

## Synoptic Meteorology I: Thermal Wind Balance

### For Further Reading

Sections 1.4.1 and 1.4.2 of *Midlatitude Synoptic Meteorology* by G. Lackmann derives the thermal wind relationship and relates the thermal wind to the mean temperature advection in a given vertical layer. Section 4.3 of *Mid-Latitude Atmospheric Dynamics* by J. Martin provides a derivation and discussion of thermal wind balance and is the primary source for the applications of thermal wind balance discussed herein.

### Deriving the Thermal Wind Relationship

Recall that the geostrophic relationship applicable on isobaric surfaces is given by:

$$fv = \frac{\partial\Phi}{\partial x} \quad (1a)$$

$$fu = -\frac{\partial\Phi}{\partial y} \quad (1b)$$

If we substitute  $v_g$  for  $v$  and  $u_g$  for  $u$  and divide both sides of (1) by  $f$ , we obtain:

$$v_g = \frac{1}{f} \frac{\partial\Phi}{\partial x} \quad (2a)$$

$$u_g = -\frac{1}{f} \frac{\partial\Phi}{\partial y} \quad (2b)$$

The *thermal wind* is defined as the vector difference in the geostrophic wind between two pressure levels  $p_1$  and  $p_0$ , where  $p_0$  is closer to the surface (and thus  $p_0 > p_1$ ). It is not an actual wind, but it is a useful construct that allows us to link the geostrophic wind (a kinematic field) to temperature (a mass field).

To obtain an expression for the thermal wind, we take the partial derivative of (2) with respect to  $p$  ( $\partial/\partial p$ ; i.e., a change in the geostrophic wind with height, consistent with how the thermal wind is defined) to obtain:

$$\frac{\partial v_g}{\partial p} = \frac{1}{f} \frac{\partial}{\partial p} \left[ \frac{\partial\Phi}{\partial x} \right] \quad (3a)$$

$$\frac{\partial u_g}{\partial p} = -\frac{1}{f} \frac{\partial}{\partial p} \left[ \frac{\partial\Phi}{\partial y} \right] \quad (3b)$$

If we commute the order of the partial derivatives on the right-hand sides of (3), then both equations contain a common  $\partial\Phi/\partial p$  term. The hydrostatic equation states that:

$$\frac{\partial p}{\partial z} = -\rho g \rightarrow \frac{\partial p}{\rho} = -g \partial z$$

If we plug in to the hydrostatic equation with the definition of the geopotential  $\Phi$  ( $\partial\Phi = g \partial z$ ) and the ideal gas law ( $\rho = p/R_d T_v$ ), we obtain the following expression:

$$\frac{\partial\Phi}{\partial p} = -\frac{R_d T_v}{p}$$

Substituting this into (3), we obtain:

$$\frac{\partial v_g}{\partial p} = \frac{1}{f} \frac{\partial}{\partial x} \left[ -\frac{R_d T_v}{p} \right] \quad (4a)$$

$$\frac{\partial u_g}{\partial p} = -\frac{1}{f} \frac{\partial}{\partial y} \left[ -\frac{R_d T_v}{p} \right] \quad (4b)$$

We can simplify (4) by taking  $R_d$  out of the derivatives as a constant. Furthermore, because we are using the form of the geostrophic wind applicable on isobaric surfaces,  $p$  is constant with respect to both  $x$  and  $y$ . We can thus extract  $p$  and put it into  $\partial p$ , where by definition,  $\partial p/p = \partial(\ln p)$  and thus  $p/\partial p = 1/\partial(\ln p)$ . Doing so, we obtain:

$$\frac{\partial v_g}{\partial \ln p} = -\frac{R_d}{f} \frac{\partial T_v}{\partial x} \quad (5a)$$

$$\frac{\partial u_g}{\partial \ln p} = \frac{R_d}{f} \frac{\partial T_v}{\partial y} \quad (5b)$$

