Synoptic Meteorology I: Finite Differences

For Further Reading

College-level Calculus texts contain extensive information regarding the mathematical definition of limits, partial derivatives, and Taylor functions and series. Sections 1.2.2 and 1.2.3 of *Mid-Latitude Atmospheric Dynamics* by J. Martin provide similar information from the perspective of their applications to the atmospheric sciences.

Partial Derivatives (or, Why Do We Care About Finite Differences?)

Apart from the ideal gas law, the equations that govern the evolution of fundamental atmospheric properties such as wind, pressure, and temperature (the *primitive equations*) contain many terms with partial derivatives. Indeed, many thermodynamic and kinematic properties of the atmosphere are typically expressed in terms of partial derivatives. We will explore many specific examples of such equations throughout both this and next semester.

Mathematically speaking, the partial derivative of some generic field f with respect to some generic variable x can be expressed as:

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} \tag{1}$$

In other words, $\frac{\partial f}{\partial x}$ is equal to the value of $\frac{\Delta f}{\Delta x}$ as Δx approaches (but does not equal) zero. Thus, for small (or finite) values of Δx, we can approximate $\frac{\partial f}{\partial x}$ by $\frac{\Delta f}{\Delta x}$. That begs the question: how do we compute $\frac{\Delta f}{\Delta x}$ from available atmospheric data?

To do so, we use what are known as *finite differences* to <u>approximate</u> the value of Δf over some finite Δx. Applied to isoplethed analyses of meteorological fields or gridded data, finite differences enable us to *compute* any quantity that depends upon one or more partial derivatives. Here, we wish to describe how finite difference approximations are obtained, the degree to which each is an approximation, and use examples to introduce how they can be applied to the atmosphere.

Developing Finite Difference Approximations

First, let us consider a generic continuous function $f(x)$, a graphical example of which is depicted below in Fig. 1. This function doesn't necessarily represent a meteorological field, but it doesn't not necessarily represent one either; it is simply a generic function. Along the curve given by $f(x)$, there are three points of interest: x_a, x_{a+1}, and x_{a-1}. The function $f(x)$ has the values $f(x_a)$, $f(x_{a+1})$, and $f(x_{a-1})$ at these three points, respectively. The distance between x_a and x_{a-1} is equal to the distance between x_a and x_{a+1}, and we can denote this distance as Δx.

We derive finite difference approximations from Taylor series, or mathematical representations of functions as infinite sums of terms that are calculated from the values of the functions' derivatives at any given point. The Taylor series expansion of $f(x)$ about $x = b$, where b is some generic point, is given by:

$$f(x) = f(b) + f'(b)(x-b) + \frac{f''(b)}{2!}(x-b)^2 + \frac{f'''(b)}{3!}(x-b)^3 + \ldots \quad (2)$$

This expansion states that $f(x)$ is equal to the value of $f(x)$ at $x = b$ plus an infinite series of higher-order terms, each of which contains an increasingly large partial derivative (primes; a single prime denotes the first partial derivative, two primes denote the second partial derivative, etc.), exponent on $x - b$, and factorial (!) order.

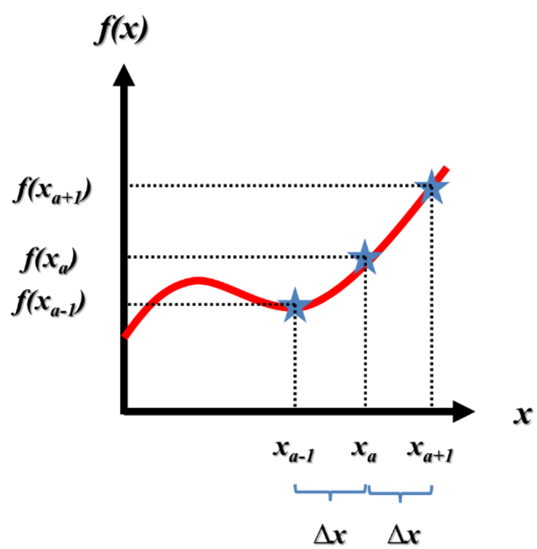

Figure 1. Graphical depiction of a generic function $f(x)$ evaluated at three points. Please see the text for further details.

