma/\beta$, there is only one interior steady state. For intermediate values of $\gamma/\beta$, multiple interior steady states can exist.

When the utility function is strictly concave, Figure 3 needs modification. The curve $F2$ increases in $\bar{z}$ at low levels of $\bar{z}$ and then decreases at high levels of $\bar{z}$. This feature of $F2$ and the following proposition are proven in Appendix C.

**Proposition 5.2.** *Assume $\gamma > \gamma_1$, where $\gamma_1$ is defined by (C.8) in Appendix C. Under the condition $\lim_{z \nearrow 1_-} F_2'(z)/F1'(z) > 1$, there exists an interior equilibrium and the number of interior equilibria will be odd. Otherwise the number of interior equilibria will be even, possibly zero. Between any two steady states, the one with a lower value of $\bar{z}$ has higher welfare $w$, higher values of $(\bar{q}, y, I)$, and lower values of $q^*(z)$ for all $z \in (\bar{z}, 1)$.*

Multiple steady states can arise in this model because of the interaction between $\bar{z}$ and search intensity. (Recall that the interior steady state is unique when search intensity is fixed.) Imagine that households believe that a low critical level $\bar{z}$ will be optimal. In this case, buyers will be constrained in only a small fraction of matches and buyers' average surplus per match will be high. Anticipating this high surplus, each household will ask the buyers to search intensively. High search intensity increases the number of matches for each household, increases the utilization of money, and so increases the expected non-pecuniary return to money. To maintain the steady state, however, the expected non-pecuniary return to money must be reduced back to the constant $(\gamma/\beta - 1)$. A low $\bar{z}$ achieves this by reducing the fraction of matches in which money relaxes the money constraint. That is, a belief of a low $\bar{z}$ can be self-fulfilling. Similarly, a belief of a high $\bar{z}$, supported by low search intensity, can be self-fulfilling.

We should emphasize that the multiplicity of steady states does not hinge on the specific way in which the matching risks are smoothed. In our model, the members share consumption. An alternative formulation is to allow agents to smooth utility, as it is done in [15]. These two formulations are the same when the utility function is linear in consumption. As Figure 3 shows, multiple steady states can arise with a linear utility function. For this reason, multiple equilibria should arise as well in the framework of [15] when search intensity is endogenized.[12]

The steady states can be ranked according to welfare, as stated in Proposition 5.2. The lower the level $\bar{z}$ is in a steady state, the higher the level of welfare. We will label the steady state with the lowest value of $\bar{z}$ as the first steady state, the steady state with the second lowest value of $\bar{z}$ as the second steady state, and so on. Between two steady states, we will refer to the one with a low $\bar{z}$ as the superior steady state and the one with a high $\bar{z}$ as the inferior steady state. Not surprisingly, a superior steady state has higher consumption of desirable goods, lower consumption of mediocre goods, and higher quality-weighted consumption. Also, buyers search more intensively in a superior steady state than in an inferior steady state, because search intensity and $\bar{z}$ must obey the negative relationship $I = F1(\bar{z})$ in all steady states.

The level $\bar{z}$ is useful not only for comparing steady states, but also for examining local properties of each steady state, as detailed in the following lemma.

**Lemma 5.3.** *In every steady state, we can write $I$, $\bar{q}$, $y$, and $w$ all as functions of $\bar{z}$, so that $I = I(\bar{z})$, $\bar{q} = \bar{q}(\bar{z})$, $y = y(\bar{z})$, and $w = w(\bar{z})$. For $\bar{z} > 0$, $\bar{q}'(\bar{z}) < 0$, $y'(\bar{z}) < 0$, and $w'(\bar{z}) < 0$. Also, the quantity of goods in a mediocre match, $q^*(z)$, increases with $\bar{z}$ for any given $z \in (\bar{z}, 1)$. However, $I'(\bar{z})$ is ambiguous.*

This lemma extends the main results, and hence the intuition, from the economy with exogenous search intensity to the current economy with endogenous search intensity. Namely, a higher $\bar{z}$ is associated with lower consumption of desirable goods, higher consumption of mediocre goods, and lower quality-weighted consumption.

---

[12] We should note that all equilibria in our model are symmetric, in the sense that all households play the same strategy in each equilibrium. If, instead, one allows different households to play different strategies, then there may be a unique asymmetric equilibrium. This is because allowing for heterogeneous strategies between households enables the economy to "convexify" between different symmetric equilibria. Lagos and Rocheteau [14] provide such an illustration using the framework in [15]. If the strategies in their model are restricted to be symmetric, then multiple equilibria are likely to emerge.

There are two new features here. The first is the dependence of search intensity on $\bar{z}$. Within each steady state, search intensity can either increase or decrease with $\bar{z}$. (This contrasts to the unambiguously negative relationship between the two variables across steady states.) The ambiguity arises because the curve $F2(z)$ may be non-monotonic. An increase in $\bar{z}$, by reducing quality-weighted consumption $y$, affects the buyer's average match surplus in two opposite directions. Although each buyer receives less from each trade when $\bar{z}$ is higher, the goods are more highly valued by the household under diminishing marginal utility of consumption. As a result, the value of goods received from trade, which is equal to $yu'(y)$, may either increase or decrease with $\bar{z}$. This generates the ambiguous association between the buyer's match surplus and $\bar{z}$. Because buyers are motivated to search by the match surplus, their search intensity may either increase or decrease with $\bar{z}$. Clearly, when the utility function is linear in consumption, search intensity always decreases with $\bar{z}$.