(5) gives us expressions for the vertical shear of the geostrophic wind. This is pretty close to how the thermal wind is defined, as a shear also represents a change, but it's not quite in the right form. To get to this form, let's multiply both sides of (5) by  $\partial(\ln p)$  and integrate from  $p_0$  to  $p_1$ :

$$\int_{p_0}^{p_1} \partial v_g = \int_{p_0}^{p_1} -\frac{R_d}{f} \frac{\partial T_v}{\partial x} \partial \ln p \quad (6a)$$

$$\int_{p_0}^{p_1} \partial u_g = \int_{p_0}^{p_1} \frac{R_d}{f} \frac{\partial T_v}{\partial y} \partial \ln p \quad (6b)$$

We note that  $R_d$  and  $f$  are both constant with respect to  $p$  and can thus be pulled out of the integrals. Likewise, if we substitute  $\bar{T}_v$  for  $T_v$ , where  $\bar{T}_v$  is the layer-mean virtual temperature, then derivative terms with respect to  $\bar{T}_v$  can be pulled out of the integrals. This is similar to what we did to obtain the hypsometric equation.

Doing so, (6) becomes:

$$v_T \equiv v_g(p_1) - v_g(p_0) = -\frac{R_d}{f} \frac{\partial \bar{T}_v}{\partial x} (\ln(p_1) - \ln(p_0)) = \frac{R_d}{f} \frac{\partial \bar{T}_v}{\partial x} \ln\left(\frac{p_0}{p_1}\right) \quad (7a)$$

$$u_T \equiv u_g(p_1) - u_g(p_0) = \frac{R_d}{f} \frac{\partial \bar{T}_v}{\partial y} (\ln(p_1) - \ln(p_0)) = -\frac{R_d}{f} \frac{\partial \bar{T}_v}{\partial y} \ln\left(\frac{p_0}{p_1}\right) \quad (7b)$$

(7) defines the *thermal wind*. It states that the change in geostrophic wind between two isobaric levels  $p_1$  and  $p_0$  is directly related to the horizontal layer-mean (between  $p_1$  and  $p_0$ ) virtual temperature gradient. As the layer-mean virtual temperature is directly proportional to thickness, we can thus state that *the thermal wind is directly related to the horizontal thickness gradient*.

Inspecting (7), we see that the meridional component of the thermal wind is a function of the zonal gradient in the layer-mean virtual temperature. Likewise, the zonal component of the thermal wind is a function of the meridional gradient in the layer-mean virtual temperature. Compare this to (2): the meridional component of the geostrophic wind is a function of the zonal geopotential height gradient, whereas the zonal component of the geostrophic wind is a function of the meridional geopotential height gradient.

In vector form, (7) can be expressed as:

$$\vec{v}_T \equiv \vec{v}_g(p_1) - \vec{v}_g(p_0) = \left[ \frac{R_d}{f} \ln\left(\frac{p_0}{p_1}\right) \right] \hat{\mathbf{k}} \times \nabla_p \bar{T}_v \quad (8)$$

where the subscript of  $p$  on the gradient operator indicates it is computed on an isobaric surface. The cross-product in (8) means that *the thermal wind is perpendicular to the thickness contours!* We can demonstrate this by applying the right-hand rule to isoplethed meteorological data. First, point your index finger upward (the  $\hat{\mathbf{k}}$  direction). Next, point your middle finger from low toward high thickness along the horizontal thickness gradient. The resulting direction in which your thumb is pointing represents the direction in which the thermal wind is blowing towards.

