Truncation Error

Learning Outcomes

Following this lecture, students will be able to:

- Understand the impacts of truncation error on the quality of finite-difference approximations.

Truncation Error

Defining **truncation error** for spatial derivatives is conceptually straightforward: it is the error in approximating a partial derivative of some function *f* with finite differences that results from truncating the Taylor series expansion of *f*. Lower-order accurate approximations truncate more terms from the Taylor series expansion and use a smaller number of grid points to calculate the finite difference. As a result, lower-order accurate finite difference approximations are associated with larger truncation error than their higher-order accurate counterparts. Truncation error also exists in time, manifest as variability between individual model timesteps that is not accounted for during the model integration.

We can consider truncation error qualitatively. Consider a generic function $f(x) = \cos x$. It has a first partial derivative with respect to x that is exactly equal to $-\sin x$. In Fig. 1a, a nine-point grid with grid points every $\pi/4$ radians is used to represent this function (blue); its first derivative with respect to x (orange); and the forward (brown), backward (green), centered (yellow), and fourth-order (black) finite difference approximations to its first derivative with respect to x. As the order of accuracy increases, the match to the true solution improves and truncation error decreases. For this example, the fourth-order finite difference approximation is almost exact, and a sixth-order finite difference approximation (not shown) is an even better match to the analytic solution.

In Fig. 1b, a seventeen-point grid with grid points every $\pi/8$ radians is used to represent the same function as above. Note that the function and its first derivative with respect to x are smoother owing to the decreased grid spacing. The decreased grid spacing also has the benefit of improving each finite difference approximation's match to the analytic solution. Here, both the centered and fourth-order approximations are nearly exact matches to the analytic solution. From this, we can deduce that the truncation error is also a function of the horizontal grid spacing relative to the wavelength of the feature being represented on that grid.

(a)

(b)

