# <u>Synoptic Meteorology I: The Geostrophic Approximation</u>

## For Further Reading

Section 1.3 of *Midlatitude Synoptic Meteorology* by G. Lackmann introduces geostrophic balance and the Rossby number. Section 3.2 of *Mid-Latitude Atmospheric Dynamics* by J. Martin provides a discussion of the geostrophic approximation and geostrophic balance. Section 4.1 of *Mid-Latitude Atmospheric Dynamics* provides a generalized discussion of how equations posed on constant height surfaces may be transformed into equations posed on isobaric surfaces.

## The Equations of Motion

In their most general form, and presented without formal derivation, the equations of motion applicable on constant height surfaces and posed in Cartesian coordinates are given by:

$$\frac{D\vec{v}}{Dt} = -2\vec{\Omega} \times \vec{v} - \frac{1}{\rho}\nabla p + \vec{g} + \vec{F}_r \tag{1}$$

From left to right, the terms of (1) represent the total derivative of the three-dimensional wind field $\vec{v}$, the Coriolis force, the pressure gradient force, effective gravity, and friction. The total derivative in the first term of (1) has the general form:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\hat{\mathbf{i}} + v\frac{\partial}{\partial y}\hat{\mathbf{j}} + w\frac{\partial}{\partial z}\hat{\mathbf{k}}$$

Equation (1) may be expanded into its component forms and recast into spherical coordinates. Presented without derivation, the corresponding equations of motion (for *u* and *v* only) are:

$$\frac{Du}{Dt} - \frac{uv\tan\phi}{a} + \frac{uw}{a} - fv + 2\Omega w\cos\phi = -\frac{1}{\rho}\frac{\partial p}{\partial x} + F_{rx} \tag{2a}$$

$$\frac{Dv}{Dt} + \frac{u^2\tan\phi}{a} + \frac{vw}{a} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y} + F_{ry} \tag{2b}$$

where $\phi$ = latitude, $\Omega$ = the rotation rate of the Earth (7.292 x $10^{-5}$ $s^{-1}$), *u* = zonal component of the wind, *v* = meridional component of the wind, *w* = vertical component of the wind, *p* = pressure, $\rho$ = density, $f = 2\Omega \sin\phi$ = Coriolis parameter, and *a* = Earth's radius (6.378 x $10^6$ m).

The second and third terms of (2) represent terms related to the curvature of the Earth. The fourth term of (2b) and fourth and fifth terms of (2a) are Coriolis terms. The two terms on the right-hand side of (2) are pressure gradient and frictional terms, respectively. These equations, alongside the corresponding equation for *w*, define the equations of motion. We note that these forms of the equations of motion are presented with height, or *z*, as the vertical coordinate.

**Scale Analysis of the Equations of Motion**

The equations of motion, as stated in their full form in (2) above, are quite complex. They describe all types and scales of atmospheric motions, including some (e.g., acoustic waves) that are either negligible or altogether irrelevant for the study of synoptic-scale motions.

We desire to simplify (2) into a form that reflects only the most important processes on the synoptic-scale. In other words, we want to keep only those terms that are large for synoptic-scale motions. We do so by performing a *scale analysis* on the terms in (2). This is done by replacing each variable with an appropriate characteristic value for that variable based upon observed values for mid-latitude synoptic-scale motions. The characteristic values appropriate for the scale analysis of (2) are given by Table 1 below.

| Variable | Characteristic Value | Description |
|---|---|---|
| $u, v$ | $U \approx 10$ m s$^{-1}$ | Horizontal velocity scale |
| $w$ | $W \approx 0.01$ m s$^{-1}$ | Vertical velocity scale |
| $x, y$ | $L \approx 10^6$ m | Horizontal length scale |
| $z$ | $H \approx 10^4$ m | Depth scale |
| $\delta p/\rho$ | $\delta P/\rho \approx 10^3$ m$^2$ s$^{-2}$ | Horizontal pressure fluctuation scale |
| $t$ | $L/U \approx 10^5$ s | Time scale |
| $f$, $2\Omega \cos \phi$ | $f_0 \approx 10^{-4}$ s$^{-1}$ | Coriolis scale |