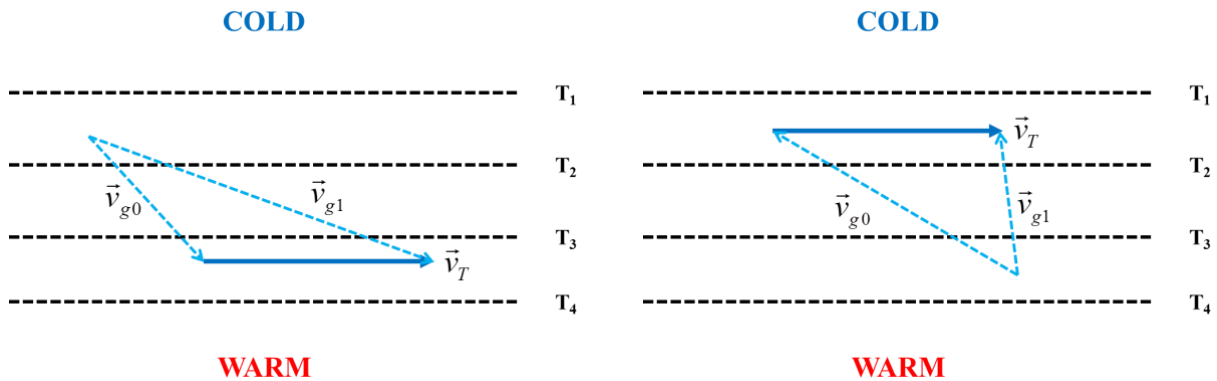
Indeed, whereas the geostrophic wind blows parallel to the isobars or isohypses, *the thermal wind “blows” parallel to lines of constant thickness!* Equivalently, we can state that the thermal wind “blows” parallel to the isotherms of layer-mean virtual temperature. Thus, if we know the thermal wind, we know how the isotherms and lines of constant thickness are spatially oriented.

The analogy to geostrophic balance can be taken one step further. Just as the Buys-Ballot law states that low pressure is to your left (in the Northern Hemisphere) when the geostrophic wind is at your back, (8) indicates that low thickness is to your left (in the Northern Hemisphere) when the thermal wind is at your back. We can also demonstrate this using the component form in (7). Consider a thermal wind directed from west to east ( $u_T > 0$ ). On the right-hand side of (7b),  $R_d$ ,  $f$ , and  $\ln(p_0/p_1)$

are all positive. As a result,  $\frac{\partial \bar{T}_v}{\partial y}$  must be negative, meaning that  $\bar{T}_v$  is colder along the positive  $y$ -axis (or to the north). This is  $90^\circ$  to the left of an observer facing downstream (here, to the east) of the thermal wind.

### Application to Horizontal Temperature Advection

If we know the geostrophic wind at two pressure levels, we can determine the thermal wind over the vertical layer between those two levels using vector subtraction. Given the thermal wind, we know how the isotherms are oriented horizontally, as described above. From this information, we can analyze how the geostrophic wind at each pressure level blows with respect to the isotherms to qualitatively assess the mean horizontal temperature advection within the vertical layer. These principles are illustrated graphically below in Fig. 1.



**Figure 1.** (left) A thermal wind  $v_T$  that is associated with cold air advection. (right) A thermal wind  $v_T$  that is associated with warm air advection. For simplicity, layer-mean temperature is depicted in place of layer-mean virtual temperature; the qualitative interpretation is unchanged between the two quantities, and the quantitative interpretation is only minimally affected. Note how the thermal wind is of identical direction and magnitude between the panels but that the sign of the horizontal temperature advection is different. This is due to how the geostrophic wind within the layer defined by the thermal wind is oriented with respect to the isotherms of layer-mean temperature.

In the left-most panel of Fig. 1, the geostrophic wind  $\vec{v}_{g0}$  at  $p_0$  (the lower level) is from the northwest. The geostrophic wind  $\vec{v}_{g1}$  at  $p_1$  (the upper level) is from the west-northwest. The geostrophic wind is turning counterclockwise with increasing distance above the ground, which we refer to as *backing* winds.

