Mesoscale Meteorology: Quasi-Geostrophic Theory

14, 16 February 2017

Wait…this is a mesoscale class…why do we care about a tenet of synoptic meteorology??

On the synoptic-scale, scale analysis of the forcing terms in the equations of motion leads us to typically neglect horizontal parcel accelerations in curved flow (the geostrophic approximation) and vertical parcel accelerations altogether (hydrostatic balance). Parcel accelerations cannot be neglected on the mesoscale, however: they are key aspects of mesoscale dynamics. So, why do we spend a week covering principles that are often violated on the mesoscale?

Mesoscale features form and evolve within the background synoptic-scale environment. Smaller-scale variations in boundary placement, topography, etc. modulate precisely when and where mesoscale phenomena occur, but such variability only matters if the larger scales are supportive of a given phenomenon in the first place. Of particular interest are the positions and intensities of troughs and ridges (and thus surface boundaries) and the sign and magnitude of vertical motion, and quasi-geostrophic theory offers several tools to help us with such assessments.

Note that these notes are not meant to be comprehensive. They assume background knowledge of the scaling assumptions that enter into geostrophic balance, for instance. They do not cover all steps within the derivations that result in the equations below. Textbooks in synoptic or dynamic meteorology cover these in more detail. Alternatively, you may wish to review primers on these topics available at:

- Geostrophic balance: http://derecho.math.uwm.edu/classes/SynI/GeosApprox.pdf
- Thermal wind: http://derecho.math.uwm.edu/classes/SynI/ThermalWind.pdf
- Intro to Q-G theory: http://derecho.math.uwm.edu/classes/SynII/QGVorticity.pdf
- Q-G height tendency: http://derecho.math.uwm.edu/classes/SynII/QGHgtTend.pdf
- Q-G omega equation: http://derecho.math.uwm.edu/classes/SynII/QGOmega.pdf
- Q-vector formulation: http://derecho.math.uwm.edu/classes/SynII/QVectors.pdf

Thermal Wind

Formal Definition

Geostrophic balance allows us to express the horizontal equations of motion as:

$$v_g = \frac{1}{f}\frac{\partial \Phi}{\partial x} \qquad u_g = -\frac{1}{f}\frac{\partial \Phi}{\partial y} \qquad \text{such that } \mathbf{v}_g = \frac{1}{f}\left(\mathbf{k} \times \nabla \Phi\right)$$

where $\Phi = gz$ is the geopotential and z is height. If we take the partial derivative of u_g and v_g with respect to pressure p, substitute with the hydrostatic relationship and ideal gas law, and integrate between two isobaric levels p_0 and p_1 (where $p_0 > p_1$, such that p_1 has higher altitude), we obtain:

$$v_T \equiv v_g\left(p_1\right) - v_g\left(p_0\right) = \frac{R_d}{f}\frac{\partial \overline{T_v}}{\partial x}\ln\left(\frac{p_0}{p_1}\right) \qquad u_T \equiv u_g\left(p_1\right) - u_g\left(p_0\right) = -\frac{R_d}{f}\frac{\partial \overline{T_v}}{\partial y}\ln\left(\frac{p_0}{p_1}\right)$$

Or, in vector form,

$$\mathbf{v}_T \equiv \mathbf{v}_g(p_1) - \mathbf{v}_g(p_0) = \frac{R_d}{f} \ln\left(\frac{p_0}{p_1}\right)\left(\mathbf{k} \times \nabla \overline{T_v}\right)$$

Here, $\mathbf{v}_T = (u_T, v_T)$ defines the *thermal wind*, and $\overline{T_v}$ is the virtual temperature averaged over the layer between p_0 and p_1. The thermal wind defines the relationship between how the geostrophic wind changes with height and the horizontal gradient of layer-mean virtual temperature (which is directly proportional to the thickness). It implies a symbiotic relationship between vertical wind shear and the horizontal layer-mean virtual temperature gradient.

From inspection of the equations above, it can be shown that the thermal wind blows *parallel to the isotherms* (of layer-mean virtual temperature) with *warm air to the right* of the wind. We can prove this with a though experiment. Consider only a north-south layer-mean virtual temperature gradient, such that $v_T = 0$, with warm air to the south. Layer-mean virtual temperature decreases with the north, such that its meridional gradient is negative. Since R_d is positive, f is positive in the Northern Hemisphere, and the natural logarithm is always positive, the leading negative on u_T ensures that $u_T > 0$, defining an eastward-directed wind since $v_T = 0$.

Application to Temperature Advection and Hodographs

If we know the geostrophic wind – or, approximately, the full wind – at two pressure levels, we can define the thermal wind over that layer using vector subtraction: $\mathbf{v}_T(p_1)$ - $\mathbf{v}_T(p_0)$. Graphically, this can be accomplished by placing the origin of both vectors at a common location and drawing a vector from the end of $\mathbf{v}_T(p_0)$ to the end of $\mathbf{v}_T(p_1)$, as depicted in Fig. 1 below. Note that this is identical to the process used to construct a hodograph.