Let us consider two cases: one where $x = x_{a+1}$ and $b = x_a$ and one where $x = x_{a-1}$ and $b = x_a$. In the first case, the distance $x - b$, or $x_{a+1} - x_a$, is equal to Δx; in the second case, the distance $x - b$, or $x_{a-1} - x_a$, is equal to $-\Delta x$. Making use of this information, we can expand (2) for these two cases:

$$f(x_{a+1}) = f(x_a) + f'(x_a)\Delta x + \frac{f''(x_a)}{2!}(\Delta x)^2 + \frac{f'''(x_a)}{3!}(\Delta x)^3 + \ldots \quad (3)$$

$$f(x_{a-1}) = f(x_a) - f'(x_a)\Delta x + \frac{f''(x_a)}{2!}(\Delta x)^2 - \frac{f'''(x_a)}{3!}(\Delta x)^3 + \ldots \qquad (4)$$

Note the similar appearance of (3) and (4) apart from the leading negative signs on the first and third order terms in (4). These arise because $x - b = -\Delta x$ for this case, as noted above.

From (3) and (4), we are interested in the value of $f'(x_a)$, or $\frac{\partial f}{\partial x}$ evaluated at x_a. We can use (3) and (4) to obtain an expression for this term; we simply need to subtract (4) from (3). Doing so, we obtain the following:

$$f(x_{a+1}) - f(x_{a-1}) = 2f'(x_a)\Delta x + \frac{2f'''(x_a)}{3!}(\Delta x)^3 + \ldots \text{(odd order terms)} \ldots \qquad (5)$$

Note how the zeroth and second order terms in (3) and (4) cancel out in this operation. If we rearrange (5) and solve for $f'(x_a)$, we obtain:

$$f'(x_a) = \frac{f(x_{a+1}) - f(x_{a-1})}{2\Delta x} - \frac{f'''(x_a)}{3!}(\Delta x)^2 + \ldots \qquad (6)$$

At this point, we wish to neglect all terms higher than the first-order term from (6). Doing so, we are left with:

$$f'(x_a) = \frac{f(x_{a+1}) - f(x_{a-1})}{2\Delta x} \qquad (7)$$

Equation (7) is what is known as a *centered finite difference*. It provides a means of calculating $\frac{\partial f}{\partial x}$ at $x = x_a$ by taking the value of f at $x = x_{a+1}$, subtracting from it the value of f at $x = x_{a-1}$, and dividing the result by the distance between the two points ($2\Delta x$). Note that x here and in later examples can be any variable; it does not have to represent the x-axis or the east-west direction. Equation (7) is equivalent if x is replaced by y, z, p, or any number of other variables.

There exist other ways for us to use (3) and (4) to get expressions for $f'(x_a)$. For instance, we can solve (3) for this term. If we do so, we obtain:

$$f'(x_a) = \frac{f(x_{a+1}) - f(x_a)}{\Delta x} - \frac{f''(x_a)}{2!}(\Delta x) - \frac{f'''(x_a)}{3!}(\Delta x)^2 - \ldots \qquad (8)$$

Neglecting all terms higher than the first order term in (8), we obtain:

$$f'(x_a) = \frac{f(x_{a+1}) - f(x_a)}{\Delta x} \tag{9}$$

Equation (9) is what is known as a *forward finite difference*. It provides a means of calculating $\frac{\partial f}{\partial x}$ at $x = x_a$ by taking the value of f at $x = x_{a+1}$, subtracting from it the value of f at $x = x_a$, and dividing the result by the distance between the two points (Δx).

Alternatively, we can solve (4) for $f'(x_a)$. If we do so, we obtain:

$$f'(x_a) = \frac{f(x_a) - f(x_{a-1})}{\Delta x} + \frac{f''(x_a)}{2!}(\Delta x) - \frac{f'''(x_a)}{3!}(\Delta x)^2 + \ldots \tag{10}$$

Neglecting all terms higher than the first order term in (10), we obtain:

$$f'(x_a) = \frac{f(x_a) - f(x_{a-1})}{\Delta x} \tag{11}$$

Equation (11) is what is known as a *backward finite difference*. It provides a means of calculating $\frac{\partial f}{\partial x}$ at $x = x_a$ by taking the value of f at $x = x_a$, subtracting from it the value of f at $x = x_{a-1}$, and dividing the result by the distance between the two points (Δx).