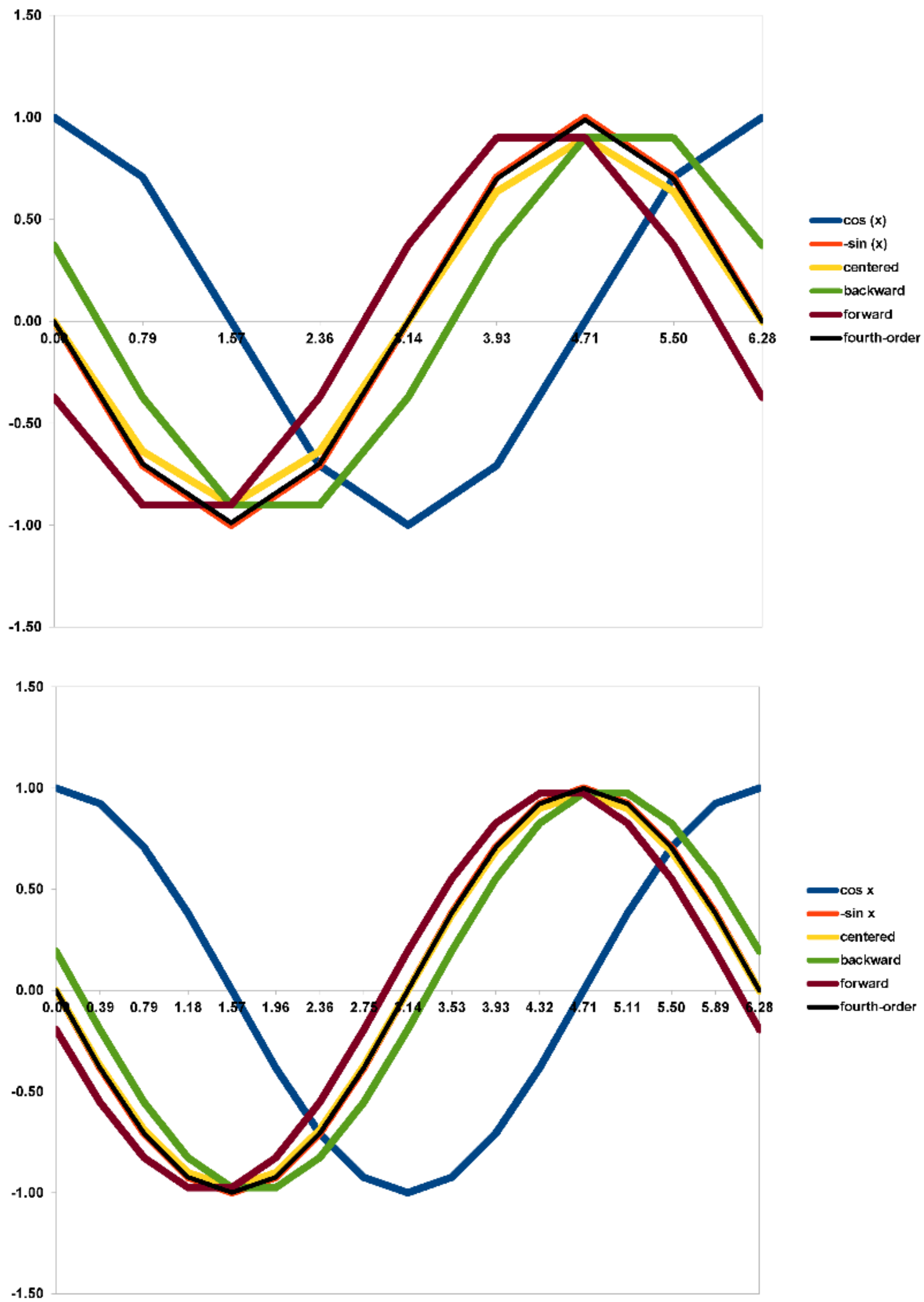

Figure 1. (a) A nine-point grid with grid points every π/4 radians is used to represent $f(x) = \cos x$ (blue); its first derivative ($-\sin x$, orange); and the forward (brown), backward (green), centered (yellow), and fourth-order (black) finite difference approximations to its first derivative. (b) As in (a), except on a seventeen-point grid with grid points every π/8 radians.

We can also consider truncation error quantitatively. For instance, let *f(x)* = *A* cos(*kx*), where *A* is amplitude, *k* is a wavenumber equal to 2π/*L*, and *L* is the wavelength. The exact value of the first partial derivative of this function with respect to *x* is *f'(x)* = –*Ak* sin(*kx*). A centered finite difference approximation to this derivative, evaluated at a generic point *x*, is given by:

$$\frac{\Delta f}{\Delta x} = \frac{A\cos\big(k(x+\Delta x)\big) - A\cos\big(k(x-\Delta x)\big)}{2\Delta x}$$

The cos terms in the approximation above can be rewritten using trigonometric identities, where $\cos(a+b) = \cos a \cos b - \sin a \sin b$ and $\cos(a-b) = \cos a \cos b + \sin a \sin b$. Doing so, we obtain:

$$\frac{\Delta f}{\Delta x} = \frac{A\big((\cos k\,x \cos k\,\Delta x - \sin k\,x \sin k\,\Delta x) - (\cos k\,x \cos k\,\Delta x + \sin k\,x \sin k\,\Delta x)\big)}{2\Delta x}$$
$$= \frac{-A \sin k\,x \sin k\,\Delta x}{\Delta x}$$

One way of computing the truncation error is to compute the ratio of the finite difference approximation to the analytic or exact solution, i.e.,

$$\frac{\Delta f / \Delta x}{\partial f / \partial x}$$

Where this ratio is approximately equal to 1, truncation error is small. Where this ratio departs from 1, truncation error is large.

Applying this to *f'(x)* and its centered finite difference approximation, we obtain:

$$\frac{\dfrac{-A \sin k\,x \sin k\,\Delta x}{\Delta x}}{-Ak \sin k\,x} = \frac{\sin k\,\Delta x}{k\Delta x}$$

Because *k* = 2π/*L*, $k\Delta x \propto \Delta x / L$. Thus, as we stated before, truncation error is a function of the horizontal grid spacing relative to the wavelength of the feature being represented on that grid.

The small angle theorem states that as sin *k*Δ*x* approaches zero, $\sin k\,\Delta x \approx k\Delta x$. Thus, when Δ*x* is small relative to the wavelength (e.g., a large number of grid points over the wavelength), sin *k*Δ*x* approaches sin (0), which is zero. Thus, for small Δ*x*, $\sin k\,\Delta x \approx k\Delta x$and the ratio between the approximate and exact solutions approaches 1. Thus, truncation error in this case is small.

The inverse is true as Δ*x* becomes large relative to the wavelength. The maximum allowable value for Δ*x* is *L*/2 (i.e., a domain with three grid points on which only a wave with wavelength 2Δ*x* may be resolved). There, $k\Delta x = 2\pi\Delta x / L = 2\pi L/2/L = \pi$, for which sin *k*Δ*x* = sin *π* = 0.

Stated equivalently, *truncation error for short wavelength features is large, whereas it is small for longer wavelength features* (for a given Δx)!

