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The value of money in a dynamic equilibrium model*

Manuel S. Santos

Department of Economics, BAC 551, Arizona State University, Tempe, AZ 85287, USA
(e-mail: msantos@asu.edu)

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Summary. This paper studies the pricing of money in an infinite-horizon economy with heterogeneous agents, incomplete financial markets and arbitrary borrowing restrictions. Purchases of the consumption good are subject to a cash-in-advance constraint. Under general conditions I show that the price of money is equal to its fundamental value, where this value is defined over all state-price processes that are compatible with the existence of no-arbitrage opportunities. This equality implies that the cash-in-advance constraint is binding infinitely often for all agents in the economy. The analysis highlights certain differences in the determination of the price of money with respect to models with money in the utility function that bear on the optimal implementation of economic policies.

Keywords and Phrases: Cash-in-advance, Incomplete financial markets, Borrowing limits, Pricing by arbitrage, Fundamental value.

JEL Classification Numbers: D52, E44, G12.

1 Introduction

This paper is concerned with the value of money in a dynamic equilibrium framework. It extends the analysis of Santos and Woodford (1992, 1997) to an economy with a cash-in-advance constraint. My main goal is to study the pricing of money –a security that provides liquidity services– in an infinite-horizon framework with heterogenous agents, incomplete financial markets and arbitrary borrowing limits. Incomplete markets are the norm in economies with frictions which may arise from

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the valuation and exchange of assets and commodities. The shadow values attached to monetary holdings are derived under conditions of no-arbitrage for trading over an infinite horizon. The analysis provides a general set of assumptions under which the price of money is equal to its fundamental value, that is, to the expected discounted value of its future liquidity services over the set of state-price processes compatible with the existence of no-arbitrage opportunities. This equality between the price of money and its fundamental value over the set of all state-price processes implies that the cash-in-advance constraint must be binding infinitely often for all agents in the economy. Therefore, the mechanism for price determination in the cash-in-advance model is quite different from those in models with money in the utility function, and it leaves a more limited scope for the influence of fiscal variables on the value of money.

Earlier work on the pricing of money focused on models with a representative agent (e.g., Lucas, 1984; Svensson, 1985; Townsend, 1987) and with money in the utility function (Tirole, 1985). These simplifying assumptions could be justified on computational grounds, but seem rather restrictive for the study of qualitative properties of equilibrium solutions. In these papers the fundamental value of money is calculated using the marginal utilities of the representative agent. These marginal utilities are only observed under specific functional forms in the model's primitives. Therefore, these studies leave unexplored the determination of the price of money in models with heterogeneous agents and financial markets frictions. In these latter economies, agents' marginal utilities may not be collinear, and hence agents may attach different valuations to the liquidity services that money provides.

A main conclusion to be drawn from Santos and Woodford (1997) is that asset pricing bubbles occur in dynamic equilibrium models under rather limited circumstances. For every equilibrium solution the price of a security in positive net supply must be equal to its fundamental value provided that the present value of aggregate wealth is finite.[1] As explained below, the assumption of a finite wealth seems quite plausible (e.g., see Abel et al., 1989), and precludes the existence of valued fiat money as a mere store of wealth in a broad class of economies comprising an infinite number of heterogeneous consumers, incomplete financial markets and arbitrary borrowing restrictions. Therefore, some typical monetary models such as the overlapping generations model (OLG) of Samuelson (e.g., see Wallace, 1980; Tirole, 1985) or models with a finite number of infinitely lived dynasties (e.g., see Bewley, 1980) would sustain rather fragile theories of money.

High-powered monetary aggregates usually earn lower returns, but may perform some other functions such as to facilitate the exchange of commodities and assets. In some economic models these trading restrictions are explicitly written down, as for instance in models with cash-in-advance constraints (e.g., Lucas and Stokey, 1987; Townsend, 1987), or in the recent search-founded theories of money (e.g., Green and Zhou, 2002; Kiyotaki and Wright, 1989; Trejos and Wright, 1995; Zhu, 2003). There are also various reduced-form models in which the transaction purpose is expressed in the form of money in the utility function or as an input of some

[1] Some further basic assumptions underlying this result are competitive behavior, homogeneous beliefs, and rational expectations. These latter assumptions are shared by a variety of monetary models.

transactions technology (e.g., Sims, 1994; Ljungqvist and Sargent, 2000, Ch. 17). In all these settings one would expect that money can be valued as any other asset so that in the absence of bubbles the price of money can be written as the expected discounted value of the stream of its future intangible returns. There is, however, an important feature in the determination of the price of money: The value of its intangible returns depends in turn on the price of money. Then, the current spot price of money will be determined by the path of all its future prices. This fact has been stressed in Tirole (1985), and it can be thought as a major source of multiplicity of monetary equilibria. Theories on the pricing of money should thus be useful for the understanding of the effects of monetary and fiscal policies and for the problem of multiplicity of self-fulfilling equilibria.

If money enters as an argument in the utility function or in a reduced-form transaction technology, its liquidity services will correspond to the utility or productivity that money delivers. In cash-in-advance economies, money will yield these intangible returns only if such constraint is binding for at least one agent in the economy at some future date. Indeed, it is shown below that if money is positively valued then the cash-in-advance constraint must be binding for an infinite sequence of dates that apply to *all* agents in the economy. I would like to draw two main conclusions from this basic result. First, this characterization is in the spirit of the *quantity theory*, as it imposes a lower bound on the price of money that depends on the output level and the available quantity of money. Increases in the supply of money must come about with readjustments in prices so that the cash-in-advance constraint will again be binding in the aggregate at an infinite sequence of dates. Second, in real-world situations money balances are usually not exhausted by all agents at any given date. This suggests that the cash-in-advance constraint is a rather crude modelization of the transaction process. In our simple framework households do not incur time costs to get money from the bank (cf., Jovanovic, 1982, for a general analysis of a Baumol-Tobin model), and the velocity of money has been exogenously fixed. But even for those enlarged settings in which the acquisition of money may be costly and visits to the bank may not be synchronized, an analogous result should still prevail for the cash-in-advance to be binding in a certain general sense for *all* agents in the economy.

Section 2 presents the model. Section 3 is concerned with the valuation of a stream of goods and monetary holdings over an infinite horizon under conditions of no-arbitrage. Section 4 establishes a generalized transversality condition in which the value of the optimal consumption plan must exhaust the household's wealth. This transversality condition is the basic tool to derive a fundamental asset pricing equation in which the price of an asset can be identified with its fundamental value over the generalized system of state-price processes. In the case of money the fundamental value is determined by the stream of future liquidity services, and a positive price will imply that the cash-in-advance constraint must be binding for all households in the economy over a subsequence of dates. Section 5 links these results to related research on the pricing of money in models with a representative agent or with money in the utility function, and explores the determination of the price of money by fiscal variables under a cash-in-advance constraint. Section 6 concludes with some final remarks. The Appendix outlines proofs of the main results.

2 The model

Our framework of analysis is an infinite-horizon economy with sequential trading, borrowing limits, and cash-in-advance. The basic features of sequential trading follow the lines of our previous paper (Santos and Woodford, 1997). Several elements of this study were shown to play a marginal role in the pricing of assets, especially under the assumption of a finite value of wealth for the economy. Hence, the present model will be highly simplified and will not consider incomplete participation –or the existence of a countably infinite number of agents– and short-lived securities. There exists but a single consumption good at all times and a fixed set of securities that conform the financial structure of the economy. There is not an explicitly defined production sector.