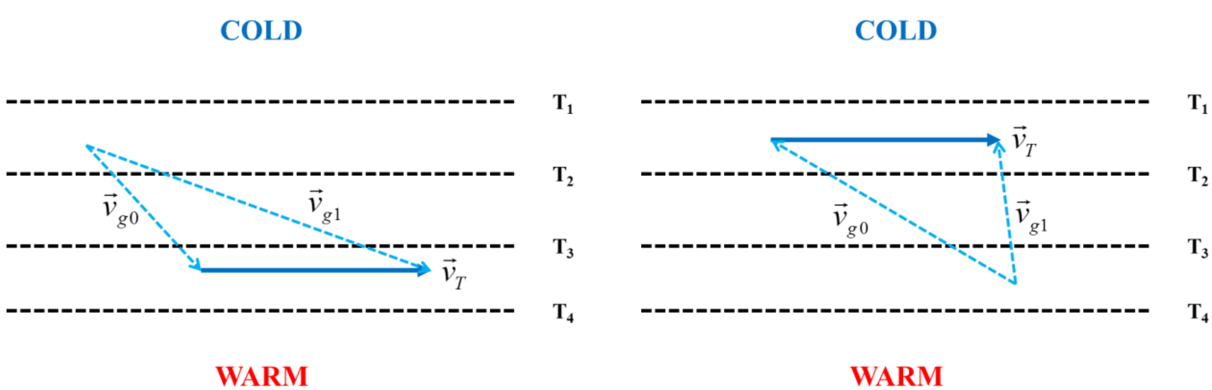

Figure 1. (left) A thermal wind $\mathbf{v}_T$ associated with cold air advection. (right) A thermal wind $\mathbf{v}_T$ associated with warm air advection. Isotherms of layer-mean T_v (or, nearly equivalently, T) are depicted by the dashed black lines parallel to $\mathbf{v}_T$. Despite identical direction and magnitude to the thermal wind in each panel, horizontal temperature advection sign differs because of differences in how the geostrophic wind varies with height (and is oriented with respect to the isotherms) in the layer between p_0 and p_1.

The left panel of Fig. 1 depicts *backing winds*, winds that turn counterclockwise with height. The right panel of Fig. 1 depicts *veering winds*, winds that turn clockwise with height. From Fig. 1, backing winds and counterclockwise-turning hodographs are associated with layer-mean cold virtual temperature advection. Veering winds and clockwise-turning hodographs are associated with layer-mean warm virtual temperature advection. We approximately state that veering winds are associated with warm air advection and backing winds are associated with cold air advection. The importance of temperature advection to synoptic and mesoscale meteorology will be made clearer later in this lecture.

Geostrophic Vorticity and Divergence

The geostrophic vorticity and divergence have the following definitions:

$$\zeta_g = \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} \qquad\qquad D_g = \frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y}$$

If we substitute in the definitions for u_g and v_g given earlier and let $f = f_0$ (a constant value of the Coriolis parameter), we obtain for ζ_g:

$$\zeta_g = \frac{\partial}{\partial x}\left(\frac{1}{f_0}\frac{\partial \Phi}{\partial x}\right) - \frac{\partial}{\partial y}\left(-\frac{1}{f_0}\frac{\partial \Phi}{\partial y}\right) = \frac{1}{f_0}\frac{\partial^2 \Phi}{\partial x^2} + \frac{1}{f_0}\frac{\partial^2 \Phi}{\partial y^2} = \nabla_p^2 \Phi$$

The subscript of p on the Laplacian operator indicates that it is evaluated on an isobaric (constant pressure) surface. This equation indicates that ζ_g is related to the Laplacian of the geopotential. Where the geopotential is a relative *minimum* (*maximum*), ζ_g is a relative *maximum* (*minimum)*. Thus, cyclonic ζ_g is maximized in the base of a trough, while anticyclonic ζ_g is maximized in the apex of a ridge.

Likewise, if we substitute in the definitions for u_g and v_g given earlier in this lecture and let $f = f_0$, we obtain for D_g:

$$D_g = \frac{\partial}{\partial x}\left(-\frac{1}{f_0}\frac{\partial \Phi}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{1}{f_0}\frac{\partial \Phi}{\partial x}\right) = 0$$

In other words, assuming a constant Coriolis parameter, the *geostrophic wind is non-divergent*!