**Table 1:** Characteristic values appropriate for synoptic-scale motions for the variables in (2).

Several important insights into ‘synoptic-scale motions’ can be drawn from the above:

- Horizontal velocity is much larger – several orders of magnitude – than vertical velocity.
- Synoptic-scale features have horizontal extent of hundreds of kilometers or more.
- These features also extend through a meaningful depth (~10 km) of the troposphere.
- Synoptic-scale motions evolve slowly, on the order of 1 day (8.64 x $10^4$ s) or longer.
- We limit ourselves to the midlatitudes given $f_0 \approx 10^{-4}$ s$^{-1}$, or its value at latitude $\phi$ = 45°N.

If we plug in the appropriate characteristic values from the table above into (2), we obtain:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $u$-mom. | $\frac{Du}{Dt}$ | $-fv$ | $2\Omega w \cos\phi$ | $\frac{uw}{a}$ | $-\frac{uv\tan\phi}{a}$ | $-\frac{1}{\rho}\frac{\partial p}{\partial x}$ | $F_{rx}$ |
| $v$-mom. | $\frac{Dv}{Dt}$ | $fu$ | | $\frac{vw}{a}$ | $\frac{u^2\tan\phi}{a}$ | $-\frac{1}{\rho}\frac{\partial p}{\partial y}$ | $F_{ry}$ |
| Scale | $U^2/L$ | $f_0 u$ | $f_0 w$ | UW/a | $U^2$/a | $\delta P/\rho L$ | $DU/H^2$ |
| Value (m s$^{-2}$) | $10^{-4}$ | $10^{-3}$ | $10^{-6}$ | $10^{-8}$ | $10^{-5}$ | $10^{-3}$ | $10^{-12}$ |

**Table 2:** Scale analysis of the equations of motion.

Note that the D in the scaling of the frictional terms is the *eddy diffusivity*, a term related to turbulent (or frictional) processes within the boundary layer. It is of order D ≈ $10^{-5}$ $m^2$ $s^{-1}$.

**The Geostrophic Approximation on Constant Height Surfaces**

From Table 2 above, it is apparent that for synoptic-scale motions, two terms are at least one order of magnitude larger than the other terms. These are the Coriolis and pressure gradient terms. If we retain only these two terms in (2), the following expressions result:

$$-fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{3a}$$

$$fu = -\frac{1}{\rho}\frac{\partial p}{\partial y} \tag{3b}$$

(3) is the *geostrophic relationship*, representing a *balance* between the horizontal pressure gradient and the Coriolis force (itself a function of the horizontal velocity) in synoptic-scale extratropical or midlatitude weather systems. In geostrophic balance (depicted in Fig. 1), the horizontal pressure gradient and Coriolis forces are of equal magnitude but opposite directions to each other:

- The horizontal pressure gradient force *always* points from high to low pressure, in contrast to the horizontal pressure gradient itself which points from low to high pressure.
- The Coriolis force points in the opposite direction of the horizontal pressure gradient force, or from low toward high pressure.

A larger horizontal pressure gradient magnitude requires a larger Coriolis force to balance it, thus necessitating a faster horizontal wind speed.

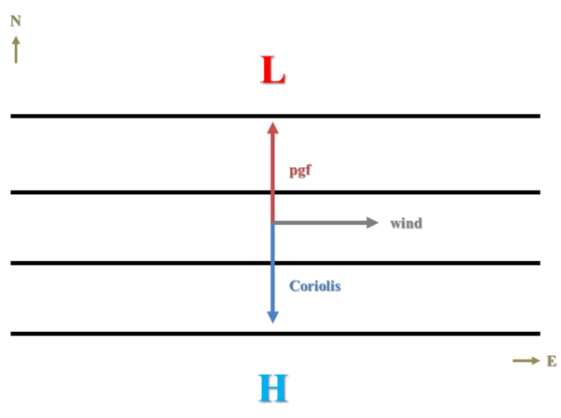


**Figure 1**. Graphical depiction of geostrophic balance for an idealized example in the Northern Hemisphere with low pressure to the north and high pressure to the south. The pressure gradient force (pgf) is denoted in red, the Coriolis force is denoted in blue, and the resultant wind is denoted in grey. Black lines denote idealized isobars, here depicted on a constant height surface.