There exists a finite set H of households and a government who hold homogeneous beliefs about the evolution of the economy. The dynamic process of revelation of information is represented by an event tree S. The term s^t indicates a generic information set or node in S that may be reached at date t, for $t = 0, 1, 2, \cdots$. Each node s^t has a unique immediate predecessor, denoted $s^t - 1$, which is dated $t - 1$. There is a unique information set s^0, the only one dated 0. Every node s^t has a finite number ξ of immediate successors. The term $s^T | s^t$, for $T \geq t$, stipulates that node s^T belongs to the subtree whose root is s^t; i.e., either $s^T = s^t$ or s^t is a predecessor of s^T.

2.1 Financial markets

At each node $s^t \in S$, there exist spot markets for the consumption good and k securities that span the horizon of the economy. These securities are specified by the current vector of prices, $q(s^t)$, and the returns they promise to deliver at future information sets. Security prices $q(s^t)$ are assumed to be non-negative, a condition implied by free disposal of securities. As explained in Santos and Woodford (1997) this rather mild assumption will simplify the analysis considerably and rules out some pathological situations in which the bubble component is negative.

For a given security j available for trading at s^t, let $\eta_j(s^{t+1}) \in R$ be the quantity of monetary units or nominal return, and $d_j(s^{t+1}) \in R$ be the quantity of the consumption good or real return, that a holder of one unit of security j will be entitled to receive at node s^{t+1}. Then, $x_j(s^{t+1}) = q_m(s^{t+1})\eta_j(s^{t+1}) + d_j(s^{t+1})$ specifies the total return to be realized by security j for a given price for money $q_m(s^{t+1})$, and $R(s^{t+1}) = q(s^{t+1}) + x(s^{t+1})$ is the vector of payoffs, whose j^{th} element is the payoff in terms of the consumption good that a unit of security j delivers at node s^{t+1}. The $\xi \times k$ matrix $V(s^t) = [R(s^{t+1})']_{s^{t+1}|s^t}$ lists the payoffs obtained at every $s^{t+1}|s^t$. Returns $x_j(s^{t+1})$ are assumed to be non-negative, and the vector of payoffs $R(s^{t+1})$ has at least one positive component.

2.2 The government

Of the above k securities, the government sector can print money and issue a bond. Also, the government collects $\theta(s^t)$ units of the consumption good from lump-sum taxation, and must finance a required quantity of public consumption $c^g(s^t)$ at each node s^t.

The government can introduce money into the economy in two ways: (a) as households transfers granted at the beginning of the trading period, and (b) as purchases of the consumption good or purchases of bonds. For simplicity, money transfers are distributed to households in accordance with their money balances. Hence, if $z_m(s^t)$ is the total amount of money held by households at the end of node s^t, then $(1 + \eta_m(s^{t+1}))z_m(s^t)$ will be the quantity of money held by these households at the beginning of node $s^{t+1}|s^t$. The bond is also a nominal security which is issued to retire old debt and to finance public spending. $z_b(s^t)$ is the outstanding stock of this security at the end of trading at s^t, and $R_b(s^{t+1})z_b(s^t) = [q_b(s^{t+1}) + q_m(s^{t+1})\eta_b(s^{t+1})]z_b(s^t)$ are the liabilities honored by the government at each $s^{t+1}|s^t$.

At every node $s^t \in S$, the government must satisfy the following budget balance

$$\begin{aligned} &R_m(s^t)z_m(s^t - 1) + R_b(s^t)z_b(s^t - 1) + c^g(s^t) \\ &= q_m(s^t)z_m(s^t) + q_b(s^t)z_b(s^t) + \theta(s^t) \end{aligned} \tag{2.1}$$

where $R_m(s^t) = (1 + \eta_m(s^t))q_m(s^t)$, and $R_b(s^t) = q_b(s^t) + q_m(s^t)\eta_b(s^t)$. For simplicity, government purchases are not subject to a cash-in-advance constraint. The supply of bonds $z_b(s^t)$ is assumed to be bounded below by some constant $\varepsilon > 0$, and the money supply $z_m(s^t) \geq \varepsilon\zeta_m(s^t)$, where $\zeta_m(s^t) = (1+\eta_m(s^t))\zeta_m(s^t-1)$ for all s^t, and $\zeta_m(s^0) = 1$. Finally, at s^0 the first two terms in the left-hand side of (2.1) should read as $R_m(s^0)z_m(s^0 - 1) = q_m(s^0)z_m(s^0 - 1)$ and $R_b(s^0)z_b(s^0 - 1) = q_b(s^0)z_b(s^0 - 1)$, where $z_m(s^0 - 1) > 0$ and $z_b(s^0 - 1) > 0$ denote the initial existing stocks of these securities.[2]

2.3 Households

Each household $h \in H$ has a preference ordering $\succeq^h$, defined on the consumption set, $X^h = \prod_{s^t \in S} R_+$. That is, it is defined for all consumption plans c^h involving non-negative consumption of the good at each node $s^t \in S$. The preference ordering $\succeq^h$ is strictly increasing on X^h.

Agent h enters the financial markets at node s^0 with an initial endowment of securities $z^h(s^0 - 1)$. In addition, at each node s^t the agent receives an endowment of $\omega^h(s^t)$ units of the consumption good and has to pay $\theta^h(s^t)$ units as lump-sump taxes. Then spot markets for the trading of the aggregate good and securities are

[2] As already pointed out, the analysis breaks down in the presence of securities in negative supply. The lower bound $\varepsilon > 0$ on the stock of bonds rules out some pathological indeterminacies in which the government may artificially inflate the price of the bond by holding negligible or negative amounts of the security at the expense of heavily taxing households. A similar condition applies for the process of the money supply that takes account of the fact that variations in this stock may also happen via household transfers. Therefore, the money supply $z_m(s^t)$ may converge to 0 only if $\zeta_m(s^t)$ converges to 0. Examples of monetary bubbles can be constructed from the literature of the optimal quantity of money (e.g., Grandmont and Younes, 1973) in which the aggregate quantity of money converges steadily to zero. Hence, some mild restriction on the evolution of the aggregate quantity of money seems necessary for the results below.

open, and purchases of the good must be paid for with money held over from the previous period and the associated monetary transfers. Agents only observe the current price vector $q(s^t)$ and form beliefs about future security prices and returns. Each household h chooses a desired quantity of consumption $c^h(s^t)$ and a k-vector of securities $z^h(s^t)$ to hold at the end of trading at s^t, subject to the following financial and liquidity constraints,

$$q(s^t)'z^h(s^t) + c^h(s^t) \leq R(s^t)'z^h(s^t - 1) + \omega^h(s^t) - \theta^h(s^t) \quad (2.2a)$$

$$q(s^t)'z^h(s^t) \geq -B^h(s^t) \quad (2.2b)$$

$$R_m(s^t)z_m^h(s^t - 1) \geq c^h(s^t) \quad (2.2c)$$

Condition (2.2a) is the standard Arrow-Radner budget constraint for an economy with sequential trading in which total revenues come from the payoffs of security holdings and the endowment of the aggregate good net of the lump-sum tax sets. These revenues can be spent to purchase the consumption good and a new portfolio of securities. Again, at s^0 the payoff from security holdings $R(s^0)'z^h(s^0 - 1)$ must be replaced by the value of the initial endowment of securities $q(s^0)'z^h(s^0 - 1)$.