At this point, it is helpful to recall basic tenets of vector addition and subtraction. Two vectors  $\vec{A}$  and  $\vec{B}$  can be added together by placing the *beginning* of  $\vec{B}$  at the *end* of  $\vec{A}$  and drawing a vector

from the beginning of  $\vec{A}$  to the end of  $\vec{B}$ . The difference  $\vec{A} - \vec{B}$  may be viewed as  $\vec{A} + (-\vec{B})$ , where  $(-\vec{B})$  is simply  $\vec{B}$  rotated by  $180^\circ$ . Equivalently,  $\vec{A} - \vec{B}$  may be obtained by placing the *beginning* of  $\vec{B}$  at the *beginning* of  $\vec{A}$  and drawing a vector from the end of  $\vec{B}$  to the end of  $\vec{A}$ .

We now return to our examples in Fig. 1. The vector difference  $\vec{v}_{g1} - \vec{v}_{g0}$ , defining the thermal wind, is obtained by placing the two vectors at a common origin, then drawing a vector from the end of  $\vec{v}_{g0}$  to the end of  $\vec{v}_{g1}$ . By definition, the layer-mean isotherms are parallel to this vector with warm air found to the right (here, the south). Because the geostrophic wind at both  $p_0$  and  $p_1$  blows from cold toward warm air, *backing of the geostrophic wind with increasing distance above the ground is associated with cold air advection*.

Similarly, in the right-most panel of Fig. 1, the geostrophic wind  $\vec{v}_{g0}$  at  $p_0$  is from the southeast. The geostrophic wind  $\vec{v}_{g1}$  at  $p_1$  is from the south-southeast. The geostrophic wind is turning clockwise with increasing distance above the ground, which we refer to as *veering* winds. The vector difference between the two is again obtained by placing the two vectors at a common origin and drawing a vector from the end of  $\vec{v}_{g0}$  to the end of  $\vec{v}_{g1}$ . By definition, the layer-mean isotherms are parallel to this vector with warm air found to the right (again, the south). Since the geostrophic wind at both  $p_0$  and  $p_1$  blows from warm toward cold air, *veering of the geostrophic wind with increasing distance above the ground is associated with warm air advection*.

A great way to put these principles into practice is to regularly look at skew- $T/\ln p$  charts from one or more location(s). If you approximate the geostrophic wind by the total wind, how it changes with height will give you an approximate idea of whether cold or warm air advection is ongoing within a given layer at a given location without needing other data. This principle is demonstrated in the accompanying “Thermal Wind Application” notes.

There is an important caveat to keep in mind when applying these principles: the thermal wind is based upon geostrophic balance, which does not hold near the ground (because of friction) and in strongly curved flows (with features where acceleration is important; i.e., hurricanes or tornadoes). The extent to which the flow departs from geostrophic balance depends on the extent to which the flow is curved (more curvature = greater departure) or affected by friction (closer to the ground or over a rougher surface = greater departure). In these situations, insights from the thermal wind are at best only qualitatively accurate and (rarely) are at worst altogether incorrect. A simple way of estimating the impact is to use a series of isohypse analyses on the isobaric levels in question to qualitatively estimate the geostrophic wind direction (if not also speed) and compare the resulting insight to that obtained using the full wind (as is often done in practice).

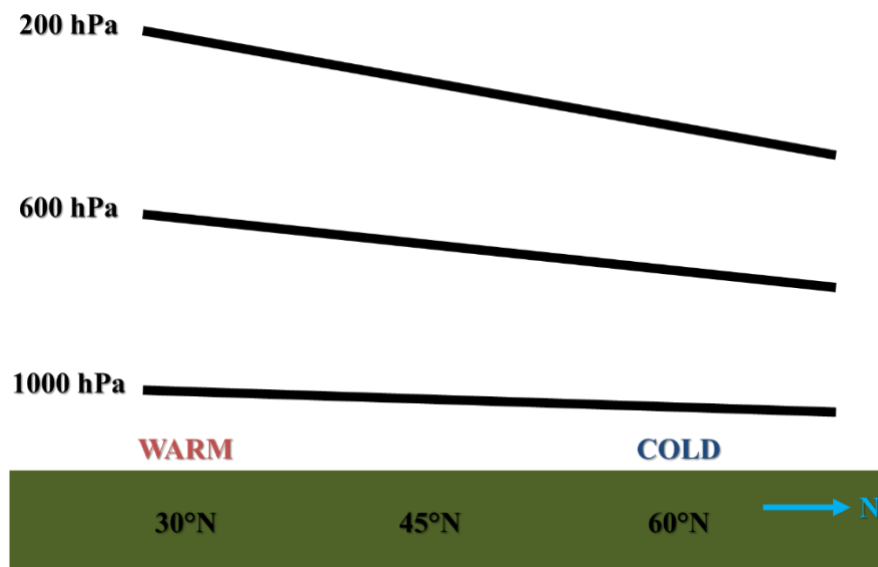
### **Further Insights from the Thermal Wind**