The second new feature is that, even when search intensity is a choice variable, welfare is still negatively associated with $\bar{z}$. To understand this, it is useful to decompose the welfare level as

$$w(\bar{z}) = [u(y) - u'(y)y] + IB(I)S - NL(I),$$

where $S$ is given by (5.3). The first term of $w$ is caused by the concavity of the utility function; the second term is the buyers' total surplus from trade; and the last term is the disutility of search intensity. Because optimal search intensity satisfies $NL'/B = S$, then

$$w'(\bar{z}) = -yu''(y)\frac{dy}{d\bar{z}} + NIL''\frac{dI}{d\bar{z}}.$$

In the special case where $u'' = 0$, only the last term remains and it is negative (see the above discussion on search intensity). So, $w'(\bar{z}) < 0$. When $u'' < 0$, search intensity may increase with $\bar{z}$, but such an increase is an attempt to mitigate, but not to eliminate or overtake, the fall in consumption caused by the increase in $\bar{z}$. That is, the direct effect of $\bar{z}$ on welfare through $y$ dominates the effect through search intensity, whatever the latter may be. Again, $w'(\bar{z}) < 0$.

### *5.3 Effects of money growth and inflation*

Consider a permanent increase in the money growth rate $\gamma$. The effects on $\bar{z}$ and search intensity are illustrated in Figure 3 when the utility function is linear in consumption. The curve $F1(z)$ turns counter-clockwise around the steady state $\bar{z} = 1$, while the curve $F2(z)$ is intact. For strictly concave utility functions, the effects of money growth on the steady states can be deduced from Lemma 5.3. We summarize the effects as follows:

**Proposition 5.4.** *Let $k = 0, 1, 2, \ldots$. In the interior steady states that are ranked $(2k + 1)^{th}$ in welfare, an increase in inflation increases $\bar{z}$, has ambiguous effects on search intensity, reduces consumption (output) and welfare. It also reduces $\bar{q}$,* *widens the range of prices, and increases prices of all goods. The opposite effects occur in the interior steady states that are ranked $2(k+1)^{th}$ in welfare, with the exception that search intensity increases. Moreover, if search intensity and $\bar{z}$ respond to money growth in the same direction, they must both increase.*

The proposition illustrates two discrepancies between our model and some informal arguments. First, the informal literature argues that inflation induces buyers to search. In our model, this is not necessarily so in the $(2k+1)^{th}$ steady state, including the most superior steady state. Search intensity necessarily increases with inflation only in some "bad" steady states.

Second, in most steady states, inflation in our model does not increase both search intensity and the range of prices. The exceptions occur in the $(2k+1)^{th}$ steady state and only when search intensity increases with inflation. To understand this result, recall that inflation widens the range of prices if and only if it reduces the quantity of goods traded in desirable matches, $\bar{q}$. This reduction in $\bar{q}$ will result in higher search intensity only if it raises buyer's average surplus in a match, which will happen only if the marginal utility of consumption for the buyer has increased significantly. Otherwise both average surplus and search intensity fall with a greater range of prices resulting from inflation. Or, in the case of the $2(k+1)^{th}$ steady state, higher inflation lowers the range of prices and raises search intensity.

Moreover, when inflation does increase search intensity and the range of prices, it reduces welfare. Thus, it is never the case in our model that inflation increases search intensity, widens the range of prices, and improves welfare all at the same time.

The optimal money growth rate depends on which steady state the economy is in. In the most superior steady state, an increase in inflation reduces welfare. In this case, the optimal money growth rate is $\gamma = \beta$ (i.e., the Friedman rule), provided that pursuing this money growth rate does not induce the economy to switch from one steady state to another steady state.[13] In the second interior steady state or, in general, in the $2(k+1)^{th}$ interior steady state, an increase in the money growth rate increases welfare. Of course, there is also a possibility that an increase in money growth can switch the economy between two steady states.

## 6 Discussion

In this section, we examine the stability of steady states and prove that it is optimal for a household to distribute money evenly among buyers. For both tasks, we simplify the analysis by assuming that the utility function is linear in consumption.

When there are multiple steady states, a natural question is which steady state is stable. The notion of stability used here is the same as in [10], which involves some "trembling" in the equilibrium. In particular, suppose that the initial value of money $(\omega_{-1})$ is different from the steady state value, for some unspecified reason. Given this initial value $\omega_{-1}$, we generate a sequence of equilibrium values of money,

[13] This optimal money growth rate is the second-best outcome in the current model, because it fails to internalize the matching externalities completely. See [5] for the general argument.