Condition (2.2b) imposes a limit on the extent to which household h can be indebted. Usually the borrowing limit $B^h(s^t)$ depends on the sequence of prices and returns $\{q(s^T), x(s^T)\}$ for all $s^T|s^t$, but a household h takes the process of borrowing limits $\{B^h(s^t)\}$ as given, just as prices are taken as given. The only general assumption regarding borrowing is that $B^h(s^t) \geq 0$ for each household h at each node s^t. Condition (2.2c) is the cash-in-advance constraint for purchases of the consumption good. This is a rather simple way to model the need for liquidity.

Finally, the following assumption will be needed for the derivation of the generalized transversality condition established in Section 4 (see Santos and Woodford, 1997, p. 45, for a counterexample to this transversality condition). This is a joint assumption on preferences and the aggregate endowment. Let $z(s^t) = \sum_{h \in H} z^h(s^t)$ be the aggregate endowment of securities and let $\hat{\omega}(s^t) = \sum_{h \in H} \omega^h(s^t) + d(s^t)'z(s^t)$ be the total supply of the consumption good at s^t. Now, write the consumption plan $c^h \in X^h$ as $c^h = (c_-^h(s^t), c^h(s^t), c_+^h(s^t))$, where $c^h(s^t)$ denotes consumption of the aggregate good at node s^t; $c_+^h(s^t)$ denotes the collection of components $\{c^h(s^T)\}$ over all $s^T|s^t \in S$, excluding s^t; and $c_-^h(s^t)$ denotes the remaining components of c^h not included in the pair $(c^h(s^t), c_+^h(s^t))$. Then, for each h, there exist $K \geq 1$ and $0 < \gamma^h < 1$ such that for every $s^t \in S$,

$$(c_-^h(s^t), c^h(s^t) + K\hat{\omega}(s^t), \gamma c_+^h(s^t)) \succ^h c^h \quad (2.3)$$

for all consumption paths satisfying $c^h(s^t) \leq 2K\hat{\omega}(s^t)$ at each $s^t \in S$, and all $\gamma \geq \gamma^h$.

The term $\succ^h$ denotes strict preference. Hence, (2.3) asserts that at every s^t the household prefers a modified consumption plan in which current consumption is increased by $K\hat{\omega}(s^t)$ units and future consumption is scaled down by a small $uniform$ amount for all nodes $s^T|s^t$. This is a mild condition that requires a certain kind of uniform impatience. In the absence of this condition a household may accumulate arbitrary amounts of wealth over small probability events at the optimal solution. See Hernandez and Santos (1996), Levine and Zame (1996) and Magill and Quinzii (1994) for further discussions of this condition in infinite-horizon economies with incomplete markets.

2.4 Equilibrium

Let an economy with sequential trading be specified by an information structure S; a set of k securities generating the infinite process of returns $\{\eta(s^t), d(s^t)\}$; a government that chooses security processes $\{z_m(s^t), z_b(s^t)\}$, and streams of tax revenues $\{\theta(s^t)\}$ and public consumption $\{c^g(s^t)\}$; a set of households H characterized by the preference orderings $\{\succeq^h\}$, the endowment streams $\{\omega^h(s^t)\}$, sequences of lump-sum taxes $\{\theta^h\}$, initial security endowments $\{z^h(s^0-1)\}$, and the sequences of functions defining the borrowing limits $\{B^h(s^t)\}$. Then, the processes $\{\bar{q}(s^t), z_m(s^t), z_b(s^t), \theta(s^t), c^g(s^t), \bar{c}^h(s^t), \bar{z}^h(s^t)\}$ define a $rational$ $expectations$ $equilibrium$ if

(i) *Governement's budget balance*: The sequence $\{z_m(s^t), z_b(s^t), \theta(s^t), c^g(s^t)\}$ satisfies (2.1) at each $s^t \in S$, where $z_m(s^0-1) = \sum_{h\in H} z_m^h(s^0-1)$, $z_b(s^0-1) = \sum_{h\in H} z_b^h(s^0-1)$, and $\theta(s^t) = \sum_{h\in H} \theta^h(s^t)$ at each $s^t \in S$.
(ii) $Optimality\ of\ households'\ behavior$: For each $h \in H$, the consumption-portfolio plan $\{\bar{c}^h(s^t), \bar{z}^h(s^t)\}$ is optimal for the preference ordering $\succeq^h$ subject to the budget constraints (2.2) being satisfied at each $s^t \in S$.
(iii) $Market\ clearing$: For every $s^t \in S$,

$$c^g(s^t) + \sum_{h\in H} \bar{c}^h(s^t) = \sum_{h\in H} \omega^h(s^t) + d(s^t)'z(s^t) \tag{2.4a}$$

$$\sum_{h\in H} \bar{z}^h(s^t) = z(s^t) \tag{2.4b}$$

This is a standard equilibrium notion for sequential markets economies, also known as an Arrow-Radner equilibrium. Note that the supply of government securities $z_m(s^t)$ and bonds $z_b(s^t)$ may change over time, but for every other security $z_j(s^t) = z_j(s^0-1)$ for all $s^t \in S$.

3 Pricing by arbitrage and the fundamental value of money

For any equilibrium price process $\{\bar{q}(s^t)\}$ the following condition must hold at each $s^t \in S$,

$$\{z{\in}R^k : [-\bar{q}(s^t)'z, \bar{q}_m(s^t)z_m, V(s^t)z]{\geq}0 \text{ with either } \bar{q}(s^t)'z{\neq}0 \text{ or } V(s^t)z{\neq}0\}{=}\emptyset \tag{3.1}$$

It should be stressed that this condition must hold in a rational expectations equilibrium. For if there is a portfolio z such that $\bar{q}(s^t)'z < 0$ and $V(s^t)z \geq 0$ then under the above conditions on non-negative returns one can construct a second portfolio $\widehat{z}$ such that $\bar{q}(s^t)'\widehat{z} = 0$ and $V(s^t)\widehat{z} > 0$.

Condition (3.1) implies the existence of a set of state prices $a(s^t) > 0$ and $\{a(s^{t+1})\}$ with $a(s^{t+1}) > 0$ for all $s^{t+1}|s^t$, and an additional shadow price $\nu(s^t) \geq 0$, such that

$$a(s^t)q_j(s^t) = \sum_{s^{t+1}|s^t} a(s^{t+1})R_j(s^{t+1}) \tag{3.2a}$$

for every security $j \neq m$, and

$$a(s^t)q_m(s^t) = \sum_{s^{t+1}|s^t} a(s^{t+1})R_m(s^{t+1}) + \nu(s^t)q_m(s^t) \tag{3.2b}$$

for security m. The existence of this generalized state-price system follows from a simple extension of the Minkowski-Farkas lemma (e.g., Duffie, 1988, p. 71). Hence, for every positive value $a(s^t)$ there exists some vector $(a(s^{t+1}), \nu(s^t))$ that solves the system of equations (3.2). This vector is unique if the $k \times (\xi + 1)$ matrix

$$\begin{bmatrix} & 0 \\ V(s^t)' : & \vdots \\ & 0 \\ & q_m(s^t) \end{bmatrix} \tag{3.3}$$

has rank equal to $\xi + 1$. This implies that there should be at least $\xi + 1$ securities, where ξ is the number of immediate successor nodes assumed to be constant over time. Note that even if the vector $(a(s^{t+1}), \nu(s^t))$ is uniquely determined, it does not mean that security markets are complete at node s^t in the usual technical sense. Indeed, in our sequential markets economy with the cash in-advance constraint, complete security markets at s^t would require the existence of at least ξ types of money so that the resulting money holdings over all the successor nodes $s^{t+1}|s^t$ would lie in a full ξ-dimensional space.