A Note on Advection and Partial Derivatives

Throughout the remainder of this and subsequent lectures, we will encounter terms of the form $-\mathbf{v} \cdot \nabla a$, where a is some generic scalar quantity. These represent *advection*, or the transport of some quantity by the wind. Consider a two-dimensional advection term in component form:

$$-\mathbf{v} \cdot \nabla a = -u\frac{\partial a}{\partial x} - v\frac{\partial a}{\partial y}$$

We can approximate the partial derivatives with centered finite difference approximations, i.e.,

$$\frac{\partial a}{\partial x} \approx \frac{a_{x+1} - a_{x-1}}{2\Delta x} \qquad \text{and} \qquad \frac{\partial a}{\partial y} \approx \frac{a_{y+1} - a_{y-1}}{2\Delta y}$$

where (x, y) denotes the location where the finite difference is evaluated, x+1 is a point along the positive x-axis a distance Δx away from x, x-1 is a point along the negative x-axis a distance Δx away from x, and y+1 and y-1 are defined similarly except relative to y. Qualitative interpretation does not require us to define Δx or Δy, just to define equally-spaced points on each side of (x, y).

Let us consider the two examples given in Fig. 2 below.

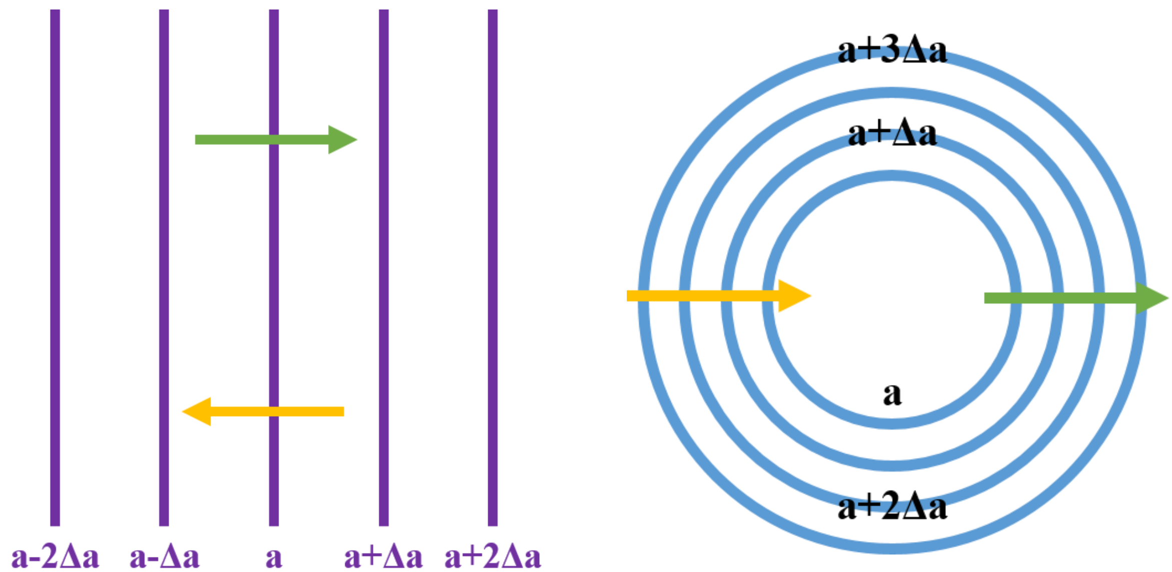

Figure 2. (left) An east-west gradient of a, with lower (higher) values to the west (east). (right) A local minimum of a. The green and gold vectors denote two wind vectors. Please see the text for further details.

Focus first on the example at left. At the location of the green arrow, v = 0 and u > 0. Let the x+1 point be located at a+Δa and the x-1 point be located at a-Δa. Thus,

$$\frac{\partial a}{\partial x} \approx \frac{a_{x+1} - a_{x-1}}{2\Delta x} = \frac{(a + \Delta a) - (a - \Delta a)}{2\Delta x} = \frac{2\Delta a}{2\Delta x}$$

Since Δa is positive and Δx is defined as positive, this term is positive. For u positive, the leading negative indicates that $-\mathbf{v} \cdot \nabla a < 0$, defining *negative advection*. For negative advection, the wind transports – or advects – lower values of a to a given location. Conversely, at the gold arrow, v = 0 and u < 0. Setting the x+1 and x-1 points at the same locations as before, we find that $\partial a/\partial x$ is the same value as in the previous example. But, u is negative, so the leading negative indicates that $-\mathbf{v} \cdot \nabla a > 0$, defining *positive advection*. For positive advection, the wind transports higher values of a to a given location.