$\{\omega_t\}_{t\geq 0}$. If this sequence converges to the value in a particular steady state, then the steady state is stable; otherwise, it is unstable.[14]

Let us start with the economy where search intensity is fixed, in which there is a unique monetary steady state. Let $\omega^s$ be the steady state value of $\omega$. To examine the dynamics of $\omega$, use (2.2) and (2.4) to solve $\bar{z}_t = \zeta(\omega_t)$. Substitute $\bar{z} = \zeta(\omega)$ to write the right-hand side of (2.9) as $G(\omega)$. Then, $\omega_{t-1} = G(\omega_t)$. Notice that $\zeta' < 0$, because a higher value of money reduces the range of matches in which the money constraint binds. With this property, we can verify that $0 < G'(\omega^s) < 1$. Thus, for any initial value $\omega_{-1} \neq \omega^s$, the sequence $\{\omega_t\}_{t\geq 0}$ generated by $\omega_t = G^{-1}(\omega_{t-1})$ diverges from the monetary steady state $\omega^s$. Such instability is also the feature of the unique monetary steady state in the overlapping generations model of money (e.g., [10]).

When search intensity is endogenous, we can also derive the mapping $G$, but it is much more difficult to determine $G'$ analytically. Numerical examples (not presented here) indicate that $0 < G'(.) < 1$ in the steady state with the highest welfare. So, the most superior steady state is unstable, just like the unique monetary steady state in the economy with fixed search intensity. By contrast, the interior steady state ranked the second in welfare has $G'(.) > 1$, and so this steady state is stable. In general, the interior steady state ranked $(2k+1)^{th}$ in welfare is unstable and the interior steady state ranked $2(k+1)^{th}$ is stable, where $k = 0, 1, 2....$

We now turn to the allocation of money among the buyers in a household. One may wonder whether a household can gain from a deviation to an uneven allocation, e.g., allocating more money and higher search intensity to some buyers than to other buyers. An extremely uneven allocation is that some buyers are given no money and not required to search. Effectively, this extreme allocation amounts to choosing $N$, the fraction of shoppers in the household – If $N$ is optimal, the extremely uneven allocation cannot be optimal. Since the current model assumes a fixed $N$, it is appropriate to exclude allocations that undo this assumption (for the optimal choice of $N$, see [18]). Thus, when examining an uneven allocation, we restrict that it allocate strictly positive amounts of money and search intensity to every buyer. Then, an uneven allocation is not optimal: Facing sellers whose production cost function is strictly convex, the quantity of goods that buyers get is a concave function of the buyer's money holdings (see (2.4)), and so an uneven allocation of money is likely to reduce expected utility.

To support our argument, suppose that a particular household deviates to an uneven allocation for one period, while other households continue to allocate money and search intensity to their buyers evenly.[15] The deviating household divides the buyers into group 1 and group 2. The size of group $j$ $(= 1, 2)$ is $n_j$, with $n_1 + n_2 =$

[14] This notion of stability is clearly different from dynamic stability in the neoclassical growth model. There, some variables like capital stocks are predetermined in the sense that their initial values are determined outside the model. Dynamic stability requires that, given the intial values of these predetermined variables, the equilibrium should converge to the steady state. This stability criterion implies trivial dynamics in our model, because none of the variables here (including $\omega$) are predetermined. Furthermore, it should be noted that stability is but one way to select among different equilbria.

[15] In the proof below, we will maintain the focus on symmetric equilibria. One should not confuse an uneven allocation within each household with an asymmetry between different households' strategies (see the earlier discussion on multiple equilibria).

$N$. The household assigns an amount $m_j/n_j$ to each buyer in group $j$, where $m_1 + m_2 = m$, and asks him to search with intensity $i_j$. Such a buyer gets a match with probability $i_j B(I)/N$. Denote $\mu_j = \Omega m_j/n_j$. An uneven allocation of money requires $\mu_1 \neq \mu_2$. We find the conditions under which the deviation is not optimal. As explained above, we restrict $0 < m_j < m$ and $i_j > 0$, for $i = 1, 2$.

Let $m_{+1}^d$ be the deviating household's money holdings at the beginning of next period, and $\omega^d$ the shadow value of such money discounted to the current period. Denote $R = \omega^d/\Omega$. A group $j$ buyer's trade decisions are characterized similarly to Lemma 2.1. That is,

$$c(\bar{q}_j) = \mu_j, u'a(\bar{z}_j) = c'(\bar{q}_j)R,$$
$$q^*(z) = c'^{-1}(u'a(z)/R), x^*(z) = c(q^*(z))/\Omega.$$

From the first two equations we solve $\bar{q}_j = \bar{q}(\mu_j)$ and $\bar{z}_j = \bar{z}(\mu_j, R)$. With these changes, we can modify the formulas in Section 5 for $y$, $w$, and the law of motion of money holdings as follows:

$$y = \frac{B(I)}{N} \sum_{j=1,2} n_j i_j \left[ \bar{q}_j J(\bar{z}_j) + \int_{\bar{z}_j}^1 a(z) q^*(z) dz \right],$$

$$w = u'y - n_1 L(i_1) - n_2 L(i_2) - IB(I) \int_0^1 c(Q(z))dz,$$

$$m_{+1}^d = m + \tau + IB(I) \int_0^1 X(z)dz - \frac{B(I)}{N\Omega} \sum_{j=1,2} n_j i_j$$
$$\times \left[ \bar{z}_j \mu_j + \int_{\bar{z}_j}^1 c(q^*(z))dz \right].$$