Proceeding inductively one can construct a generalized state-price process $\{a(s^t), \nu(s^t)\}$ for the entire information structure such that (3.2) holds at all nodes $s^t \in S$. Let $A(s^t)$ denote the set of such processes for the subtree with root s^t. Every generalized state-price process $\{a(s^t), \nu(s^t)\}$ provides an obvious way of evaluating a future stream of consumption goods and money holdings. Thus, for any non-negative stream $\{\omega(s^T), \kappa(s^T)\}$, specifying quantities of the consumption good $\omega(s^T) \geq 0$ and monetary holdings $\kappa(s^T) \geq 0$ with $\kappa(s^T) = (1 + \eta_m(s^T))\kappa(s^T - 1)$, at each $s^T|s^t$ for all $T > t$, any non-negative price process $\{q(s^T)\}$, and any $(a, \nu) \in A(s^t)$, one can define the present value at s^t of the subsequent stream as

$$v_{(\omega,\kappa)}(s^t; q; a, \nu) \equiv \frac{1}{a(s^t)} \sum_{T=t}^{\infty} \sum_{s^T|s^t} [a(s^{T+1})\omega(s^{T+1}) + \nu(s^T)q_m(s^T)\kappa(s^T)] \tag{3.4}$$

Note that $v_{(\omega,\kappa)}(s^t; q; a, \nu)$ may vary with $\{a(s^t), \nu(s^t)\}$.

One main justification for these valuations is provided by the following result stating that $v_{(\omega,\kappa)}(s^t; q; a, \nu)$ describes the feasible range of values for self-financing portfolios whose returns dominate and are dominated by (ω, κ) when trading is allowed over an increasing sequence of dates so as to encompass the infinite horizon of the economy. Let $\{\omega(s^t), \kappa(s^t)\}$ be a non-negative stream of resources and monetary holdings, and assume that $\{q(s^t)\}$ is a sequence of security prices. Then, at any node s^t let us define

$$\underline{\pi}^T_{(\omega,\kappa)}(s^t) \equiv \sup q(s^t)' z(s^t)$$

where the supremum is over all plans $\{z(s^r)\}$ for the nodes $s^r|s^t$, such that

$$R(s^{r+1})' z(s^r) \leq \omega(s^{r+1}) + q(s^{r+1})' z(s^{r+1}) \tag{3.5a}$$

$$\kappa(s^{r+1}) \geq R_m(s^{r+1}) z_m(s^r) \tag{3.5b}$$

for all r with $T > r \geq t$, and $q(s^T)' z(s^T) \leq 0$. That is, $\underline{\pi}^T_{(\omega,\kappa)}(s^t)$ represents the least upper bound over all portfolio plans $\{z(s^r)\}$ whose returns and monetary holdings are dominated by the stream $\{\omega(s^r), \kappa(s^r)\}$ for all $s^r|s^t$, with $T \geq r > t$. In other words, $\underline{\pi}^T_{(\omega,\kappa)}(s^t)$ is the maximum amount that can be borrowed at node s^t against the flow of income $\{\omega(s^r)\}$ for all nodes $s^r|s^t$ with $T \geq r > t$, without exceeding the liquidity holdings $\{\kappa(s^r)\}$, such that the debt will be repaid at all terminal nodes $s^T|s^t$ of a fixed date T.

Similarly, at node s^t we may define

$$\overline{\pi}^T_{(\omega,\kappa)}(s^t) \equiv \inf q(s^t)' z(s^t)$$

where the infimum is over all plans $\{z(s^r)\}$ for the nodes $s^r|s^t$ with $r \geq t$, such that

$$R(s^{r+1})' z(s^r) \geq \omega(s^{r+1}) + q(s^{r+1})' z(s^{r+1}) \tag{3.5c}$$

$$R_m(s^{r+1}) z_m(s^r) \geq \kappa(s^{r+1}) \tag{3.5d}$$

for all r with $T > r \geq t$, and $q(s^T)' z(s^T) \geq 0$. That is, $\overline{\pi}^T_{(\omega,\kappa)}(s^t)$ represents a greatest lower bound for the amount of wealth needed at node s^t in order to dominate the flow of income $\{\omega(s^r)\}$ and liquidity holdings $\{\kappa(s^r)\}$ for all $s^r|s^t$ with $T \geq r > t$.

Now, let

$$\underline{\pi}_{(\omega,\kappa)}(s^t) \equiv \lim_{T\to\infty} \underline{\pi}^T_{(\omega,\kappa)}(s^t) \tag{3.6a}$$

and

$$\overline{\pi}_{(\omega,\kappa)}(s^t) \equiv \lim_{T\to\infty} \overline{\pi}^T_{(\omega,\kappa)}(s^t) \tag{3.6b}$$

It follows that $\underline{\pi}_{(\omega,\kappa)}(s^t)$ and $\overline{\pi}_{(\omega,\kappa)}(s^t)$ are the tightest lower and upper valuations at s^t for the subsequent stream of goods and monetary holdings $\{\omega(s^r), \kappa(s^r)\}$ for all $s^r|s^t$ with $r > t$, that can be derived from the values of all self-financing portfolios by extending gradually the trading horizon of the economy to the infinite future. Note that this construction avoids unlimited financing with no resources by rolling over all debts at every node, or what it is known as Ponzi schemes.

Proposition 3.1. Let $\{\omega(s^r), \kappa(s^r)\}$ be a non-negative process, specifying an infinite stream of resources $\{\omega(s^r)\}$ and monetary holdings $\{\kappa(s^r)\}$, for all $s^r|s^t$, $r > t$. Then

$$\overline{\pi}_{(\omega,\kappa)}(s^t) \equiv \sup_{(a,\nu)\in A(s^t)} v_{(\omega,\kappa)}(s^t; q; a, \nu) \tag{3.7a}$$

$$\underline{\pi}_{(\omega,\kappa)}(s^t) \leq \inf_{(a,\nu)\in A(s^t)} v_{(\omega,\kappa)}(s^t; q; a, \nu) \tag{3.7b}$$

Furthermore, assume that $\underline{\pi}_{(\omega,\kappa)}(s^r) < +\infty$ for all $s^r|s^t$, $r > t$. Then

$$\underline{\pi}_{(\omega,\kappa)}(s^t) \equiv \inf_{(a,\nu)\in A(s^t)} v_{(\omega,\kappa)}(s^t; q; a, \nu) \tag{3.7c}$$

This proposition is a natural extension of well-known results for finite-horizon economies. The values $v_{(\omega,\kappa)}(s^t; q; a, \nu)$, over all (a, ν) in $A(s^t)$, are within the range of feasible limits obtained by pricing the bundle of goods and monetary holdings (ω, κ) from self-financing portfolios, after successive truncations of the horizon; moreover, $\underline{\pi}_{(\omega,\kappa)}(s^t)$ and $\overline{\pi}_{(\omega,\kappa)}(s^t)$ are the tightest lower and upper limits for these values.