Finite Differences as Approximations

We do not need to neglect the higher-order terms in obtaining any of the above expressions for $f'(x_a)$; we have done so here primarily for simplicity. If we were to retain the higher-order terms, we would obtain more accurate approximations for $f'(x_a)$. This highlights a key point: all finite differences are *approximations*. All finite differences are associated with what is known as *truncation error*, which is determined by the power of Δx on the first term that is neglected in obtaining the finite difference approximation.

For instance, consider our centered finite difference given by Equation (7). In obtaining (7), the first term that we neglected in (6) included a $(\Delta x)^2$ term. As a result, we say this finite difference is "second-order–accurate." In contrast, consider our forward and backward finite differences, given by Equations (9) and (11), respectively. In obtaining each equation, the first terms that we neglected in (8) and (10) included a (Δx) term. As a result, we say that these finite differences are "first-order–accurate." The higher the order of accuracy, the more accurate the finite difference.

In synoptic meteorology, where exact values for partial derivatives are often not necessary, we typically utilize the centered finite difference. Forward and backward finite differences are rarely utilized except along the edges of the data, where the -1 and +1 points may not exist. Higher-order finite differences, typically fourth- or higher-order–accurate, are necessary for numerical weather prediction models given chaos theory, which states that very small differences in data can lead to very large forecast differences.

Finally, note that our finite difference approximations inherently approximate the partial derivative over a finite distance Δx. As a result, we must take care to limit the horizontal distance over which we calculate finite differences when approximating partial derivatives in synoptic calculations.

A Finite Difference Approximation for Second Derivatives

While the first partial derivative of some field provides a measure of its *slope*, sometimes we are interested in evaluating the second partial derivative of some field. Recall from calculus that the second partial derivative of a field provides a measure of its *concavity*; positive second partial derivatives infer that a field is concave up (or convex), while negative second partial derivatives infer that a field is concave down. Applied to meteorology, a field that is convex represents a local minimum, whereas a field that is concave down represents a local maximum; i.e., the second partial derivative has the opposite sign of the field itself.

We can obtain a finite difference approximation for the second partial derivative by adding (3) and (4). Doing so, we obtain:

$$f(x_{a+1}) + f(x_{a-1}) = 2f(x_a) + 2\frac{f''(x_a)}{2!}(\Delta x)^2 + \ldots \tag{12}$$

If we solve (12) for $f''(x_a)$ and truncate the higher-order terms, we obtain:

$$f''(x_a) = \frac{f(x_{a+1}) + f(x_{a-1}) - 2f(x_a)}{(\Delta x)^2} \tag{13}$$

Equation (13) provides a fourth-order–accurate means of evaluating $\frac{\partial^2 f}{\partial x^2}$, or $f''(x_a)$, by adding the value of f at x_{a+1} to the value of f at x_{a-1}, subtracting two times the value of f at x_a, and dividing the result by the square of the distance between points $(\Delta x)^2$.

We can use (13) to prove the second partial derivative's mathematical definition. For example, let us consider a hypothetical example where the 2-m temperature is 78°F in Madison and Waukesha and 82°F in Johnson Creek. These cities are approximately evenly spaced from each other between Madison in the west and Waukesha in the east. Let Madison be the *a*-1 location, Johnson Creek be the *a* location, and Waukesha be the *a*+1 location. (13) indicates that we should first add Madison's and Waukesha's temperatures together, giving us 156°F. From this, we subtract double Johnson Creek's temperature, or 164°F. Consequently, the numerator is positive. Since the denominator is inherently positive, (13) would be negative in this case. This implies a concave down field, which we previously stated represents a local maximum – just as we see in the given temperatures!