Similarly, we can modify the optimality condition for search intensity, (5.3), as:

$$\frac{NL'(i_j)}{B(I)} = S_j \equiv u' \left[ \bar{q}_j J(\bar{z}_j) + \int_{\bar{z}_j}^1 a(z) q^*(z) dz \right] - R \left[ \bar{z}_j \mu_j + \int_{\bar{z}_j}^1 c(q^*(z))dz \right]. \tag{6.1}$$

After substituting $\bar{q}_j = \bar{q}(\mu_j)$ and $\bar{z}_j = \bar{z}(\mu_j, R)$), we can write $S_j = S(\mu_j, R)$, and hence the solution to (6.1) as $i_j = i(\mu_j, R)$.

For the deviation to be optimal, $m_1$ must be optimal under the constraint $m_2 = m - m_1$, and $m$ must satisfy an envelope condition similar to (5.1). Because $0 < m_1 < m$, we express these requirements as follows:

$$\frac{N}{B(I)} \lambda = T(\mu_j, R) \equiv i(\mu_j, R) \left[ \frac{J(\bar{z}(\mu_j, R))}{a(\bar{z}(\mu_j, R))} - \bar{z}(\mu_j, R) \right], \ j = 1, 2, \tag{6.2}$$

where $\lambda = \omega_{-1}^d/(\beta\omega^d) - 1$. Moreover, the optimality condition for $n_1$, under the constraint $n_2 = N - n_1$, is

$$f(\mu_1, R) = f(\mu_2, R), \tag{6.3}$$

where $f(\mu, R) \equiv [iL'(i) - L(i)]_{i=i(\mu,R)} - R\mu\lambda$.

The uneven allocation of money and search intensity is not optimal when (6.2) has at most two solutions for $\mu$. If (6.2) has at most one solution, then clearly $\mu_1 = \mu_2$ and $i_1 = i_2$. Suppose that (6.2) has only two solutions, $\mu_1$ and $\mu_2$, with $\mu_1 > \mu_2$. Computing the derivative $i_\mu(\mu, R)$ from (6.1), we obtain

$$f_\mu(\mu, R) = \frac{RB}{N}\left[T(\mu, R) - \frac{N}{B}\lambda\right].$$

By (6.2), it is clear that $f_\mu(\mu_1, R) = f_\mu(\mu_2, R) = 0$. Because $\mu_1$ and $\mu_2$ are the only two solutions to (6.2), then either $T(\mu, R) > \frac{N}{B}\lambda$ or $T(\mu, R) < \frac{N}{B}\lambda$ for all $\mu \in (\mu_2, \mu_1)$. If $T(\mu, R) > \frac{N}{B}\lambda$ for all $\mu \in (\mu_1, \mu_2)$, then $f_\mu(\mu, R) > 0$ and $f(\mu_1, R) > f(\mu_2, R)$. If $T(\mu, R) < \frac{N}{B}\lambda$ for all $\mu \in (\mu_1, \mu_2)$, then $f(\mu_1, R) < f(\mu_2, R)$. Either way, (6.3) is violated and so the uneven allocation is not optimal.

With the functional forms $c(q) = q^\psi$ and $L(i) = i^\xi$ that we used in Example 5.1, the function $T(., R)$ is either monotone or having one hump. Thus, (6.2) indeed has at most two solutions for $\mu$, and the deviation to an uneven allocation of money and search intensity is not optimal.

## 7 Conclusion

This paper has explored the relationship between inflation, welfare and price dispersion in a model where money's use is built up from microfoundations. The model has enabled us to study the efficiency with which money is able to allocate heterogeneous goods across heterogeneous households. In doing so, we have found that raising the money growth rate lowers the ability for money to allocate goods efficiently, as higher money growth lowers the real value of money which constrains households in their purchases of the goods they most desire. To offset this loss, households choose to substitute into mediocre goods. The lower efficiency in the allocation of goods lowers welfare. Furthermore, the price of the most desirable goods rises more than the least desirable goods so that the range of prices widens, i.e. higher price dispersion can be associated with higher inflation.

Another channel through which inflation and price dispersion can interact is the buyers' search intensity. With endogenous search intensity, we find that the economy can exhibit multiple equilibria. Furthermore, in the high welfare equilibrium, an increase in the growth rate of money can increase search intensity only if an increase in the inefficiency in the allocation of goods associated with higher inflation raises the surplus to the buyer in a match. In this case, the range of prices also widens. However, the increase in search intensity cannot overcome the inefficiency from higher inflation, so that higher money growth necessarily lowers welfare in the equilibrium with the highest level of welfare. On the other hand, if the economy is in an inferior equilibrium, then higher money growth can raise search intensity and welfare. In this case, however, money growth shrinks the range of prices.