Let us consider the example at right, with a westerly wind blowing through a local minimum of a. At the location of the green arrow, v = 0 and u > 0. If we let the x+1 point be located at a+2Δa and the x-1 point be located at a, we obtain:

$$\frac{\partial a}{\partial x} \approx \frac{a_{x+1} - a_{x-1}}{2\Delta x} = \frac{(a + 2\Delta a) - (a)}{2\Delta x} = \frac{2\Delta a}{2\Delta x}$$

This quantity, as in the other example, is positive. For $u > 0$, the leading negative again indicates that $-\mathbf{v} \cdot \nabla a < 0$. At the gold arrow, $v = 0$ and $u > 0$. If we let the x+1 point be located at a and the x-1 point be located at a+2Δa, we obtain:

$$\frac{\partial a}{\partial x} \approx \frac{a_{x+1} - a_{x-1}}{2\Delta x} = \frac{(a) - (a + 2\Delta a)}{2\Delta x} = \frac{-2\Delta a}{2\Delta x}$$

This quantity is negative. For $u > 0$, the leading negative indicates that $-\mathbf{v} \cdot \nabla a > 0$. The findings for this example can be interpreted in light of how the wind transports the local minimum. Areas downwind see the local minimum transported toward them, thus decreasing values by advection, while areas upwind see the local minimum transported away from them, thus increasing values by advection.

Together, these examples let us state the following guidelines, independent of wind direction:

- Wind that blows from lower toward higher values defines negative advection.
- Wind that blows from higher toward lower values defines positive advection.
- Wind blowing through a local minimum will have negative advection downwind (where the wind blows toward) and positive advection upwind (where the wind blows from).
- Wind blowing through a local maximum will have positive advection downward and negative advection upwind.

As we are often interested in qualitative rather than quantitative guidance, these guidelines help us quickly interpret the *sign* of advection. For temperature-related variables, warm advection is analogous to positive advection. For vorticity-related variables, cyclonic advection is analogous to positive advection in the Northern Hemisphere.

In the following, we will also encounter partial derivatives of the sort $\partial a/\partial p$, where pressure p is the vertical coordinate. Thus, these terms describe vertical variation in a quantity a. These can be approximated using centered finite differences as before, i.e.,

$$\frac{\partial a}{\partial p} \approx \frac{a_{p+1} - a_{p-1}}{2\Delta p}$$

where the p+1 point is defined above and the p-1 point is defined below the level at which the finite difference is evaluated. However, for this definition, note that Δp is *negative* because p decreases with height: p at p+1 is smaller than p at p-1. Thus, if a increases with height, $\partial a/\partial p$ is negative, while if a decreases with height, $\partial a/\partial p$ is positive.

Quasi-Geostrophic Height Tendency Equation

Formal Definition

It can be shown that the quasi-geostrophic forms of the vorticity and thermodynamic equations can be written in terms of two unknowns: vertical motion ω (partial derivative of pressure with time, so positive values indicate downward motion) and geopotential Φ. The quasi-geostrophic height tendency equation is obtained by manipulating these equations to eliminate ω. Doing so, one obtains the following equation:

$$\nabla^2\chi + f_0^2\frac{\partial}{\partial p}\left[\frac{1}{\sigma}\frac{\partial\chi}{\partial p}\right] = f_0\left(-\mathbf{v}_g\cdot\nabla\left(\zeta_g + f\right)\right) - f_0^2\frac{\partial}{\partial p}\left[\frac{h}{\sigma}\left(-\mathbf{v}_g\cdot\nabla\theta\right)\right]$$
$$- f_0 K\zeta_g - \frac{f_0^2 R}{c_p}\frac{\partial}{\partial p}\left(\frac{1}{\sigma p}\frac{dQ}{dt}\right)$$

This is a partial differential equation describing the local time-rate of change of the geopotential, where $\chi = \partial\Phi/\partial t$, on an isobaric surface. It is written with pressure p as the vertical coordinate. In the above, σ defines static stability and is assumed to be constant in the vertical, $\zeta_g + f$ defines the geostrophic absolute vorticity, h is a function of pressure and is positive-definite, K is a frictional coefficient and is positive-definite, and dQ/dt is the diabatic heating rate and is positive for diabatic warming. All other variables have their standard meteorological meaning.

It contains four forcing terms on the right-hand side:

- Geostrophic absolute vorticity advection
- Differential (in the vertical) potential temperature advection
- Friction
- Differential (in the vertical) diabatic heating

The left-hand side of the equation expresses χ in terms of its second partial derivative in x, y, and p. Where a variable is a local min (max), the second partial derivative is a local max (min). As a result, we can express the quasi-geostrophic height tendency equation as:

$$\chi \propto -f_0\left(-\mathbf{v}_g\cdot\nabla\left(\zeta_g + f\right)\right) + f_0^2\frac{\partial}{\partial p}\left[\frac{h}{\sigma}\left(-\mathbf{v}_g\cdot\nabla\theta\right)\right] + f_0 K\zeta_g + \frac{f_0^2 R}{c_p}\frac{\partial}{\partial p}\left(\frac{1}{\sigma p}\frac{dQ}{dt}\right)$$

A positive (negative) geopotential height tendency χ corresponds to rising (falling) heights.

Interpretation

We can interpret the forcing terms to the quasi-geostrophic height tendency equation as follows, focusing on falling heights. The opposite holds for rising heights.