These bounds determine the range of variation of the fundamental value of a security. Let $\{x_j(s^t)\}$ be the process of dividends generated by one unit of security j, for $j \neq m$. Then, it follows from (3.4)-(3.7) that for each $s^t \in S$,

$$q_j(s^t) \geq \overline{\pi}_{(x_j,0)}(s^t) \geq \underline{\pi}_{(x_j,0)}(s^t) \tag{3.8}$$

These weak inequalities also hold for $j = m$, but it is worth pointing out that

$$v_{(0,\zeta_m)}(s^t; q; a, \nu) = \frac{1}{a(s^t)} \sum_{T=t}^{\infty} \sum_{s^T|s^t} \nu(s^T) q_m(s^T) \zeta_m(s^T) \tag{3.9}$$

where $\zeta_m(s^T) = (1 + \eta_m(s^T))\zeta_m(s^T - 1)$ for all $s^T|s^t$, $T > t$, and $\zeta_m(s^t) = 1$. Observe that money transfers are not counted as nominal returns of the security, since these transfers add on to the stock of money.

If $q_j(s^t) = \underline{\pi}_{(x_j,0)}(s^t)$ in (3.8), then Proposition 3.1 implies that $q_j(s^t) = v_{(x_j,0)}(s^t; q; a, \nu)$ for all $(a, \nu) \in A(s^t)$. Hence, the price of the security is always equal to its fundamental value. But if $q_j(s^t) > \underline{\pi}_{(x_j,0)}(s^t)$ then by Proposition 3.1 there exists some $(a, \nu) \in A(s^t)$ such that $q_j(s^t) > v_{(x_j,0)}(s^t; q; a, \nu)$. In this latter case the price of the security is greater than its fundamental value, and so there exists a bubble on security j for some (a, ν).

The ability of a household to borrow against the value of its future income is summarized by the process of borrowing limits $\{B^h(s^t)\}$. An important case is when

$$B^h(s^t) = \underline{\pi}_{(\omega^h-\theta^h,0)}(s^t) \text{ for all } s^t \in S \tag{3.10}$$

As already discussed, this is the maximum permissible amount that can be borrowed against the process $\{\omega^h(s^t) - \theta^h(s^t)\}$ over all paths in the information structure S, so that a household h will never get into a state of bankruptcy; i.e., the debt could be repaid under all contingencies. This budget constraint has the following properties.

Proposition 3.2. Assume that the process of borrowing limits $\{B^h(s^t)\}$ satisfies (3.10). Assume that (c^h, z^h) is budget feasible under constraints (2.2). Then,

(i) The plan (c^h, z^h) satisfies

$$\sum_{t=0}^{\infty}\sum_{s^t} a(s^t)c^h(s^t) + \sum_{t=0}^{\infty}\sum_{s^t} \nu(s^t)q_m(s^t)z_m^h(s^t) \tag{3.11a}$$

$$\leq \sum_{t=0}^{\infty}\sum_{s^t} a(s^t)[\omega^h(s^t) - \theta^h(s^t)] + a(s^0)q(s^0)'z^h(s^0-1)$$

for all $(a,\nu) \in A(s^0)$ such that $\sum_{t=0}^{\infty}\sum_{s^t} a(s^t)[\omega^h(s^t) - \theta^h(s^t)] < +\infty$.

(ii) The plan (c^h, z^h) satisfies

$$\lim_{T\to\infty} \sum_{s^T|s^t} a(s^T)q(s^T)'z^h(s^T) \geq 0 \tag{3.11b}$$

for each $s^t \in S$, and for all $(a,\nu) \in A(s^0)$ such that $\sum_{t=0}^{\infty}\sum_{s^t} a(s^t)[\omega^h(s^t) - \theta^h(s^t)] < +\infty$.

Moreover, if $(\bar{c}^h, \bar{z}^h)$ is an optimal plan then $\underline{\pi}_{(\omega^h-\theta^h,0)}(s^t) < +\infty$ for every s^t. Hence, by Proposition 3.1 it follows that $\sum_{t=0}^{\infty}\sum_{s^t} a(s^t)[\omega^h(s^t) - \theta^h(s^t)] < +\infty$ for some $(a,\nu) \in A(s^0)$.

If sequential markets are not complete, or the process of borrowing limits $\{B^h(s^t)\}$ does not satisfy (3.10) for all households, then there may be no common set $(a,\nu) \in A(s^0)$ for which $\sum_{t=0}^{\infty}\sum_{s^t} a(s^t)[\omega^h(s^t) - \theta^h(s^t)] < +\infty$ for all h (e.g., see Santos and Woodford, 1997, Examples 4.2 and 4.4). Therefore, in the present framework the existence of a finite number of households does not entail that the equilibrium aggregate wealth is finite valued. There exist, however, some well-known conditions under which this latter result holds:

(i) *A household owns a fraction of the aggregate endowment*: Let $\omega = \sum_{h\in H}\omega^h$. Assume that there is $h \in H$ such that

$$\omega^h(s^t) \geq \varepsilon\omega(s^t) \text{ for all } s^t \in S \tag{3.12a}$$

for some $\varepsilon > 0$. If there exists some $(a,\nu) \in A(s^0)$ such that $\sum_{t=0}^{\infty}\sum_{s^t} a(s^t)\omega^h(s^t) < +\infty$, then $\sum_{t=0}^{\infty}\sum_{s^t} a(s^t)\omega(s^t) < \frac{1}{\varepsilon}\sum_{t=0}^{\infty}\sum_{s^t} a(s^t)\omega^h(s^t) < +\infty$.

(ii) *The interest rate is consistently greater than the rate of output growth in the economy*: If the state-price process $\{a(s^t)\}$ converges fast enough to zero, then $\sum_{t=0}^{\infty}\sum_{s^t} a(s^t)\omega(s^t) < +\infty$. In stochastic frameworks this condition may be hard to check, since there is no simple recognizable asset whose payoffs always dominate the growth of output. Hence, the following more restrictive condition seems of practical relevance.

(iii) *The endowment allocation is bounded by a portfolio-trading plan*: Assume that there exists a hypothetical self-financing portfolio $\tilde{z}$ such that

$$x(s^t)'\tilde{z}(s^t-1) \geq \varepsilon\omega(s^t) \;\text{ for all }\; s^t \in S \tag{3.12b}$$

for some $\varepsilon > 0$. Then (3.7)-(3.8) imply that $\sum_{t=1}^{\infty}\sum_{s^t} a(s^t)\omega(s^t) < \frac{1}{\varepsilon}a(s^0)q(s^0)'\tilde{z}(s^0) < +\infty$ for all feasible state-price processes $\{a(s^t)\}$.