## Appendix

### *A. Proof of Lemma 3.1*

Temporarily denote the right-hand side of (3.3) as $RHS(y, \bar{z})$. From the definition of $Q(y, z)$ in (3.2), we have:

$$Q_1(y, z) = \frac{u''(y)a(z)}{c''(q^*)} < 0, \quad Q_2(y, z) = \frac{u'(y)a'(z)}{c''(q^*)} < 0. \tag{A.1}$$

Using these results, we can calculate:

$$RHS_1(y, \bar{z}) = \frac{u''}{c''(\bar{q})} \left[a(\bar{z})J(\bar{z}) + c''(\bar{q})K\right] < 0,$$
$$RHS_2(y, \bar{z}) = \frac{u'}{c''(\bar{q})} a'(\bar{z})J(\bar{z}) \leq 0, \ \ = 0 \text{ only if } \bar{z} = 0,$$

where

$$K \equiv \int_{\bar{z}}^{1} \frac{a^2(z)}{c''\left(q^*(z)\right)} dz > 0. \tag{A.2}$$

The above properties imply that $[y - RHS(y, \bar{z})]$ is strictly increasing in $y$ and $\bar{z}\ (> 0)$. Fix $\bar{z}$ and examine $[y - RHS(y, \bar{z})]$ as a function of $y$. For existence and uniqueness of the solution for $y$, it is necessary and sufficient that this function crosses 0 only once. Under the assumptions that $c'(\infty) = \infty$ and $u'(0)$ is sufficiently large, we have $Q(0, \bar{z}) > 0$, and so $y - RHS(y, \bar{z}) < 0$ at $y = 0$. Also, $\lim_{y \to \infty} [y - RHS(y, \bar{z})] > 0$. Therefore, there exists a unique solution for $y$. The solution $Y(\bar{z})$ clearly satisfies $Y'(\bar{z}) \leq 0$, with equality only for $\bar{z} = 0$. □

### *B. Proofs of Propositions 4.1 and 4.2*

For Proposition 4.1, it is apparent from (3.1) that $d\bar{z}/d\gamma > 0$. Because $Y'(\bar{z}) < 0$ and $d\bar{z}/d\gamma > 0$, we have $\frac{dy}{d\gamma} < 0$. Then $\frac{dq^*(z)}{d\gamma} = \frac{a(z)u''}{c''(q^*(z))} \left(\frac{dy}{d\gamma}\right) \geq 0$ ($= 0$ only when $u'' = 0$). Moreover,

$$\frac{d\bar{q}}{d\gamma} = \frac{\left(1 - Ku''\right) u'a'(\bar{z})}{\left(1 - Ku''\right) c''(\bar{q}) - u''a(\bar{z})J(\bar{z})} \left(\frac{d\bar{z}}{d\gamma}\right) < 0.$$

From Equation (3.4), since $\bar{q}$ is decreasing in $\bar{z}$ and $q^*$ increasing in $\bar{z}$, then the price $p(z)$ is increasing in $\gamma$. The range of prices, $\Delta p = \bar{p}$, also increases because the reduction in $\bar{q}$ raises $\bar{p}$. However, it can be shown that the standard deviation of prices responds to the increase in $\gamma$ ambiguously. This completes the proof of Proposition 4.1.

Turn to Proposition 4.2. For the equilibrium steady state to be efficient, $\bar{z}$ must be equal to the efficient value, which is 0. This requires $\gamma = \beta$. On the other hand, if $\gamma = \beta$, then $\bar{z} = 0$. In this case, the equation for $y$ in the equilibrium (i.e., (3.3)) is identical to the equation for $y^o$ (i.e., (4.2)), and so $y = y^o$. Since other equilibrium variables are only functions of $(y, \bar{z})$, and since $(y, \bar{z})$ are equal to the efficient values, the values of those variables are efficient.

For any $\gamma > \beta$, the dependence of $(y, \bar{z})$ on $z$ established in Proposition 4.1 implies $\bar{z} > \bar{z}^o$ and $y < y^o$. Measure the level of welfare per period in the steady state by $w$. Then,

$$w = (1 - \beta)v = u(y) - \bar{z}c(\bar{q}) - \int_{\bar{z}}^{1} c(q^*(z))dz.$$

Using $c'(\bar{q}) = u'(y)a(\bar{z})$ and the definition of $K$ in (A.2), we have:

$$\frac{dw}{d\gamma} = u'(y)\bar{Z}'(\gamma)Y'(\bar{z})\left(1 - Ku''\right)\left[1 - \frac{\bar{z}a(\bar{z})}{J(\bar{z})}\right].$$

Clearly, $dw/d\gamma < 0$ iff $J(\bar{z}) > \bar{z}a(\bar{z})$. The function $[J(z) - za(z)]$ is an increasing function of $z$ and, when $z = 0$, it is equal to 0. Thus, $J(\bar{z}) > \bar{z}a(\bar{z})$ for all $\bar{z} > 0$, i.e., for all $\gamma > \beta$. □

## *C. Proofs for Section 5*

In this appendix, we will omit the argument $\bar{z}$ in $a(\bar{z})$, $a'(\bar{z})$, and $J(\bar{z})$. When the argument is $z$ rather than $\bar{z}$, we will specify it explicitly. Similarly, abbreviate $c'(\bar{q})$ as $c'$ and $c''(\bar{q})$ as $c''$.