- Cyclonic geostrophic absolute vorticity advection on an isobaric surface results in falling geopotential height on that isobaric surface. In general, this term only controls the motion of the trough/ridge pattern; it generally does not influence trough/ridge amplitude.

- Cold advection decreasing with height, or warm advection increasing with height, over a finite pressure layer centered on an isobaric surface results in falling geopotential height on that isobaric surface. This term can influence both trough/ridge motion and amplitude.
- Friction acts to cause heights to fall near the surface for anticyclonic geostrophic absolute vorticity. This term acts only near the surface and is of comparatively small magnitude.
- Diabatic cooling decreasing with height, or diabatic warming increasing with height, over a finite pressure layer centered on an isobaric surface results in falling geopotential height on that isobaric surface. This term is important when diabatic processes are important but is negligible otherwise.

While the quasi-geostrophic height tendency equation can be used to predict geopotential height, it is instead often used as a diagnostic equation. It is typically applied in the middle troposphere.

The differential potential temperature advection term can be evaluated using thermal wind or by evaluating potential temperature advection on isobaric levels above and below the isobaric level on which the quasi-geostrophic height tendency equation is evaluated. The ability to use thermal wind for this evaluation means that, if you approximate the geostrophic wind with the full wind, the local geopotential height tendency can be evaluated from a vertical sounding!

Direct measurements of diabatic heating rate are typically not available. Instead, we can infer the sign and relative magnitude of this term using other data and knowledge of atmospheric physics. For instance, the presence of clouds above and a relatively dry layer below a given isobaric level infers condensation and latent warming in the cloud and potential evaporation and latent cooling in the layer below.

Quasi-Geostrophic Omega Equation

<u>Formal Definition</u>

If one manipulates the quasi-geostrophic vorticity and thermodynamic equations to eliminate Φ instead of ω, a diagnostic equation for ω – the quasi-geostrophic omega equation – is obtained:

$$\sigma\nabla^2\omega + f_0^2\frac{\partial^2\omega}{\partial p^2} = -f_0\frac{\partial}{\partial p}\left(-\mathbf{v}_g\cdot\nabla\left(\zeta_g + f\right)\right) - h\nabla^2\left(-\mathbf{v}_g\cdot\nabla\theta\right) + f_0\frac{\partial}{\partial p}\left(K\zeta_g\right) - \frac{R}{pc_p}\nabla^2\left(\frac{dQ}{dt}\right)$$

This is a partial differential equation for vertical motion ω on an isobaric surface. It contains four forcing terms on the right-hand side:

- Differential (in the vertical) geostrophic absolute vorticity advection
- Potential temperature advection
- Differential (in the vertical) friction
- Diabatic heating

The left-hand side of the equation expresses ω in terms of its second partial derivative in x, y, and p. Thus, as we did with the quasi-geostrophic height tendency equation, we can express the quasi-geostrophic omega equation in terms of a proportionality:

$$\omega \propto f_0 \frac{\partial}{\partial p}\left(-\mathbf{v}_g \cdot \nabla\left(\zeta_g + f\right)\right) + h\nabla^2\left(-\mathbf{v}_g \cdot \nabla\theta\right) - f_0 \frac{\partial}{\partial p}\left(K\zeta_g\right) + \frac{R}{pc_p}\nabla^2\left(\frac{dQ}{dt}\right)$$

<u>Interpretation</u>

Using the same definitions introduced with the quasi-geostrophic height tendency equation, we can interpret the forcing terms to the quasi-geostrophic omega equation as follows, focusing on ascent ($\omega < 0$). The opposite holds for descent.

- Cyclonic geostrophic absolute vorticity advection that increases with height over a finite pressure layer centered on an isobaric surface results in ascent across the isobaric surface.
- Warm advection on an isobaric surface results in ascent across the isobaric surface.
- Because $K \sim 0$ above the surface, cyclonic geostrophic absolute vorticity near the surface results in lower tropospheric ascent. This is known as Ekman pumping. This occurs only in the planetary boundary layer.
- Diabatic warming on an isobaric surface results in ascent across the isobaric surface.

Vertical motion is typically maximized in the middle troposphere, and so the quasi-geostrophic omega equation is typically applied on middle tropospheric isobaric surfaces (700-300 hPa). It is only a diagnostic equation; there are no time derivatives in its formulation.

Because of the relationship between geostrophic absolute vorticity and geopotential from earlier, there is a symbiotic link between the quasi-geostrophic height tendency and omega equations. If the geopotential height on an isobaric surface falls, then the geostrophic absolute vorticity on that isobaric surface increases. This implies greater cyclonic geostrophic absolute vorticity advection on that isobaric surface and, depending on its vertical structure, a potential for greater differential cyclonic geostrophic absolute vorticity advection and thus stronger forcing for ascent.