In the Northern Hemisphere, the Coriolis force always points perpendicular and to the right of the wind, as depicted in Fig. 1. We can prove this from (2). If we retain only the total derivative terms and terms that involve the Coriolis parameter *f*, we obtain:

$$\frac{Du}{Dt} = fv \tag{4a}$$

$$\frac{Dv}{Dt} = -fu \tag{4b}$$

The total derivatives on the left-hand side of (4) represent *accelerations* (i.e., changes in $u$ and $v$ with time following the motion), such that (4) quantifies parcel accelerations resulting exclusively from the Coriolis force.

Consider air that is moving from south to north, such that $v > 0$. In this case, for $f > 0$ (as is always true in the Northern Hemisphere), the left-hand side of (4a) is positive. This means that, following the motion, the air parcel accelerates to the east ($u > 0$), or 90° to the right of the air's motion from south to north. Similar arguments can be made for any wind direction. Thus, *the Coriolis force is always directed 90° to the right of the wind in the Northern Hemisphere*.

**How is Geostrophic Balance Achieved?**

Consider air that is initially at rest. Because the Coriolis force is directly proportional to the wind speed, the Coriolis force is initially 0. Now consider a synoptic-scale horizontal pressure gradient, with lower pressure to the north and higher pressure to the south (Fig. 2). In this case, the horizontal pressure gradient force is directed from south to north. This causes the air to accelerate and, thus, begin to move. As the air begins to move, the Coriolis force becomes non-zero. Very quickly, the horizontal pressure gradient force and Coriolis force begin to balance, and the air is deflected to the east (or right).

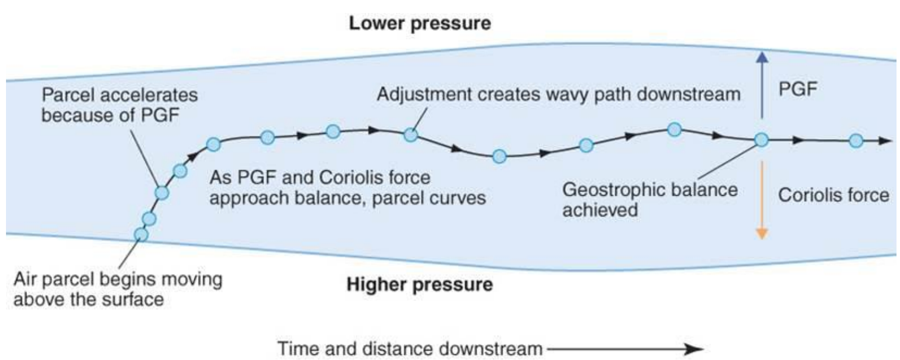


**Figure 2**. Schematic illustrating how geostrophic balance is initially achieved, what happens when geostrophic balance is disrupted, and the restoration of geostrophic balance through geostrophic adjustment. Figure obtained from Fig. 6-17 of *Meteorology: Understanding the Atmosphere*, 4th Ed., by S. Ackerman and J. Knox.

It is possible for geostrophic balance to be disrupted. This can occur through external forcings that change the horizontal pressure gradient and/or Coriolis forces, such as diabatic heating, frictional dissipation, or localized instabilities. Since these are typically localized, short-lived forcings, the atmosphere generally attempts to restore the geostrophic balance disrupted by these forcings. The process by which this occurs is known as *geostrophic adjustment*. During geostrophic adjustment, both the wind and pressure (and, by extension, mass) fields adjust to one another such that balance may again be achieved. Balance is typically restored through gravity and/or inertia-gravity waves, the characteristics and dynamics of which are beyond the scope of this class.

The wavy path followed by the air parcel in Fig. 2 after initially achieving geostrophic balance is an example of geostrophic adjustment. The horizontal pressure gradient and Coriolis forces adjust to each other, resulting in a temporarily wavy flow until balance is established.