Our model economy does not consider an explicit production sector, but production can be easily accommodated in the present framework as the analysis focusses on a given equilibrium security price process $\{\bar{q}(s^t)\}$. For such a process one can take as given the production and investment plans of the firms, and aggregate the value of all these productive units into a single share. Then, $\{d(s^t)\}$ defines the aggregate process of gross returns net of investment of all capital goods, and $\{\omega(s^t)\}$ would correspond to the income commanded by raw materials, raw labor, and innate skills. In advanced economies the share of income of physical and human capital net of the investment rate is consistently greater than zero (e.g., see Abel et al., 1989). Hence, condition (3.12b) is usually met by some aggregate measure of the capital stock, and so the stream of future incomes generated by these economies must have a finite present value.

4 Main results

All main results in this section should be understood to hold for a given rational expectations equilibrium $\{\bar{q}(s^t), z_m(s^t), z_b(s^t), \theta(s^t), c^g(s^t), \bar{c}^h(s^t), \bar{z}^h(s^t)\}$. The main tool in this analysis is the following generalized transversality condition.

Proposition 4.1. Let $\sum_{t=0}^{\infty}\sum_{s^t} a(s^t)\omega(s^t) < +\infty$ for $(a,\nu) \in A(s^0)$. Then,

$$\lim_{T\to\infty}\sum_{s^T} a(s^T)\bar{q}(s^T)'\bar{z}^h(s^T) \leq 0 \text{ for every } h \in H \tag{4.1}$$

Hence, if for some feasible state process $\{a(s^t)\}$ the aggregate endowment ω is finite valued, then at the optimal plan the value of every household's portfolio holdings over all nodes at date T must converge to zero as T goes to ∞, when these portfolio holdings are evaluated under the given state-price process $\{a(s^t)\}$. Note that condition (4.1) entails that (3.11a) must hold with equality for every equilibrium allocation. Let $\Delta_m(s^t) = z_m(s^t) - (1+\eta_m(s^t))z_m(s^t-1)$ at each s^t.

Corollary 4.2. Let $\sum_{t=0}^{\infty}\sum_{s^t} a(s^t)\omega(s^t) < +\infty$ for $(a,\nu) \in A(s^0)$. Then

$$\bar{q}_b(s^0)\hat{z}_b(s^0-1) + \sum_{t=0}^{\infty}\sum_{s^t} a(s^t)c^g(s^t) = \sum_{t=0}^{\infty}\sum_{s^t} a(s^t)[\bar{q}_m(s^t)\Delta_m(s^t) + \theta(s^t)] \tag{4.2}$$

Corollary 4.2 is a mere consequence of the transversality condition (4.1) and equilibrium identities (2.4). The present value of taxation and money creation net of money transfers to households must finance the initial public debt and the flow of public consumption. New bond issues at each $s^t \neq s^0$ will serve to refinance consumption and current debt liabilities, but these additional debt issues will eventually have to be repaid in their full amount. It should be stressed that (4.1)-(4.2) are necessary conditions for the existence of an equilibrium solution under the assumption of a finite aggregate wealth. These conditions rule out the existence of asset pricing bubbles for securities in positive net supply.

Theorem 4.3. Let $\overline{\pi}_{(\omega,0)}(s^0) < +\infty$. Then

$$\bar{q}_j(s^t) = \underline{\pi}_{(x_j,0)}(s^t) \text{ for every security } j \neq m \text{ such that } z_j(s^t) > 0 \tag{4.3a}$$

and

$$\bar{q}_m(s^t) = \underline{\pi}_{(0,\zeta_m)}(s^t) \tag{4.3b}$$

for every $s^t \in S$.

Say that for household h the cash-in-advance constraint (2.2c) is binding $from$ node s^t if $\bar{R}_m(s^{t+1})\bar{z}_m^h(s^t) = \bar{c}^h(s^{t+1})$ for some $s^{t+1}|s^t$. In other words, constraint (2.2c) is binding from node s^t if the household cannot reduce her end-of-period holdings $\bar{z}_m^h(s^t)$ without violating this constraint at some successor node $s^{t+1}|s^t$.

Theorem 4.4. Assume that $\overline{\pi}_{(\omega,0)}(s^0) < +\infty$. Let $\hat{S}(s^t) = \{s^T|s^t\text{: } Constraint\ (2.2c)\ is\ binding\ from\ node\ s^T\ for\ all\ h \in H\}$. Then for every $s^t \in S$ the set $\hat{S}(s^t)$ contains an infinite number of elements.

Theorem 4.3 extends the asset pricing analysis of Santos and Woodford (1997) to an economy with liquidity constraints. Theorem 4.4 is new and it implies that the cash-in-advance constraint must be binding infinitely often for all agents in the economy. The next two examples present some simple deterministic economies in which the fundamental value of money is not uniquely defined over the set of generalized state-price processes. In all equilibria, $\overline{\pi}_{(\omega,0)}(s^0) = +\infty$, and so Theorem 4.3 fails to hold.

Example 4.1. This is the simplest setting with a cash-in-advance constraint. There is a representative consumer with a utility function $V(c) = \sum_{t=0}^{\infty}\beta^t u(c_t)$, where $0 < \beta < 1$ and $u : R_+ \to R$ is a $\mathcal{C}^1$ increasing concave function. The objective is maximized subject to the constraints

$$q_t M_t + c_t \leq \omega_t + q_t M_{t-1} \tag{4.4a}$$

$$c_t \leq q_t M_{t-1} \tag{4.4b}$$

for $t = 0, 1, \cdots$. Assume that the initial money endowment $M_{-1} = 1$, and $\omega_t = \omega$ for all $t \geq 0$. Then a unique stationary equilibrium exists with $q_t = q^* = 1/\omega$ and $M_t = 1$, for each t. Moreover, utility maximization implies the asset pricing equation

$$\lambda_t q_t = \beta \lambda_{t+1} q_{t+1} + \beta \nu_{t+1} q_{t+1} \tag{4.5}$$

where $\lambda_{t+1} > 0$, $\nu_{t+1} \geq 0$ are the Lagrange multipliers associated to constraints (4.4a) and (4.4b), respectively.[3] In this equilibrium, $\lambda_t = \lambda^* > 0$, $\nu_t = \nu^* > 0$, for all $t \geq 0$. Hence,

$$q^* = \frac{\beta \nu^* q^*}{\lambda^*(1-\beta)} \tag{4.6}$$

Thus, the price of money equals its fundamental value. However (λ^*, ν^*) is not the only feasible state-price process. This is because the financial structure is incomplete, since the consumer is facing simultaneously two constraints at every $t = 0, 1, \cdots$. Hence, one can appeal to arbitrage considerations to argue that the right pricing formula for money in our economy is

$$q^* = \hat{\lambda} q^* + \hat{\nu} q^* \tag{4.7}$$

with the restriction $\hat{\lambda} > 0$, $\hat{\nu} \geq 0$. The stationary state-price process $(\hat{\lambda}, \hat{\nu}) = (1, 0)$ yields a zero fundamental value, whereas the remaining stationary state-price processes give full value. A value $\hat{\lambda}$ in the range $0 < \hat{\lambda} < 1$ may be interpreted as that valued money could coexist in equilibrium with productive assets.