Under the constraint of thermal wind balance, as manifest by (7) and (8), the following are true:

- Where there is a change in geostrophic wind (speed and/or direction) with height, there must be a horizontal gradient in layer-mean temperature.
- Where there is a horizontal gradient in layer-mean temperature, there must be a change in geostrophic wind (speed and/or direction) with height.

The term *vertical wind shear* is commonly used to refer to changes in wind speed and/or direction with height. Recall from our discussion of (5) that the thermal wind can be viewed as the vertical wind shear of the geostrophic wind. Because of (7), thermal wind balance can also be viewed as the balance between vertical wind shear and horizontal gradients of layer-mean virtual temperature or thickness. In general, if one were to analyze synoptic-scale observations, one would find that this balance holds quite frequently in the midlatitudes.

We can also apply thermal-wind-related concepts to understanding Earth's general circulation, particularly in the midlatitudes. Consider that, in an annual average, latitudes closer to the Equator receive more insolation than do latitudes closer to the poles. As a result, again in an annual average, it is warmer throughout the troposphere closer to the Equator – say, at 30°N – than it is closer to the poles – say, at 60°N. This means that there exists a horizontal layer-mean temperature gradient. From the hypsometric equation, it also means that the thickness between two isobaric surfaces located near the surface (e.g., 1000 hPa) and the tropopause (e.g., 200 hPa) is greater near the Equator than near the poles. This is graphically depicted in Fig. 2.



**Figure 2.** Annually averaged vertical cross-section between 30°N and 60°N of three selected isobaric surfaces (200, 600, and 1000 hPa; black lines) and the relative layer-mean temperature between the 1000 hPa and 200 hPa isobaric surfaces at 30°N (**WARM**) and 60°N (**COLD**).

The north-south gradient in thickness – i.e., layer-mean virtual temperature, or to an approximation layer-mean temperature – means that the thermal wind for this layer is directed from west to east, such that warm air is found  $90^\circ$  to the right of an observer with the thermal wind at their back. This defines the vector difference in the geostrophic wind within the layer being considered. Note that a non-zero thermal wind defines what is known as a *baroclinic* atmosphere.

We can go a step further, however. Recall that geostrophic balance is defined as the balance between the horizontal pressure gradient force and the Coriolis force. In Fig. 2, the horizontal (here, north-south) pressure gradient is larger at higher altitudes than it is closer to the surface. This implies that the horizontal pressure gradient force is greater at higher altitudes, thus requiring a greater Coriolis force to maintain geostrophic balance. Because the magnitude of the Coriolis force is directly proportional to the horizontal wind speed, this implies a faster horizontal wind speed at higher altitudes than closer to the surface.

Furthermore, the horizontal pressure gradient is directed from north to south at all altitudes, such that the horizontal pressure gradient force is directed from south to north. The Coriolis force, under the constraint of geostrophic balance, is thus directed from north to south. Because we know that the Coriolis force is directed  $90^\circ$  to the right of the wind, we know that the geostrophic wind is directed from west to east at all altitudes. Because the horizontal wind speed – or magnitude of the geostrophic wind vector – is larger at higher altitudes, this implies westerly geostrophic winds that increase with increasing distance above ground. This is a defining characteristic of the midlatitudes and is frequently observed on the synoptic- to planetary-scales.