**The Geostrophic Relationship on Isobaric Surfaces**

We obtained (3) through a scale analysis of the equations of motion applicable on constant height surfaces. Consequently, (3) is applicable only on constant height surfaces. We wish to now obtain a form of (3) applicable on isobaric surfaces. To do so, a coordinate transformation from the *z* to the *p* vertical coordinate is necessary. First, consider an idealized example of two isobaric surfaces that vary in height only in the *x*-direction (Fig. 3):

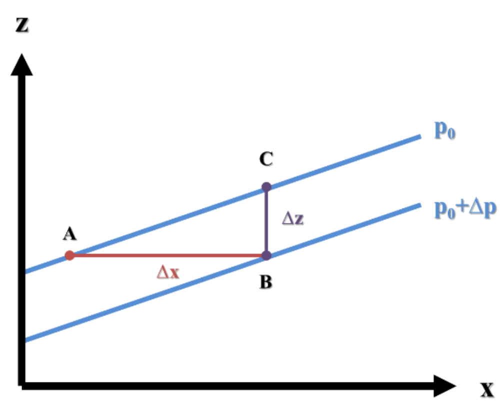


**Figure 3**. Idealized depiction of two isobaric surfaces, $p_0$ and $p_0+\Delta p$, that vary in height only within the *x*-direction. The distances $\Delta x$ and $\Delta z$ represent the distance between these two isobaric surfaces in the *x*- and *z*- directions, respectively.

Consider the change in pressure moving from point A, where $p = p_0$, to point B, where $p = p_0 + \Delta p$. This can be represented in terms of a partial derivative, where:

$$p_B - p_A = \left(\frac{\partial p}{\partial x}\right)_z \Delta x \tag{5a}$$

The subscript $z$ in (5a) denotes that this partial derivative is evaluated on a constant height surface, which is appropriate because $z$ is identical at points A and B.

Likewise, consider the change in pressure as one moves from point B, where $p = p_0 + \Delta p$, to point C, where $p = p_0$. This can also be represented in terms of a partial derivative, where:

$$p_C - p_B = \left(\frac{\partial p}{\partial z}\right)_x \Delta z \tag{5b}$$

The subscript $x$ in (5b) denotes that this partial derivative is evaluated at a constant value of $x$, which is appropriate because $x$ is identical at points B and C.

Because $p_A = p_C$, we can substitute $p_A$ for $p_C$ in (5b). If we then multiply the resulting equation by -1, an expression for $p_B - p_A$ results. Equating this expression to (5a), we obtain:

$$\left(\frac{\partial p}{\partial x}\right)_z \Delta x = -\left(\frac{\partial p}{\partial z}\right)_x \Delta z \tag{6}$$

If we divide (6) by $\Delta x$, we obtain:

$$\left(\frac{\partial p}{\partial x}\right)_z = -\left(\frac{\partial p}{\partial z}\right)_x \frac{\Delta z}{\Delta x} \tag{7}$$

Upon inspecting Fig. 3, we can see that $\Delta z/\Delta x$ is equivalent to an isobaric surface's slope. Taking the limit of this term as $\Delta x$ approaches 0 allows us to substitute for it with a partial derivative (from the fundamental theorem of calculus):

$$\left(\frac{\partial p}{\partial x}\right)_z = -\left(\frac{\partial p}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_p \tag{8}$$

In (8), the subscript of $p$ indicates that the partial derivative is evaluated on an isobaric surface. If we substitute the hydrostatic equation into the right-hand side of (8), we obtain:

$$\left(\frac{\partial p}{\partial x}\right)_z = \rho g \left(\frac{\partial z}{\partial x}\right)_p \tag{9a}$$

If we repeat the same process, except for pressure varying in the $y$-direction, we obtain:

$$\left(\frac{\partial p}{\partial y}\right)_z = \rho g \left(\frac{\partial z}{\partial y}\right)_p \tag{9b}$$