As with the quasi-geostrophic height tendency equation, potential temperature advection can be evaluated using thermal wind or from a spatial analysis on the chosen isobaric level. The ability to use thermal wind for this evaluation means that, if you approximate the geostrophic wind with the full wind, the sign and relative magnitude of vertical motion can be evaluated from a vertical sounding!

The opposite of Ekman pumping is known as Ekman suction. Inferences of the sign and relative magnitude of diabatic heating rate follow from those for the quasi-geostrophic height tendency equation; e.g., strong latent warming in thunderstorms provides further forcing for ascent.

The Q-Vector Form of the Quasi-Geostrophic Omega Equation

<u>Formal Definition</u>

The two primary forcing terms of the quasi-geostrophic omega equation, differential geostrophic absolute vorticity advection and potential temperature advection, can and often do have opposite sign to each other. This is the primary motivation for developing an alternative equation, one that does not suffer from this problem: the **Q**-vector equation. As we will find, it also has properties that make it attractive for assessing frontogenesis, a topic which we will discuss next week.

To obtain the **Q**-vector equation, one starts with the quasi-geostrophic horizontal momentum and thermodynamic equations. Substitutions from thermal wind and geostrophic divergence, coupled with considerable manipulation, are required to obtain the **Q**-vector equation:

$$\sigma\nabla^2\omega + f_0^2\frac{\partial^2\omega}{\partial p^2} = -2\nabla\cdot\mathbf{Q} \qquad \text{or, proportionally, } \omega \propto 2\nabla\cdot\mathbf{Q}$$

where **Q** = (Q_1, Q_2) and:

$$Q_1 = -\frac{R}{p}\left(\frac{\partial\mathbf{v}_g}{\partial x}\cdot\nabla T\right) \qquad Q_2 = -\frac{R}{p}\left(\frac{\partial\mathbf{v}_g}{\partial y}\cdot\nabla T\right)$$

Thus, **Q**-vector convergence $\nabla\cdot\mathbf{Q} < 0$ is associated with forcing for synoptic-scale ascent, while **Q**-vector divergence $\nabla\cdot\mathbf{Q} > 0$ is associated with forcing for synoptic-scale descent.

<u>Interpretation</u>

Most analyses of the **Q**-vector equation make use of automated routines to computing **Q**-vectors from gridded analyses of atmospheric data. However, one can estimate **Q**-vectors as follows. Let us perform an axis transformation, defining the *x*-axis to be parallel to an isotherm (of *T*, in an approximate sense) with warm air to the right. The *y*-axis is defined perpendicular and to the left of the *x*-axis. An example is provided in Fig. 3 below.

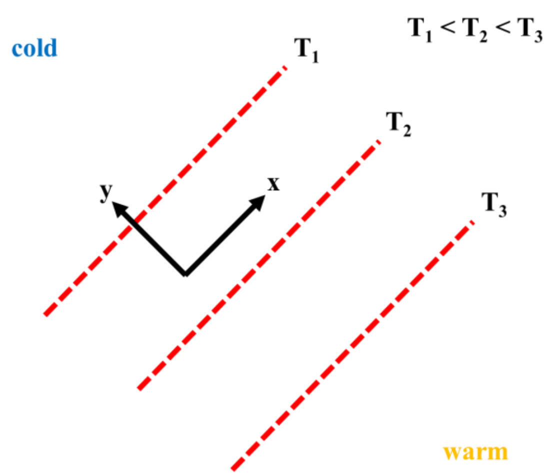

Figure 3. Idealized depiction of the axis transformation described above. North (east) is to the top (right) of the figure; however, the new *x*-axis is defined parallel to the isotherms with warm air to the right, with the new *y*-axis perpendicular to the left of the *x*-axis.

In this coordinate system, $\partial T/\partial x$ is zero. This allows us to simplify Q_1 and Q_2. If we do so and apply the non-divergence of the geostrophic wind, we obtain:

$$\mathbf{Q} = -\frac{R}{p}\left\|\frac{\partial T}{\partial y}\right\|\left(\mathbf{k}\times\frac{\partial\mathbf{v}_g}{\partial x}\right)$$

R_d and p are both positive constants on a given isobaric surface. These terms, along with the magnitude of the cross-isotherm temperature gradient, control the magnitude but not direction of the **Q** vector. If we are primarily interested in the **Q** vector's direction, then we can approximate the above with:

$$\mathbf{Q} \approx \left(-\mathbf{k} \times \frac{\partial \mathbf{v_g}}{\partial x} \right), \text{ where } \frac{\partial \mathbf{v}_g}{\partial x} \approx \frac{\mathbf{v}_g(x+1) - \mathbf{v}_g(x-1)}{2\Delta x}$$

To evaluate **Q**, we first find the vector change in $\mathbf{v}_g$ *along the isotherm* (the rotated *x*-axis). Next, we apply the **-k** x operator, which indicates a 90° clockwise rotation (i.e., to the right) of this vector. Let us apply this to the idealized trough-ridge pattern given in Fig. 4 below.