*C.1. The relationship $I = F2(\bar{z})$ and the proof of Proposition 5.2* The relationship $I = F2(\bar{z})$ arises from (5.2) and (5.3). Recall that $q^* = Q(y, z)$ and $\bar{q} = Q(y, \bar{z})$, where $Q$ is defined by (3.2). Then, (5.2) becomes:

$$\frac{y}{IB(I)} = Q(y, \bar{z})J(\bar{z}) + \int_{\bar{z}}^{1} a(z)Q(y, z)dz. \tag{C.1}$$

Similar to Lemma 3.1, this equation yields a unique solution for $y$ for given $\bar{z}$ and $I$. Denote this solution as $y = Y(\bar{z}, I)$. Substitute $y = Y(\bar{z}, I)$ and $q^* = Q(y, z)$ into (5.3):

$$\frac{L'(I)N}{B(I)} = S\left(Y(\bar{z}, I), Q(Y(\bar{z}, I), \bar{z})\right), \tag{C.2}$$

where $S(y, \bar{q})$ is a buyer's average surplus per trade, given as follows:

$$S(y, \bar{q}) = \left[u'(y)\,\bar{q}J - \bar{z}c(\bar{q})\right] + \int_{\bar{z}}^{1} \left[a(z)\,Q(y, z)u'(y) - c(Q(y, z))\right] dz. \tag{C.3}$$

(Notice that the critical level $\bar{z}$ also appears in this expression independently, but its marginal effect on $S$ is zero once its effects through $(y, \bar{q})$ are fixed.) The Equation (C.2) involves only $\bar{z}$ and $I$, and so it determines a relationship between $I$ and $\bar{z}$. This is the relationship $I = F2(\bar{z})$ used in Section 5.

To prove Proposition 5.2, we first find the proper domain of $F1(\bar{z})$ by requiring that the matching rates for a buyer and a seller, $IB/N$ and $IB/(1-N)$, be bounded above by one. Express this requirement as $IB(I) \leq \min\{N, 1-N\}$. Define $I_H$ by $I_H B(I_H) = \min\{N, 1-N\}$, and $z_L$ by $F1(z_L) = I_H$. Then,

$$\frac{J(z_L)}{a(z_L)} - z_L = \left(\frac{\gamma}{\beta} - 1\right) \max\left\{1, \frac{N}{1-N}\right\}. \tag{C.4}$$

The proper domain of $F1(\bar{z})$ is $\bar{z} \in [a_L, 1]$ and the range is $[0, I_H]$. Notice that $[J(z)/a(z) - z]$ approaches $0$ when $z \to 0$. Thus, $z_L > 0$ for all $\gamma > \beta$, and $z_L \to 0$ when $\gamma \to \beta$.

Second, we explore the features of the two relationships, $I = F1(\bar{z})$ and $I = F2(\bar{z})$. The function $F1(z)$ is always decreasing, as is clear from (5.1). However, $F2(z)$ may be a non-monotonic function. To see this, calculate:

$$Y_1 \equiv \frac{\partial Y}{\partial \bar{z}} = \frac{a'(\bar{z})u'(y)J(\bar{z})}{\frac{c''(\bar{q})}{IB} - u''(y)\left[a(\bar{z})J(\bar{z}) + Kc''(\bar{q})\right]} \leq 0,$$
$$\text{“} = \text{” only if } \bar{z} = 0, \tag{C.5}$$

$$Y_2 \equiv \frac{\partial Y}{\partial I} = \frac{c''(\bar{q})y(1-\eta)/I}{c''(\bar{q}) - IBu''(y)[a(\bar{z})J(\bar{z}) + Kc''(\bar{q})]} > 0. \tag{C.6}$$

Here, $\eta = -B'I/B \in (0, 1)$ and $K > 0$ is defined in (A.2). Expressing $S$ in (C.3) as a function of $(\bar{z}, I)$ by writing $(y, \bar{q})$ as functions of $(\bar{z}, I)$, we can verify that $S$ is a decreasing function of $I$. Also, $S$ increases in $\bar{z}$ (i.e., $F2'(z) > 0$) iff

$$u'\left(1 - \frac{\bar{z}a}{J}\right) < (-u'')\left[y - IBKu'\left(1 - \frac{\bar{z}a}{J}\right)\right]. \tag{C.7}$$

This condition is clearly violated when $u'' = 0$; thus, $F2(z)$ is always negatively sloped if utility is linear in consumption. Also, $F2'(\bar{z}) < 0$ when $\bar{z} \to 1$, because $y \to 0$ in that case. However, if $u'' < 0$ and $\bar{z} \to 0$, then $\bar{z}a/J \to 1$ and so (C.7) is satisfied. Thus, $F2'(\bar{z}) > 0$ when $u'' < 0$ and $\bar{z}$ is close to 0.