Example 4.2. There are two infinitely-lived agents with an identical objective $V(c) = \sum_{t=0}^{\infty} \beta^t u(c_t)$, and subject to the above constraints (4.4). The utility function $u : R_+ \to R$ is $\mathcal{C}^1$, increasing, concave, and such that $u'(\omega) < \beta u'(0)$, where ω is the aggregate endowment. Agent 1's endowment $\omega^1 = (\cdots, \omega_t^1, \cdots)$ is $\omega_t^1 = \omega > 0$ if t is odd and $\omega_t^1 = 0$ if t is even. Agent 2's endowment $\omega^2 = (\cdots, \omega_t^2, \cdots)$ is $\omega_t^2 = 0$ if t is odd and $\omega_t^2 = \omega > 0$ if t is even. Both agents are endowed with initial money holdings, which sum up to unity, and such quantities will be specified below to guarantee the existence of a symmetric stationary equilibrium. This equilibrium is constructed from the following system of equations:

$$c_1 + c_2 = \omega \ \text{ and } \ u'(c_1) = \beta u'(c_2)$$
$$M_1 + M_2 = 1, \quad q^* M_1 = \omega, \ \text{ and } \ q^* M_2 = c_2$$

In this equilibrium, at even periods the cash-in-advance constraint is not binding for agent 1, and at odd periods such constraint is not binding for agent 2. Hence, at each time period there are both a set of multipliers (λ, ν) with $\nu = 0$ and a set of multipliers (λ', ν') with $\nu' > 0$. The fundamental value of money will again depend on the selected process of state prices. Moreover, if an asset in zero net

[3] Note that the shadow value ν_{t+1} is defined in a slightly different way than in (3.2). But as pointed out in the Appendix below, both formulations are mathematically equivalent.

supply is introduced to complete the market, such asset must necessarily pay a zero rate of interest in order to preserve the original equilibrium. Hence, one may be led to conclude that money has a zero fundamental value. Money, on the other hand, is modelled here in the same basic way as in example 4.1.

5 Further discussion of the results and related issues in monetary theory

The foregoing examples illustrate that the fundamental value of money may not be uniquely defined. Indeed, in these economies sequential markets are incomplete, and so there could be a continuum of generalized state-price processes that are compatible with pricing by arbitrage. A main contribution of the analysis of the preceding section (e.g., Theorem 4.3) is to show that this indeterminacy in the valuation of the stream of liquidity services ceases to exist if the aggregate wealth is finite valued. It is not clear how the assumption of a finite valued wealth may be relaxed. As is well known (cf. Shell, 1971), unbounded wealth results in a failure of Walras's law, and general equilibrium techniques break down. Then, bubbles may arise in these economies which implies that security prices cannot be identified with the present value of their dividends.

The fundamental equation (4.3b) for the pricing of money has been previously derived for models with a single household (e.g., Lucas, 1984; Svensson, 1985; Townsend, 1987), in which the liquidity services are weighted using the marginal utilities of this representative agent. Theorem 4.3 establishes the following more general result: The price of money equals the present value of its future liquidity services under *all* generalized state-price processes compatible with the existence of no-arbitrage opportunities. Hence, this theorem extends the pricing of money in an unambiguous way to economies with heterogeneous agents that may assign different valuations to the liquidity services that money provides.

The above examples also highlight some important differences with respect to models with money in the utility function. Under fairly general conditions Tirole (1985) shows that in models with money in the utility function the price of money must be equal to its fundamental value even if the value of aggregate wealth is infinity. This result suggests that the equivalence between models with cash-in-advance and money in the utility function does not really hold for the valuation of its liquidity services. The equivalence between these two approaches requires that the utility function must necessarily be non-separable in consumption and money balances, and must be non-differentiable.[4]

As is well known, models with money in the utility function may generate pathological inflationary equilibria in which the price of money converges to zero. Scheinkman (1980) provides some Inada-type conditions to rule out these equilibria. In our simple cash-in-advance economy these equilibria are precluded by construction; i.e, if the economy gets demonetized then it is not possible to consume the aggregate good. But in weaker versions of the cash-in-advance constraint such as in the canonical OLG model and in some models with cash-credit goods

[4] Models in which the utility function is differentiable and separable in money balances are usually simpler to analyze and compute, but one should be aware of their different equilibrium properties.

(e.g., Lucas and Stokey, 1987; Woodford, 1994) the equilibrium price of money may also converge to zero. For the canonical OLG model Santos (1990) shows that these inflationary equilibria cannot occur if the offer curve of the representative agent is separated from the endowment allocation. The plausibility of all these sufficient conditions –which place a lower bound on the equilibrium price of money– is an empirical question, but as discussed below the use of paper currency seems to be quite rooted.

Equalities (4.3) are the basis for theories of asset price determination in which the price of an asset is identified with the present value of the stream of future dividends. This result is useful for empirical purposes, since asset price determination can then be dissociated from agents' beliefs on future prices. These beliefs are simply identified with the present value of future returns. For the case of money, however, this argument does not seem so compelling. Thus, from (4.3b) let

$$\bar{q}_m(s^0) = \frac{1}{a(s^0)} \sum_{t=0}^{\infty} \sum_{s^t} \nu(s^t) \bar{q}_m(s^t) \zeta_m(s^t) \tag{5.1}$$

for every $(a, \nu) \in A(s^0)$. This equation is not particularly enlightening for the determination of the price of money, since it may contain a continuum of solutions. The spot price of money is simply written as an expected value of future prices. Hence, several authors (cf., Sims, 1994; Woodford, 1995; Kocherlakota and Phelan, 1998; Ljungqvist and Sargent, 2000) have discussed the usefulness of some other conditions for the determination of the price of money. Let us rewrite the government budget constraint (4.2) in the following form

$$\bar{q}_b(s^0) z_b(s^0 - 1) = \sum_{t=0}^{\infty} \sum_{s^t} a(s^t) [\bar{q}_m(s^t) \Delta_m(s^t) + \theta(s^t)] - \sum_{t=0}^{\infty} \sum_{s^t} a(s^t) c^g(s^t) \tag{5.2}$$

As already pointed out, this equation reads that the equilibrium value of the initial public debt must be counterbalanced by future net public revenues. Hernandez and Santos (1996, Prop. 3.4) show that every other nominal security must be in zero net supply. Hence, the government bond is the only nominal security that can be in positive net supply since it can be backed by future taxation and monetary creation. Also, by Theorem 4.3 the following additional asset pricing equation holds for the government bond

$$\bar{q}_b(s^0) = \frac{1}{a(s^0)} \sum_{t=1}^{\infty} \sum_{s^t} [a(s^t) \bar{q}_m(s^t) \eta_b(s^t)] \tag{5.3}$$

Assuming zero money creation, one can think that $\bar{q}_b(s^0)$ is pinned down by (5.2) and then $\bar{q}_m(s^0)$ is obtained from (5.3). [5] It should be stressed, however, that (5.2) is not a structural equation for the determination of $\bar{q}_b(s^0)$. This equation follows from the transversality condition (4.1) as a necessary condition that any

[5] This mechanism for price determination is often called the fiscal theory of the price level (e.g., see Woodford, 1995).

equilibrium solution must satisfy. Hence, it seems difficult to establish causality links among the various terms of this equation. The government may always resort to money creation, but taxation and public consumption may need to be maintained within certain bounds so that the public debt is sustainable. Besides, standard proofs of existence of competitive equilibria usually assume that each individual agent satisfies Walras' law at all price vectors (cf., Mas-Colell, 1992). A general proof of existence of monetary equilibria for this class of economies may require that the sequence of government decision variables $\{z_m(s^t), z_b(s^t), \theta(s^t), c^g(s^t)\}$ must obey (5.2) over all price processes $\{q(s^t)\}$, rather than at the equilibrium solution $\{\bar{q}(s^t)\}$. Available arguments on the determination of the price level by fiscal policy variables have been restricted to simple examples.