The expressions in (9) allow us to convert the horizontal pressure gradient terms evaluated on a constant height surface in (3) to their equivalent forms evaluated on an isobaric surface. If we plug (9) into (3), we obtain:

$$fv = g\frac{\partial z}{\partial x} \tag{10a}$$

$$fu = -g\frac{\partial z}{\partial y} \tag{10b}$$

Or, making use of the definition of the geopotential that we introduced in an earlier lecture, (10) can be written as:

$$fv = \frac{\partial \Phi}{\partial x} \tag{11a}$$

$$fu = -\frac{\partial \Phi}{\partial y} \tag{11b}$$

Note that there is no reference to time in (3), (10), or (11). Thus, the geostrophic relationship is a *diagnostic relationship*; it can only be used to diagnose the horizontal velocity as a function of the pressure or height field at a given time. This stands in contrast to a *prognostic relationship*, or one that involves a reference to time (such as through a partial derivative with respect to time), which would permit predicting the velocity field's evolution.

**The Geostrophic Wind**

To express geostrophic balance, $u$ in (3), (10), and (11) is replaced by $u_g$ while $v$ is replaced by $v_g$. These two variables, $u_g$ and $v_g$, define what is known as the *geostrophic wind*. Given that density is not routinely measured and that most meteorological analysis is conducted on isobaric surfaces, either (10) or (11) are used to evaluate the geostrophic wind from observations.

The geostrophic wind is a stable, slowly evolving flow. Contrast this with highly curved, rapidly accelerating, or near-surface flows, which can evolve more rapidly (i.e., the terms neglected in the scale analysis of (2) are no longer negligibly small). These situations are ones in which geostrophic balance explicitly does not hold and thus are situations where departures from geostrophic balance are most common.

Because the pressure gradient and Coriolis forces are of equal magnitude but in opposite directions under the constraint of geostrophic balance, the geostrophic wind blows *parallel* to the isobars on constant height surfaces (Fig. 1) or *parallel* to the lines of constant geopotential height on isobaric surfaces. This relationship between the wind and the isobars or constant height contours is one of the defining characteristics of geostrophic balance. An application of this relationship is the *Buys-*

*Ballot law*, which states that low pressure or low height is to your left when the geostrophic wind is at your back.

In the midlatitudes, the geostrophic wind approximates the true horizontal velocity to within ~10%. How closely the wind parallels the isobars or isohypses is a measure for how close the atmosphere is to geostrophic balance.

An example of evaluating the geostrophic wind from observations is given Fig. 4.

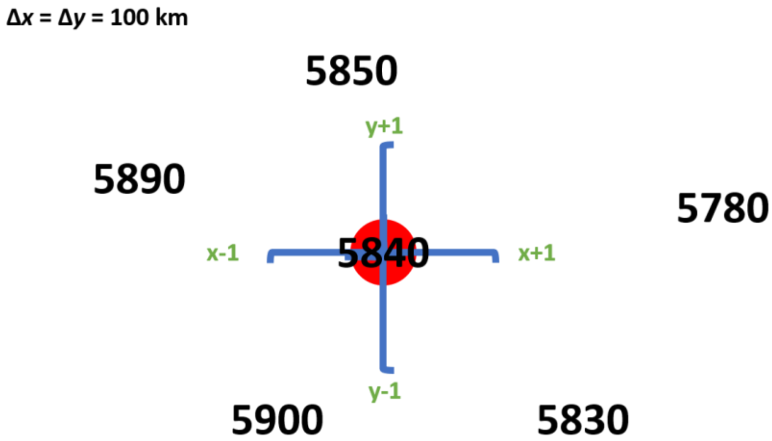


**Figure 4**. Hypothetical 500 hPa geopotential height (*z*; units: m) observations.

In this example, hypothetical 500 hPa geopotential height observations are given, and we wish to use the observations to evaluate the geostrophic wind at the location of the red dot. First, we must establish the *x*- and *y*-coordinate axes needed for the calculation; these are depicted in blue and, given the definition of the partial derivative, cover only a finite distance (100 km in each direction from the red dot).