Third, we compare the values of $F1(z)$ and $F2(z)$ at the two endpoints of the domain $[z_L, 1]$. Restrict attention to $\gamma > \beta$, so that $z_L > 0$. Temporarily denote the right-hand side of (C.2) as $RHS(\bar{z}, I)$ and denote

$$D(z) = \left.\frac{L'(I)N}{B(I)}\right|_{I=F2(z)} - \left.\frac{L'(I)N}{B(I)}\right|_{I=F1(z)}$$
$$= RHS(z, F2(z)) - \left.\frac{L'(I)N}{B(I)}\right|_{I=F1(z)}.$$

Because $L'(I)N/B(I)$ is an increasing function of $I$, then $F2(z) < F1(z)$ iff $D(z) < 0$.

Consider $z = z_L$. Since $z_L$ is an increasing function of $\gamma$ (see (C.4)), it is meaningful to define

$$\gamma_1 = \inf\{\gamma : D(z_L) < 0, \gamma \geq \beta\}. \tag{C.8}$$

Because $F1(z_L) = I_H$ by definition, then $D(z_L) < 0$ iff $RHS(z_L, F2(z_L)) < L'(I_H)N/B(I_H)$. Notice that $I_H$ does not depend on $\gamma$. When $\gamma \to \infty$, $z_L \to 1$, and so $RHS(z_L) \to 0 < L'(I_H)N/B(I_H)$. Thus, $\gamma_1 < \infty$. Moreover, $D(z_L) < 0$ for all $\gamma > \gamma_1$.

Next, consider $z = 1$. Note that $Q(y, 1) = 0$, because $a(1) = 0$ and $c'(0) = 0$. Then, $RHS(1, I) = 0$ for all $I > 0$. This implies $F2(1) = 0$ under the assumption $L'(0) = 0$. Also, because $[J(z)/a(z) - z]$ approaches $\infty$ as $z \to 1$ and because $\lim_{I\to 0} IB(I) = 0$, (5.1) implies $F1(1) = 0$. Therefore, $F1(1) = F2(1)$. That is, $\bar{z} = 1$ is always a steady state.

Now, consider $z = 1 - \varepsilon$, where $\varepsilon > 0$ is an arbitrarily small number. Because $F2(1) = F1(1)$, then $F2(1-\varepsilon) > F1(1-\varepsilon)$ iff $F2'(1) < F1'(1)$. Since $F1'(1) < 0$, this condition is equivalent to $F2'(1)/F1'(1) > 1$. When this condition holds, there are an odd number of interior steady states because $D(z_L) < 0$ and $D(1-\varepsilon) > 0$. Similarly, the number of interior steady states is even (possibly zero) when $F2'(1)/F1'(1) < 1$.

Finally, compare two steady states, one of which has a higher value of $\bar{z}$ than the other. As stated later in Lemma 5.3, $(w, \bar{q}, y)$ are all decreasing functions of $\bar{z}$, and so the lower the value of $\bar{z}$ in a steady state, the higher the values of $(w, \bar{q}, y)$. Also, for any given $z \in (\bar{z}, 1)$, $q^*(z)$ is decreasing in $\bar{z}$, and so a steady state with a lower $\bar{z}$ has higher values of $q^*(z)$. To compare the levels of search intensity between two steady states, note that $I = F1(\bar{z})$ in all steady states. Since $F1' < 0$, then a steady state with a lower $\bar{z}$ has higher search intensity. □

*C.2. Proofs of Lemma 5.3 and Proposition 5.4* The Equation (C.2) expresses search intensity as $I = F2(\bar{z})$. Substituting this function into $y = Y(\bar{z}, I)$, we express $y$ as a function of $\bar{z}$:

$$y = y(\bar{z}) \equiv Y(\bar{z}, F2(\bar{z})).$$

Then, $\bar{q} = Q(y(\bar{z}), \bar{z})$, which is a function of only $\bar{z}$. Substituting $I = F2(\bar{z})$ and $y = y(\bar{z})$ into the expression for welfare, we can write the level of welfare as $w = w(\bar{z})$.

Search intensity does not necessarily increase with $\bar{z}$, because $I = F2(\bar{z})$ and the function $F2$ is not necessarily an increasing function. Differentiating $y = y(\bar{z})$, $\bar{q} = \bar{q}(\bar{z})$ and $w = w(\bar{z})$, and substituting $dI = F2'(\bar{z})d\bar{z}$, we have:

$$dy = \frac{(d\bar{z})}{A}\left[Y_1 N\frac{L''B - L'B'}{B^2} + Y_2\frac{u'^2 a'}{c''}(J - \bar{z}a)\right],$$

$$d\bar{q} = \frac{(d\bar{z})}{A}\frac{u'a'}{c''}\left[\left(\frac{L''B - L'B'}{B^2}N\right)\left(1 + \frac{au''}{a'u'}Y_1\right) - Y_2 u''\frac{y}{IB}\right],$$

$$dw = \frac{(d\bar{z})}{A}\left[Y_1 u'' y\frac{NL'B'}{B^2} + \frac{a'(u')^2}{c''}(J - \bar{z}a)\left[NIL''\left(1 + \frac{au''}{a'u'}Y_1\right) - Y_2 y u''\right]\right],$$

where

$$A \equiv \frac{L''B - L'B'}{B^2}N - u''Y_2\left[\frac{y}{IB} + \frac{u'a}{c''}(J - \bar{z}a)\right] > 0,$$

$$1 + \frac{au''}{a'u'} Y_1 = \frac{(1 - IBKu'')c''}{c'' - IBu''(aJ + Kc'')} > 0.$$