Just as for the finite difference approximation for the first partial derivative, (13) is equivalent if *x* is replaced by *y*, *z*, *p*, or any number of other variables. Likewise, just as for the finite difference approximate for the first partial derivative, higher-order accurate finite difference approximations for the second partial derivative can also be obtained.

A Refresher on Vector Notation

As partial derivatives are found in nearly all aspects of synoptic meteorology, it is useful to close by reminding ourselves of some of their basic properties, particularly as it relates to vectors.

For instance, horizontal temperature advection can be written as:

$$advection = -u\frac{\partial T}{\partial x} - v\frac{\partial T}{\partial y} = -\vec{\mathbf{v}} \cdot \nabla T \qquad (14)$$

The first representation is written in what is known as *component* notation, whereas the second is in what is known as *vector* notation (here, assuming that the gradient operator applies only in the horizontal direction). The two are equivalent to each other because of the definitions of the gradient operator and dot product, i.e.,

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} \qquad \mathbf{a} \cdot \mathbf{b} = \left(a_x\mathbf{i} + a_y\mathbf{j}\right) \cdot \left(b_x\mathbf{i} + b_y\mathbf{j}\right) = a_x b_x + a_y b_y$$

where we have again considered only the horizontal directions in this analysis. For the dot product, **a** is simply $-\vec{\mathbf{v}} = -u\mathbf{i} - v\mathbf{j}$ and **b** is simply $\frac{\partial T}{\partial x}\mathbf{i} + \frac{\partial T}{\partial y}\mathbf{j}$.

There are other quantities that can be written in component or vector notation that we will consider in more detail later in the semester. For example, consider the divergence of the horizontal wind:

$$divergence = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \nabla \cdot \vec{\mathbf{v}} \qquad (15)$$

This is a relatively straightforward application of the gradient operator and dot product, the proof of which is left to the student. We can also consider the vertical component of the vorticity of the horizontal wind:

$$vorticity = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \mathbf{k} \cdot \left(\nabla \times \vec{\mathbf{v}}\right) \qquad (16)$$

This involves application of the gradient operator, dot product, and cross-product, where the cross-product of two vectors is given by:

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & 0 \\ b_x & b_y & 0 \end{vmatrix}$$

For this cross-product, **a** is simply ∇ and **b** is simply $-\vec{\mathbf{v}} = -u\mathbf{i} - v\mathbf{j}$. Thus,

$$\nabla\times\vec{\mathbf{v}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 \\ u & v & 0 \end{vmatrix} = \mathbf{i}\begin{vmatrix} \frac{\partial}{\partial y} & 0 \\ v & 0 \end{vmatrix} - \mathbf{j}\begin{vmatrix} \frac{\partial}{\partial x} & 0 \\ u & 0 \end{vmatrix} + \mathbf{k}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ u & v \end{vmatrix}$$
$$= \mathbf{i}(0-0) - \mathbf{j}(0-0) + \mathbf{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

Taking the dot product of this result with **k** retains only the **k**-component of the vector, giving us the vorticity as defined in (16).

Together, the gradient operator, dot product, and cross-product – and, specifically, the divergence, advection, and vorticity applications thereof – represent the most commonly used vector properties in synoptic meteorology. We will introduce others as warranted later in this and next semesters.

Finally, we will often consider total derivatives, or terms like $\frac{D(\ )}{Dt}$. In their component forms, in Cartesian coordinates and excluding the vertical dimension, the total derivative is given by:

$$\frac{D(\ )}{Dt} = \frac{\partial(\ )}{\partial t} + u\frac{\partial(\ )}{\partial x} + v\frac{\partial(\ )}{\partial y}$$

The first right-hand–side term represents the local change in the variable, whereas the remaining right-hand-side terms are equal to the negative of the advection of the variable. In some cases, the total derivative in an equation is written in component form, with the local change term remaining on one side of the equality and the advection terms being moved to the other side (acquiring their leading negative sign in so doing). Together, the total derivative is meteorologically interpreted as the change in the variable *following the motion* (i.e., tracking along with the air and not fixed to a specific location, and thus intrinsically representing a Lagrangian reference frame).