We next interpolate between observations (or, if available, isopleths) to obtain the geopotential heights at each location. Doing so, we obtain 5865 m at *x*-1, 5820 m at *x*+1, 5885 m at *y*-1, and 5845 m at *y*+1. Assuming that we are at 38°N, where $f = 8.98 \times 10^{-5}\ s^{-1}$, we obtain the geostrophic wind as follows:

$$v_g = \frac{1}{f}\frac{\partial \Phi}{\partial x} = \frac{g}{f}\frac{z_{x+1} - z_{x-1}}{2\Delta x} = \frac{9.81 m s^{-2}}{8.98 x 10^{-5} s^{-1}}\left(\frac{5820m - 5865m}{2*100000m}\right) = \text{-24.58 m s}^{-1}$$

$$u_g = -\frac{1}{f}\frac{\partial \Phi}{\partial y} = \frac{g}{f}\frac{z_{y+1} - z_{y-1}}{2\Delta y} = -\frac{9.81 m s^{-2}}{8.98 x 10^{-5} s^{-1}}\left(\frac{5845m - 5885m}{2*100000m}\right) = \text{21.85 m s}^{-1}$$

The magnitude of the geostrophic wind is equal to $\sqrt{\left(u_g\right)^2+\left(v_g\right)^2}$ , or 32.89 m s$^{-1}$. The geostrophic wind direction is equal to $\frac{180^0}{\pi}\arctan(-u,-v)$ , or 318.37°, slightly north of due northwest (315°).

**The Ageostrophic Wind**

In defining the geostrophic relationship above, we neglected the total derivative terms on the left-hand side of (2). If we instead keep these terms (as well as the Coriolis and pressure gradient terms) while continuing to neglect the rest, we obtain:

$$\frac{Du}{Dt} = fv - \frac{1}{\rho}\frac{\partial p}{\partial x} \tag{12a}$$

$$\frac{Dv}{Dt} = -fu - \frac{1}{\rho}\frac{\partial p}{\partial y} \tag{12b}$$

However, (3) – written in terms of $u_g$ and $v_g$ – provides us expressions for the horizontal pressure gradient terms that appear in (12). If we substitute for those terms in (12), we obtain:

$$\frac{Du}{Dt} = fv - fv_g = f\left(v - v_g\right) \tag{13a}$$

$$\frac{Dv}{Dt} = -fu + fu_g = -f\left(u - u_g\right) \tag{13b}$$

The terms $v - v_g$ and $u - u_g$ define the *ageostrophic wind*, where $v - v_g = v_{ag}$ and $u - u_g = u_{ag}$.

What does (13) mean? Acceleration is tied to the ageostrophic wind, representing the departure of the total wind from the geostrophic wind. Acceleration is important on the synoptic scale primarily near jets, localized wind-speed maxima into which air accelerates and out of which air decelerates.

Because the geostrophic wind approximates the full wind to within ~10%, the ageostrophic wind is approximately one order of magnitude smaller than the geostrophic wind. Given a typical scaling of $U$ ~ 10 m s$^{-1}$, the characteristic scale for the ageostrophic wind is ~1 m s$^{-1}$.

**The Rossby Number**

As stated before, the acceleration terms (the total derivatives) are typically one order of magnitude smaller than the Coriolis and pressure gradient terms. Just how close this is to being true for any given weather situation can be assessed by determining the ratio between the characteristic scales of the acceleration and Coriolis terms. This defines the *Rossby number*, named after the famous meteorologist Carl-Gustav Rossby, and is given by:

$$Ro = \frac{\left\|\frac{D\vec{\boldsymbol{v}}}{Dt}\right\|}{\|2\Omega \times \vec{\boldsymbol{v}}\|} \Rightarrow \frac{\frac{U^2}{L}}{f_o U} = \frac{U}{f_0 L} \tag{14}$$

Note that the $U$, $f_0$, and $L$ in (14) are not the same as their values in Table 1. Rather, they represent values specific to the synoptic-scale weather conditions being assessed.

For $Ro \approx 0.1$, the magnitude of the acceleration term is one order of magnitude smaller than the magnitude of the Coriolis term. This describes geostrophic balance. For $Ro \approx 1$ or $Ro > 1$, the magnitude of the acceleration term is comparable to or exceeds the magnitude of the Coriolis term. Geostrophic balance does *not* hold in these situations, requiring us to consider other balances such as gradient wind balance or cyclostrophic balance for such scenarios.