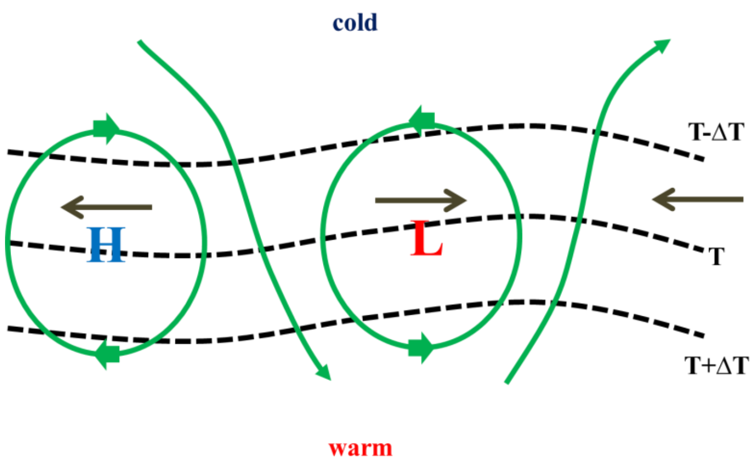

Figure 4. Idealized trough-ridge pattern, as defined by the green streamlines, with a ridge (H) to the west and a trough (L) to the east. Dashed black lines indicate isotherms, with colder air to the north. Solid dark grey arrows indicate **Q**-vectors at three locations, ridge, trough, ridge from west to east.

Consider the **H** in Fig. 4. Here, the new *x*-axis is pointed just slightly south of due east, along the isotherm. Let the *x*+1 point lie at the intersection of isotherm T with the green streamline to the east and the *x*-1 point lie at the intersection of isotherm T with the green streamline to the west. In this case, the geostrophic wind at *x*+1 is out of the north and the geostrophic wind at *x*-1 is out of the south. If we place them at a common origin and draw a vector from the end of that at *x*-1 to the end of that at *x*+1, we obtain a vector pointed from north to south. Application of the **-k** x operator rotates this vector 90° clockwise, such that it points toward the west.

Consider the **L** in Fig. 4. Here, the new *x*-axis is pointed just slightly south of due east, along the isotherm. Let the *x*+1 point lie at the intersection of isotherm T with the green streamline to the east and the *x*-1 point lie at the intersection of isotherm T with the green streamline to the west. In this case, the geostrophic wind at *x*+1 is out of the south and the geostrophic wind at *x*-1 is out of the north. If we place them at a common origin and draw a vector from the end of that at *x*-1 to the end of that at *x*+1, we obtain a vector pointed from south to north. Application of the **-k** x operator rotates this vector 90° clockwise, such that it points toward the east.

If we evaluate **Q**-vector divergence at a point between the **H** and **L**, we get positive divergence, such that $\omega > 0$, indicating forcing for descent. The opposite is true downstream of the **L**, ahead of the next **H**: negative divergence (convergence), such that $\omega < 0$, indicating forcing for ascent.

Connection to Frontogenesis

A key advantage of the **Q**-vector formulation versus the quasi-geostrophic omega equation is its connection to frontogenesis, indicating how the magnitude of the horizontal temperature gradient (a metric of frontal strength) changes with time. The quasi-geostrophic thermodynamic equation can be manipulated to show that:

$$\frac{d_g}{dt}\left(\frac{R}{p}\nabla T\right) = Q_1\hat{\mathbf{i}} + Q_2\hat{\mathbf{j}}$$

Here, the total derivative is written with a subscript of *g*, indicating that the wind that enters into its definition is the geostrophic rather than the full wind. This equation indicates that the rate of change of the horizontal temperature gradient following the geostrophic wind is a function of the **Q**-vector.

Perhaps a more useful expression, however, would be one for the *magnitude* of the horizontal temperature gradient, i.e.,

$$\frac{d_g}{dt}\left(\|\nabla T\|\right)$$

This definition can be expanded to show that:

$$\frac{d_g}{dt}\left(\|\nabla T\|\right) = \frac{1}{\|\nabla T\|}\frac{p}{R}\left(\nabla T \cdot \mathbf{Q}\right)$$

This indicates that the rate of change of the magnitude of the horizontal temperature gradient is related to the orientation of the temperature gradient (always pointed from cold to warm air) and the **Q**-vector. If these two vectors are perpendicular to each other, their dot product is zero and the magnitude of the horizontal temperature gradient does not change following the flow. If these two vectors are parallel to and in the same direction as each other, their dot product is positive and the magnitude of the horizontal temperature gradient increases following the flow. Finally, if these two vectors are parallel to but in opposite directions from each other, their dot product is negative and the magnitude of the horizontal temperature gradient decreases following the flow.