Analogous expressions to the one above can be obtained for any finite differencing scheme. For the forward, backward, and fourth-order finite difference approximations, these take the form:

$$\frac{-\cos k\Delta x}{k\Delta x \tan kx} + \frac{1}{k\Delta x \tan kx} + \frac{\sin k\Delta x}{k\Delta x} \qquad \text{(forward)}$$

$$-\frac{1}{k\Delta x \tan kx} + \frac{\cos k\Delta x}{k\Delta x \tan kx} + \frac{\sin k\Delta x}{k\Delta x} \qquad \text{(backward)}$$

$$\frac{\sin k\Delta x(4-\cos k\Delta x)}{3k\Delta x} \qquad \text{(fourth-order)}$$

All three are dependent on $k\Delta x$, or the horizontal grid spacing relative to the wavelength. The forward and backward approximations are also dependent on kx, or the position along the wave.

A graph of the ratios for the centered and fourth-order finite difference approximations is provided in Fig. 2. Depicted along the x-axis is wavenumber n, equal to the wavelength L divided by the horizontal grid spacing Δx. For smaller values of n, a feature's wavelength is small relative to the model grid spacing and thus has greater truncation error. In this sense, model resolution can be defined relative to the truncation error; e.g., the wavelength of the smallest feature that can be represented on the model grid with a minimum of truncation error (e.g., where the ratio is greater than or equal to ~0.95). For the fourth-order approximation, this appears to be ~8Δx, whereas for the centered approximation, this appears to be ~13Δx.

Furthermore, for any given n, the fourth-order finite difference approximation is associated with less truncation error than the centered approximation. Again, this is particularly evident at smaller wavelengths. Thus, the horizontal grid spacing for a given model simulation must be chosen in light not just of the smallest features desired to be resolved on the model grid but also the order of accuracy used by the finite difference approximations available in the model.

As we will see through the remainder of our discussion of numerical methods, short wavelengths – those below which the model can reasonably resolve, whether defined considering truncation error or some other means – pose a particular challenge to model accuracy and stability. Later, we will consider methods for addressing this in numerical model simulations.

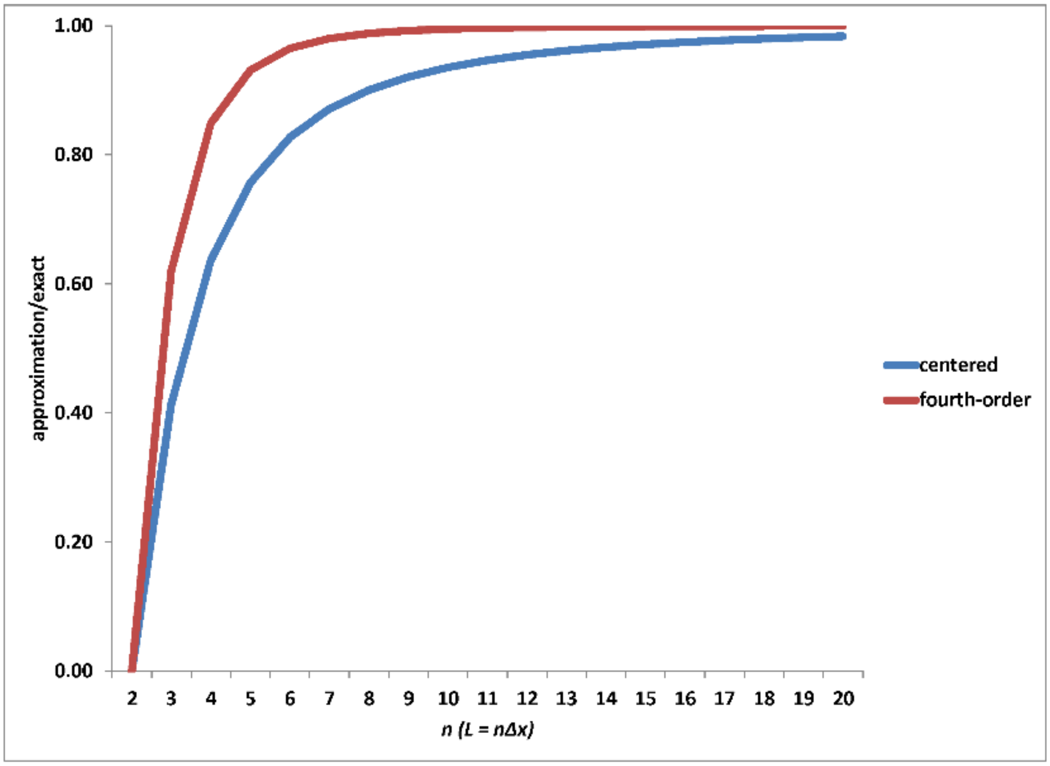

Figure 2. The ratio of the value of the finite difference approximation of the first partial derivative of $f(x) = A \cos kx$ to its exact value, as a function of the number of grid points n used to resolve the wave, for the centered (blue) and fourth-order (red) finite difference approximations. Adapted from Warner (2011), their Fig. 3.22.