Because $Y_1 \leq 0, Y_2 > 0, B' < 0, a' < 0$, and $u'' \leq 0$, then $dy$, $d\bar{q}$ and $dw$ all have the same sign, which is opposite to the sign of $d\bar{z}$. Moreover, for $z \in (\bar{z}, 1)$, the quantity of goods in the match is $q^*(z) = Q(y, z)$. Since $Q$ decreases with $y$ and $y$ decreases with $\bar{z}$, $q^*(z)$ increases with $\bar{z}$ for any given $z \in (\bar{z}, 1)$. This completes the proof of Lemma 5.3.

For Proposition 5.4, differentiating the equation $F1(\bar{z}) = F2(\bar{z})$ yields:

$$\frac{d\bar{z}}{d\gamma} = \frac{\partial F1/\partial\gamma}{F2'(\bar{z}) - F1'(\bar{z})}.$$

Recall that $\partial F1/\partial\gamma > 0$. In the steady state with the highest welfare, $F2'(\bar{z}) > F1'(\bar{z})$, and so $d\bar{z}/d\gamma > 0$. In the interior steady state ranked the second in welfare, $F2'(\bar{z}) < F1'(\bar{z})$, and so $d\bar{z}/d\gamma < 0$. In general, $d\bar{z}/d\gamma > 0$ in the interior steady state ranked $(2k+1)^{th}$ in welfare and $d\bar{z}/d\gamma < 0$ in the interior steady state ranked $2(k+1)^{th}$, where $k = 0, 1, 2, ....$ The responses of $I$, $y$ and $w$ can then be deduced from Lemma 5.3. Finally, suppose that $dI/d\gamma$ and $d\bar{z}/d\gamma$ have the same sign. Because $dI/d\gamma = F2'(\bar{z})d\bar{z}/d\gamma$, we have $F2'(\bar{z}) > 0 > F1'(\bar{z})$. From the above formula for $d\bar{z}/d\gamma$, we have $d\bar{z}/d\gamma > 0$, and so $dI/d\gamma > 0$. □

## References


1. Benabou, R.: Search, price setting, and inflation. Review of Economic Studies **55**, 353–373 (1988)
2. Benabou, R.: Inflation and efficient in search markets. Review of Economic Studies **59**, 299–329 (1992)
3. Berentsen, A., Rocheteau, G.: On the efficiency of monetary exchange: how divisibility of money matters. Journal of Monetary Economics **49**, 1621–1649 (2002)
4. Berentsen, A., Rocheteau, G.: Money and the gains from trade. International Economic Review **44**, 263–297 (2003)
5. Berentsen, A., Rocheteau, G., Shi, S.: Friedman meets Hosios: efficiency in search models of money. Manuscript, University of Toronto (2001)
6. Burdett, K., Judd, K.: Equilibrium price dispersion. Econometrica **51**, 955–970 (1983)
7. Camera, G., Corbae, D.: Money and price dispersion. International Economic Review **40**, 985–1008 (1999)
8. DeBelle, G., Lamont, O.: Relative price variability and inflation: evidence from US cities. Journal of Political Economy **105**, 132–152 (1997)
9. Engineer, M., Shi, S.,: Asymmetry, imperfectly transferable utility, and fiat money in improving the terms of trade. Journal of Monetary Economics **41**, 153–183 (1998)
10. Grandmont, J.M.: On endogenous competitive business cycles. Econometrica **53**, 995–1045 (1985)
11. Head, A., Kumar, K.: Price dispersion, inflation, and welfare. International Economic Review (forthcoming)
12. Kiyotaki, N., Wright, R.: A contribution to the pure theory of money. Journal of Economic Theory **53**, 215–235 (1991)
13. Lach, S., Tsiddon, D.: The behavior of prices and inflation: an empirical analysis of disaggregated price data. Journal of Political Economy **100**, 349–89 (1992)
14. Lagos, R., Rocheteau, R.: Superneutrality and the welfare effects of inflation. Manuscript, New York University (2003)
15. Lagos, R., Wright, R.: A unified framework for monetary theory and policy analysis. Manuscript, University of Pennsylvania (2002)

16. Molico, M.: The distribution of money and prices in search equilibrium. Ph.D. dissertation, University of Pennsylvania (1998)
17. Reinsdorf, M.: New evidence on the relation between inflation and price dispersion. American Economic Review **84**, 720–731 (1994)
18. Shi, S.: A divisible search model of fiat money. Econometrica **65**, 75–102 (1997)
19. Shi, S.: Search, inflation, and capital accumulation. Journal of Monetary Economics **44**, 81–103 (1999)
20. Shi, S., Wang, W.: The variability of velocity of money in a search model. Manuscript, University of Toronto (2001)