Applying these principles to Fig. 4, we find that the **Q**-vector is approximately perpendicular to the horizontal temperature gradient at each location where the **Q**-vector was evaluated. Thus, the magnitude of the horizontal temperature gradient does not change with time in this example. Let us consider a different example from an intense nor'easter that developed in late March 2014, as in Fig. 5 below.

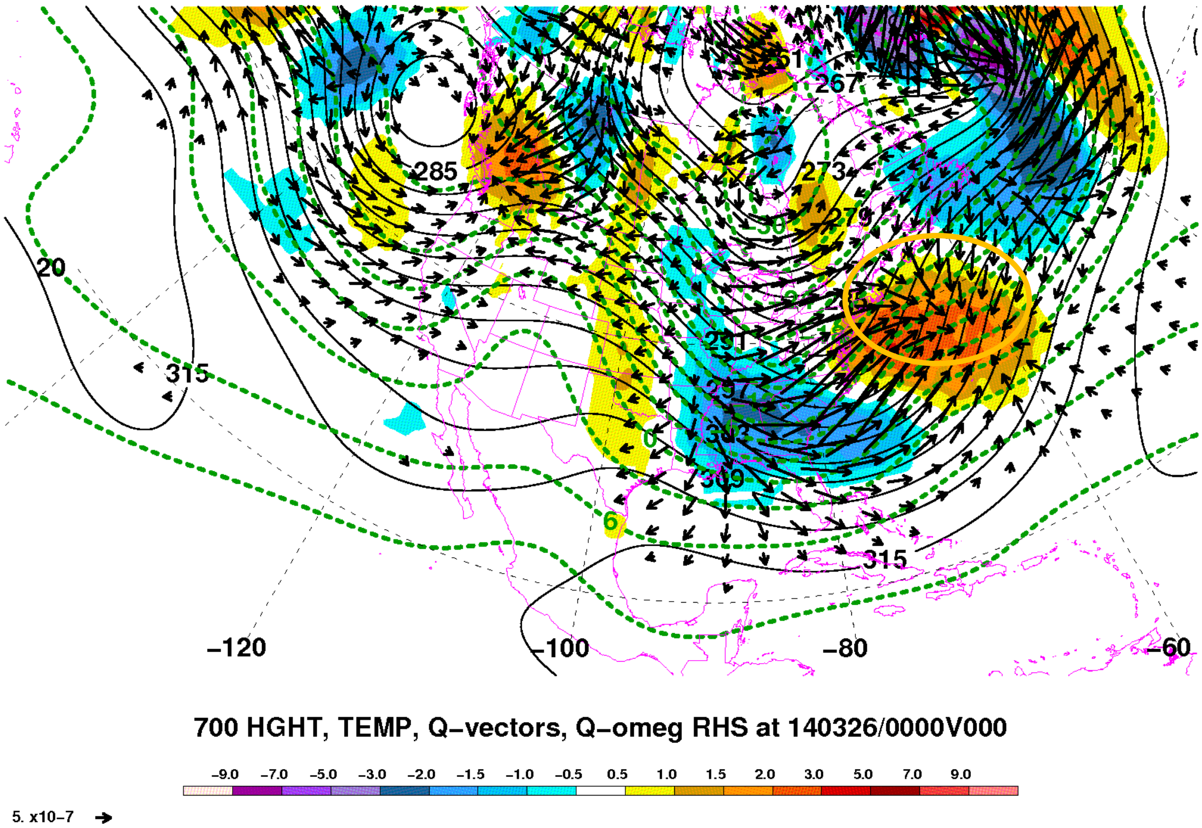

Figure 5. 700 hPa geopotential height (black lines every 3 dam = 30 m), temperature (green dashed lines every 3°C), **Q**-vectors (black vectors; reference vector at bottom left, units: Pa m^{-1} s^{-1}), and **Q**-vector convergence (shaded with warm colors denoting forcing for ascent; units: 10^{-12} Pa m^{-2} s^{-1}), derived from 1° GFS model analysis data valid at 0000 UTC 26 March 2014. Figure obtained from http://www.atmo.arizona.edu/~tgalarneau/realtime/diagnostics.html.

Focus on the orange-circled area just off of the northeast United States coastline. Here, **Q**-vectors point from cold to warm air – parallel to and in the same direction as the horizontal temperature gradient. This depicts a frontogenetical situation. South of the orange-circled area, the opposite is indicated: **Q**-vectors that point from warm to cold air – parallel to and in the opposite direction as the horizontal temperature gradient. This depicts frontolysis: weakening of a front with time.

Fig. 5 illustrates that forcing for both vertical motion and frontal evolution can be identified with **Q**-vectors. Areas of frontogenesis often overlap with regions of ascent. While the release of upright (CAPE) or slantwise instability may be important, banded precipitation in the cold sector of mid-latitude cyclones often occurs where strong frontogenesis and ascent overlap, particularly in the 850-500 hPa layer. In next week’s lecture, we will more formally define frontogenesis and revisit some of these concepts.