These ideas have been highly debated in the literature (e.g., see *opt. cit.*). The following point, however, seems to be missing from previous arguments derived from models with money in the utility function. Equation (5.1) may contain multiple solutions, but Theorems 4.3-4.4 show that if the price of money is positive then an infinite number of shadow prices ν must always be positive, and at these nodes the cash-in-advance constraint is binding for all agents in the economy. For a deterministic model and for an alternative timing of the cash-in-advance constraint (cf., Lucas and Stokey, 1987) at those nodes in which all agents are cash-constrained the price of money is univocally determined by the level of output. Further changes in the quantity of money must be accommodated by increases in the price level so that the cash-in-advance constraint is still binding for an infinite sequence of nodes. This equilibrium behavior is therefore consistent with the quantity theory of money, rather than with a fiscal theory of the price level. In this cash-in-advance model equation (5.1) plays a key role in the determination of the price of money.

6 Final comments

This paper presents some fairly strong results on the determination of the price of money in a general class of economies with heterogeneous agents, incomplete financial markets and arbitrary borrowing limits. Money provides liquidity services which are modelled by a simple cash-in-advance constraint. In these economies agents may have different shadow values for holding money. Under general conditions I show that the price of money is equal to its fundamental value over *all* feasible state-price processes. This equality between the price of money and its fundamental value implies that the cash-in-advance constraint must be binding infinitely often for *all* agents.

These results can be extended in several directions. First, note that condition (3.1) may hold under other trading restrictions such as short-sale constraints and reserve requirements (cf., Gimenez, 2000; Michel and Wigniolle, 2003). Indeed, as illustrated in the Appendix below the shadow value attached to the price of money may be associated with various other intangible returns attributed to monetary holdings. Second, as already mentioned there are some general cash-in-advance models (e.g., Lucas and Stokey, 1987; Prescott, 1987; Woodford, 1994) in which money is not the only asset serving the transactions purpose. In these latter models the velocity of money depends on assets returns. This may lead to some further scope for fiscal variables in the determination of the value of money, but from our discussion in Section 5 it seems that our qualitative results should prevail in these more general settings provided that the economy does not get demonetized. Third, these results should be of interest to analyze some further qualitative properties of the solution such as the existence of monetary equilibria in these economies (see Bloise, Dreze and Polemarchakis, 2003, for a recent proof of existence in a framework of complete markets).

At the heart of monetary theory is the determination of the price of money. This basic question bears on the proper conduct of monetary policy. A significant recent event is the successful launching of the euro by the European Central Bank (ECB). The Maastricht Treaty and the ailing Stability Pact were conceived to pave the way for a sound financial system in which the public debt of member countries would be sustainable. Fiscal policy coordination was not thought to have a primary influence in inflation. Indeed, the ECB is fully independent, and solely responsible for the managing of inflation in the euro zone. I broadly interpret these arrangements as supportive of models that bear some connection with the quantity theory of money. Another salient feature of the ECB is the issuance of large euro notes, which is in line with the protracted use of paper currency in modern economies (e.g., see Doyle, 2000; Rogoff, 2002). Innovations in financial markets and in information technologies may give more prominence to other types of money. But in spite of the development of electronic payments it is not evident that paper currency will soon disappear from transactions. Finally, there seems to be considerable variability in the rate of inflation across countries in the euro zone. Puente (2003) argues that the data is consistent with a cash-in-advance model that includes non-tradeable goods, asymmetric productivity shocks, and limited labor mobility. In his model the central bank can target the average rate of inflation for the monetary union, but not its cross-country variance.

7 Appendix

The above results follow from properties of asset prices and budget constraints over the set of generalized state-price processes. These properties have been analyzed in the aforementioned papers of Hernandez and Santos (HS) and Santos and Woodford (SW). Hence, here I will briefly indicate how these arguments may be extended to the cash-in-advance model.

By a simple extension of the Minkowski-Farkas it follows from (3.1) that there exists a non-negative vector of shadow values $\{\hat{\nu}(s^{t+1}|s^t)\}$ such that

$$a(s^t)q_m(s^t) = \sum_{s^{t+1}|s^t} [a(s^{t+1})R_m(s^{t+1}) + \hat{\nu}(s^{t+1})(1+\eta_m(s^{t+1}))q_m(s^{t+1})] \tag{7.1}$$

Note that (3.2b) follows from (7.1) after rewriting $\nu(s^t) = \sum_{s^{t+1}|s^t} \hat{\nu}(s^{t+1})(1+\eta_m(s^{t+1}))q_m(s^{t+1})/q_m(s^t)$.

Proposition 3.1 is an extended version of SW, Proposition 2.2. There are two main differences. First, Proposition 3.1 includes money holdings; this extension is easily accommodated in the present analysis. Second, $\overline{\pi}_{(\omega,\kappa)}(s^t)$ is defined in a slightly different way than in SW. This new definition is useful to illustrate that $\overline{\pi}_{(\omega,\kappa)}(s^t)$ and $\underline{\pi}_{(\omega,\kappa)}(s^t)$ can be obtained as limits of corresponding valuations in finite horizon economies. Hence, equality (3.7a) does not follow from the previous paper. But note that $\overline{\pi}_{(\omega,\kappa)}(s^t)$ is less than or equal to the corresponding upper valuation in SW. Then, Proposition 2.2 in SW implies that $\overline{\pi}_{(\omega,\kappa)}(s^t) \leq \sup_{(a,\nu)\in A(s^t)} v_{(\omega,\kappa)}(s^t; q; a, \nu)$. The other direction of this weak inequality follows directly from the definition of $\overline{\pi}_{(\omega,\kappa)}(s^t)$. Proposition 3.2 is a reformulation of some equivalence results established in HS, Proposition 3.3 , and in SW, Proposition 2.3.

The proof of Proposition 4.1 amounts to an extension of the perturbation argument of Lemma 3.8 of SW to encompass the cash-in-advance constraint. This extension is non-trivial, since revenues from the sales of assets and commodities cannot be used for immediate consumption. A rigorous proof of this proposition was kindly provided to me by Miguel A. Iraola, and will appear in his thesis at Universitad Carlos III de Madrid. Corollary 4.2 just follows from (2.4) and (4.1). Also, Theorem 4.3 is easily obtained from (4.1), e.g., see SW, Theorem 3.1.

Finally, the proof of Theorem 4.4 can be established by the following counter inductive argument: If the theorem is not true, then there is a date T such that for every node $s^r|s^t$ with $r \geq T$ there exists some agent $h \in H$ such that the cash-in-advance constraint is not binding from s^r. Hence, there must be a feasible state-price vector $(a(s^r), a(s^{r+1}), \nu(s^r))$ satisfying (3.2) with $\nu(s^r) = 0$. Therefore, there exists a generalized state-price process $(a, \nu) \in A(s^t)$ such that $\nu(s^r) = 0$ for all $s^r|s^t$ with $r \geq T$. Then, (4.4b) implies that $q(s^T) = 0$ for all $s^T|s^t$. Proceeding by induction we get $q_m(s^t) = 0$. This contradiction proves the theorem.

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